Journal of Combinatorial Theory, Series B **86**, 413–419 (2002)
doi:10.1006/jctb.2002.2136

Ramsey Properties of Families of Graphs

Ronald Graham

Department of Computer Science and Engineering, UCSD, La Jolla, California 92093-0114
E-mail: graham@ucsd.edu

Tomasz Łuczak[1]

Department of Discrete Mathematics, Adam Mickiewicz University, 60-769 Poznań, Poland
E-mail: tomasz@amu.edu.pl

Vojtěch Rödl

Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322
E-mail: rodl@mathcs.emory.edu

and

Andrzej Ruciński

Department of Discrete Mathematics, Adam Mickiewicz University, 60-769 Poznań, Poland
E-mail: rucinski@amu.edu.pl

Received October 10, 2001; published online July 29, 2002

For a graph F and natural numbers $a_1, \ldots, a_r$, let $F \to (a_1, \ldots, a_r)$ denote the property that for each coloring of the edges of F with r colors, there exists i such that some copy of the complete graph K_{a_i} is colored with the ith color. Furthermore, we write $(a_1, \ldots, a_r) \to (b_1, \ldots, b_s)$ if for every F for which $F \to (a_1, \ldots, a_r)$ we have also $F \to (b_1, \ldots, b_s)$. In this note, we show that a trivial sufficient condition for the relation $(a_1, \ldots, a_r) \to (b_1, \ldots, b_s)$ is necessary as well. © 2002 Elsevier Science (USA)

1. INTRODUCTION

Following the well-known arrow notation, we write $F \to (G_1, \ldots, G_r)$, if for each coloring of the edges of F with r colors there exists $i \in [r] = \{1, \ldots, r\}$, and a copy of G_i with all the edges colored by the ith color; by $F \to (a_1, \ldots, a_r)$ we mean $F \to (K_{a_1}, \ldots, K_{a_r})$. The property is obviously

[1] To whom correspondence should be addressed.

symmetric with respect to permutations of the sequences $(G_1, \ldots, G_r)$ and $(a_1, \ldots, a_r)$. Hence, in what follows, we will rather speak of multisets of graphs (or integers).

If $F = K_a$, instead of $F \to (a_1, \ldots, a_r)$ we simply write $a \to (a_1, \ldots, a_r)$. For two multisets of integers $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ we put

$$(a_1, \ldots, a_r) \to (b_1, \ldots, b_s) \tag{1}$$

if for all graphs F such that $F \to (a_1, \ldots, a_r)$ we also have $F \to (b_1, \ldots, b_s)$. Note that $F \to (a)$ if and only if $F \supseteq K_a$, and so $(a) \to (b_1, \ldots, b_s)$ if and only if $a \to (b_1, \ldots, b_s)$. Consequently, relation (1) can be viewed as a generalization of the arrow notation $a \to (b_1, \ldots, b_s)$.

The aim of this note is to characterize those pairs of multisets of integers, $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$, for which (1) holds. (The corresponding problem for vertex colorings is solved in [2].)

Note that if for $i = 1, \ldots, r$, we have $a_i \to (b_i^{(1)}, b_i^{(2)}, \ldots, b_i^{(s_i)})$, then also

$$(a_1, \ldots, a_r) \to (b_1^{(1)}, \ldots, b_1^{(s_1)}, \ldots, b_r^{(1)}, \ldots, b_r^{(s_r)}).$$

Inspired by this observation we write

$$(a_1, \ldots, a_r) \Rightarrow (b_1, \ldots, b_s)$$

if there exists a partition $A_1 \cup \cdots \cup A_r$ of the set $\{1, \ldots, s\}$ such that for every $i = 1, \ldots, r$, we have either $A_i = \emptyset$ or $a_i \to (b_j \colon j \in A_i)$. Clearly, as we have already noticed, if $(a_1, \ldots, a_r) \Rightarrow (b_1, \ldots, b_s)$ then also $(a_1, \ldots, a_r) \to (b_1, \ldots, b_s)$. The main result of this note states that the reverse implication holds as well.

Theorem 1.1. *Let $a_1, \ldots, a_r \geqslant 2$, and $b_1, \ldots, b_s \geqslant 2$, be two multisets of integers. Then $(a_1, \ldots, a_r) \to (b_1, \ldots, b_s)$ if and only if $(a_1, \ldots, a_r) \Rightarrow (b_1, \ldots, b_s)$.*

Let us mention the following consequence of the above theorem. If we have both $(a_1, \ldots, a_r) \to (b_1, \ldots, b_s)$ and $(b_1, \ldots, b_s) \to (a_1, \ldots, a_r)$, we denote this fact by writing $(a_1, \ldots, a_r) \leftrightarrow (b_1, \ldots, b_s)$.

Corollary 1.1. *Let $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r \geqslant 3$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_s \geqslant 3$. Then $(a_1, \ldots, a_r) \leftrightarrow (b_1, \ldots, b_s)$ if and only if $r = s$ and $a_i = b_i$ for $i = 1, 2, \ldots, r$.*

Thus, the relation $(a_1, \ldots, a_r) \to (b_1, \ldots, b_s)$ between multisets of integers forms a partial order.

Example. Consider multisets (5,5) and (3,3,3). Clearly, $(3,3,3) \not\Rightarrow (5,5)$, and by Theorem 1.1 we have $(3,3,3) \not\to (5,5)$, which should not be surprising: there exist graphs F, e.g., $F = K_{17}$, such that $F \to (3,3,3)$ but $F \not\to (5,5)$. But we also have $(5,5) \not\Rightarrow (3,3,3)$, and thus Theorem 1.1 says that there exists a graph F such that $F \to (5,5)$ but $F \not\to (3,3,3)$.

2. PROOF OF THE MAIN RESULT

Let $\mathbf{G} = (G_1, \ldots, G_s)$ denote a family of edge-disjoint graphs with the same vertex set, denoted further by $V(\mathbf{G})$. We call $\mathbf{G}$ an *s-structure*. For instance, an s-coloring of the edges of a graph naturally yields such an s-structure, with some G_i possibly empty.

We say that an s-structure $\tilde{\mathbf{G}} = (\tilde{G}_1, \ldots, \tilde{G}_s)$ is a copy of an s-structure $\mathbf{G} = (G_1, \ldots, G_s)$ in an s-structure $\mathbf{F} = (F_1, \ldots, F_s)$ if $\tilde{G}_i \subseteq F_i$ for $i = 1, \ldots, s$, and there exists a bijection $f : V(\tilde{\mathbf{G}}) \to V(\mathbf{G})$ which is simultaneously an isomorphism between $\tilde{G}_i$ and G_i for all $i = 1, \ldots, s$. Furthermore, we write $\mathbf{F} \to (\mathbf{G})_r$ if any coloring of the edges of the graph $F = F_1 \cup \cdots \cup F_s$ with r colors leads to a copy $\tilde{\mathbf{G}}$ of $\mathbf{G}$ in $\mathbf{F}$ such that each $\tilde{G}_i$, $i = 1, \ldots, s$, is monochromatic (different $\tilde{G}_i$'s may be monochromatic in different colors). Finally, let $\omega(G)$ denote the clique number of G.

The following theorem is a special case of a much more general result of Nešetřil and Rödl [4].

THEOREM 2.1. *For every $r \geqslant 2$ and every s-structure $\mathbf{G}$ there exists an s-structure $\mathbf{F}$ such that $\mathbf{F} \to (\mathbf{G})_r$ and $\omega(F_i) = \omega(G_i)$ for each $i = 1, \ldots, s$.*

Proof of Theorem 1.1. As we have already observed, the fact that

$$(a_1, \ldots, a_r) \Rightarrow (b_1, \ldots, b_s)$$

implies

$$(a_1, \ldots, a_r) \to (b_1, \ldots, b_s)$$

follows directly from the definition of the relation "$\Rightarrow$". In order to prove the reverse implication, let us assume that

$$(a_1, \ldots, a_r) \not\Rightarrow (b_1, \ldots, b_s)$$

i.e., that for every partition $S = A_1 \cup \cdots \cup A_r$ of $\{1, \ldots, s\}$ there is an index i_S, $1 \leqslant i_S \leqslant r$, such that

$$a_{i_S} \not\to (b_j\colon\ j \in A_{i_S}). \tag{2}$$

Our goal is to find a graph F such that $F \to (a_1, \ldots, a_r)$ but $F \not\to (b_1, \ldots, b_s)$.

In order to do that, for every partition S as above, let $\mathbf{G}_S = (G_1^S, \ldots, G_s^S)$ denote an s-structure with $s - |A_{i_S}|$ graphs G_i^S empty, which is obtained from the complete graph $K_{a_{i_S}}$ by partitioning its edges into graphs G_j^S, $j \in A_{i_S}$, in such a way that $\omega(G_j^S) < b_j$ for all $j \in A_{i_S}$ (the existence of such a partition follows from (2)). Let $\mathbf{G} = (G_1, \ldots, G_s)$ be an s-structure which is a disjoint union of all $\mathbf{G}_S$, i.e., $V(\mathbf{G}) = \bigcup_S V(\mathbf{G}_S)$ and for each $j = 1, \ldots, s$, $G_j = \bigcup_S G_j^S$, where the summations are taken over all possible partitions S of the set $\{1, \ldots, s\}$ into r parts, with some of them possibly empty. Note that $\omega(G_j) = \max_S \omega(G_j^S) < b_j$.

Furthermore, let $\mathbf{F}$ be an s-structure such that $\mathbf{F} \to (\mathbf{G})_r$ and $\omega(F_i) = \omega(G_i)$ for $i = 1, \ldots, s$, whose existence is guaranteed by Theorem 2.1. Finally, set $F = F_1 \cup \cdots \cup F_s$.

Note that $\omega(F_j) < b_j$ for all $j = 1, \ldots, s$ and so $F \not\to (b_1, \ldots, b_s)$. Now, let $c : E(F) \to [r]$ be an arbitrary coloring of the edges of F with r colors. Since $\mathbf{F} \to (\mathbf{G})_r$, there exists a copy $\tilde{\mathbf{G}}$ of $\mathbf{G}$ in which each $\tilde{G}_j$, $j = 1, \ldots, s$, is monochromatic. Consider the partition $S' = A_1 \cup \cdots \cup A_r$ of $\{1, \ldots, s\}$ where $A_i = \{j\colon c(\tilde{G}_j) = i\}$, and the s-structure $\tilde{\mathbf{G}}_{S'} = (\tilde{G}_j^{S'} : j \in A_{i_{S'}})$ corresponds to that partition. By the definition of S', we have $c(\tilde{G}_j^{S'}) = i_{S'}$ for all $j \in A_{i_{S'}}$ and we obtain a copy of $K_{a_{i_{S'}}}$, all of whose edges are colored by color $i_{S'} \in [r]$. Consequently, $F \to (a_1, \ldots, a_r)$, and so

$$(a_1, \ldots, a_r) \not\to (b_1, \ldots, b_s).$$

This completes the proof of Theorem 1.1. ∎

Example. We will show that $(5, 5) \not\to (3, 3, 3)$, i.e., we will illustrate the proof of Theorem 1.1 for $a_1 = a_2 = 5$ and $b_1 = b_2 = b_3 = 3$. Let $S = A_1 \cup A_2$ be a partition of $\{1, 2, 3\}$. For each such partition S we need to find an index $i_S \in \{1, 2\}$, and a 3-structure $\mathbf{G}_S = (G_1^S, G_2^S, G_3^S)$, which is obtained by dividing the edges of $K_{a_{i_S}} = K_5$, such that the graphs G_j^S, $j \notin A_{i_S}$, are empty, and $\omega(G_j^S) < 3$, for all $j \in A_{i_S}$.

Clearly, it is enough to choose i_S such that $|A_{i_S}| \geqslant 2$, and as the first two of the graphs G_j^S, $j \in A_{i_S}$, take two edge-disjoint pentagons contained in K_5. Since there are eight possible partitions of $\{1, 2, 3\}$ into two sets, $\mathbf{G}$ is a disjoint union of eight 3-structures. However, in fact, we can omit identical patterns, so in this very symmetric example one can take as $\mathbf{G}$ the union of just three 3-structures: $G_1 = (C_5, C_5, \emptyset)$, $G_2 = (C_5, \emptyset, C_5)$, and $G_3 = (\emptyset, C_5, C_5)$.

When, as in the conclusion of the proof, each $\tilde{G}_i$ from a copy $\tilde{\mathbf{G}}$ of $\mathbf{G}$ is monochromatic under a 2-coloring of F, it follows by the pigeonhole principle that in at least one of the 3-structures $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3$, both copies of C_5 are colored with the same color. Consequently, there is a monochromatic K_5 in F.

Proof of Corollary 1.1. Note first that, since $a_1, \ldots, a_r \geqslant 3$, for each $i = 1, \ldots, r$,

$$(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r) \not\to (a_1, \ldots, a_r). \tag{3}$$

Indeed, take a minimal graph F with the property $F \to (a_1, \ldots, a_r)$ and let F' be obtained by deleting any edge from F. Then, clearly, $F' \not\to (a_1, \ldots, a_r)$. On the other hand, each coloring of F' with r colors, avoiding K_{a_j} in the jth color for $j = 1, \ldots, i-1,\ i+1, \ldots, r$, must use the ith color at least $\binom{a_i}{2} - 1 > 0$ times. Hence $F' \to (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r)$.

Now, let $(a_1, \ldots, a_r) \to (b_1, \ldots, b_s)$. By Theorem 1.1, we have $(a_1, \ldots, a_r) \Rightarrow (b_1, \ldots, b_s)$. Note that by (3), in the corresponding partition all sets A_i must be nonempty. Hence $s \geqslant r$ and, by symmetry, $r \leqslant s$, so that $r = s$.

Because $(a_1, \ldots, a_r) \leftrightarrow (b_1, \ldots, b_r)$, we must have $a_1 = b_1$, and so $A_1 = \{b_1\}$. Thus, $(a_2, \ldots, a_r) \leftrightarrow (b_2, \ldots, b_r)$, and the statement follows by induction. ∎

3. FINAL REMARKS AND OPEN PROBLEMS

One can generalize the arrow notation and write

$$(G_1, \ldots, G_r) \to (H_1, \ldots, H_s),$$

if for each graph F such that $F \to (G_1, \ldots, G_r)$ we also have $F \to (H_1, \ldots, H_s)$. In particular, $(G) \to (H)$ means simply that $G \supseteq H$. Let $[s] = A_1 \cup \cdots \cup A_r$. It is easy to see that if

$$(G_i) \to (H_j)_{j \in A_i}, \tag{4}$$

for all $i = 1, \ldots, r$, then also

$$(G_1, \ldots, G_r) \to (H_1, \ldots, H_s). \tag{5}$$

We call (4) a decomposition of relation (5). Theorem 1.1 states that if all graphs $G_1, \ldots, G_r$ and $H_1, \ldots, H_s$ are complete, then the relation $(G_1, \ldots, G_r) \to (H_1, \ldots, H_s)$ can be decomposed into a set of relations of type (4), where, perhaps, some $A_i = \emptyset$, i.e., some graphs G_i can be omitted.

Problem 1. Under what conditions on graphs $G_1, \ldots, G_r$ and $H_1, \ldots, H_s$, does relation (5) imply a decomposition of type (4)?

One can use a stronger version of Theorem 2.1 from [4] to extend our Theorem 1.1 to graphs which are not complete but are, in a way, "very highly connected." For instance, a result analogous to Theorem 2.1 holds for 3-chromatically connected graphs, i.e., graphs which cannot be disconnected by removing a set of vertices which induces a bipartite subgraph (see [3]). Other classes of graphs for which Theorem 1.1 clearly holds are discussed e.g., in [4, 5].

This is just one of many basic open problems related to Ramsey properties of families of graphs. We next mention another two. They may be viewed as relaxations of Problem 1.

PROBLEM 2. Under what conditions on graphs $G_1, \ldots, G_r$ and $H_1, \ldots, H_s$, does relation

$$(G_1, \ldots, G_r) \leftrightarrow (H_1, \ldots, H_s) \tag{6}$$

imply that $r = s$ and the multisets $G_1, \ldots, G_r$ and $H_1, \ldots, H_s$ are identical?

Corollary 1.1 states that the implication in Problem 2 is satisfied by complete graphs on at least three vertices. Undoubtedly, it holds for a much wider family of graphs, in particular for all "highly connected" graphs. However, it is not true for stars.

Indeed, let S_k denote the star of k rays. Then, the pigeonhole principle and Petersen's theorem (*any graph F with maximum degree* $2k$ *can be decomposed into k subgraphs of maximum degrees at most* 2) imply that

$$F \to (S_{2k_1+1}, \ldots, S_{2k_s+1})$$

if and only if $\Delta(F) > 2\sum_{i=1}^{s} k_i$. Hence, for instance,

$$(S_7, S_7, S_7, S_7) \leftrightarrow (S_9, S_9, S_9).$$

This also shows that the implication in Problem 1 cannot be true for all connected graphs.

The simplest, nontrivial case of Problems 1 and 2 is when $r = s = 2$ and $G_1 = G_2 = G$ and $H_1 = H_2 = H$. Then, the properties discussed in these problems are essentially equivalent to the implication

$$(G, G) \to (H, H) \Rightarrow G \supseteq H. \tag{7}$$

PROBLEM 3. Under what conditions on graphs G and H, does (7) hold?

As before, it is true when both G and H are complete. In fact, Theorem 2.1 implies much more: for every K_k-free graph G there is a K_k-free graph F for which $F \to (G, G)$. In other words, for $H = K_k$,

$$(G, G) \to (H) \Rightarrow G \supseteq H. \tag{8}$$

Such strong statements have been extensively studied in Ramsey Theory for numerous families of "highly connected" graphs (see [5] for many results in this direction).

However, let us emphasize that Problems 1–3 above are of a somewhat different flavor. For example, unlike (8), implication (7) can hold for graphs which are not 2-connected. Indeed, consider the "whisk graph" K_3^+ which is a "triangle with a tail," i.e., the unique graph on four vertices with the degree sequence 3221. Trivially, if $F \to (K_3, K_3)$ then F contains K_3^+, and thus (8) fails with $G = K_3$ and $H = K_3^+$. On the other hand, it is not hard to see that for $H = K_3^+$, (7) remains valid. (It follows from the fact that any graph G not containing a copy of K_3^+ is a union of vertex disjoint triangles and triangle-free components.)

Finally, we note that (7) fails when H is a star. Indeed, a result of Kurek [1] states that,

$$\inf\left\{\max_{F'\subseteq F} \frac{|E(F')|}{|V(F')|}\colon F \to (K_k, K_k)\right\} = \frac{1}{2}(R(k) - 1),$$

where $R(k)$ is the Ramsey number. Thus, the maximum degree of any graph F for which $F \to (K_k, K_k)$, grows at least exponentially with k. In particular, for $k \geqslant 3, \Delta(F) \geqslant 2k - 1$, and consequently $(K_k, K_k) \to (S_k, S_k)$. Alternatively, it follows by the local lemma that for such graphs F, their maximum degree $\Delta(F)$ grows superlinearly with k, which leads to the same conclusion.

ACKNOWLEDGMENTS

The second, third and fourth authors were supported in part by KBN Grant 2 P03A 021 17, NSF Grant 0071261, and KBN Grant 2 P03A 032 16, respectively.

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