A summary of
An upper bound for the Tuŕan number $t_3(n, 4)$,
Journal of Comb. Theory, (A), **87** (1999), 381-389.
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Turán problems deals with the inevitable occurrence of some specified structure when the edge density in a graph exceeds certain threshold. For a graph H, let $t(n, H)$ denote the Turán number of H, which is defined to be the largest integer m such that there is a graph G on n vertices and m edges which does not contain H as a subgraph. The problem of interest is to determine $t(n, H)$ for a given graph H.

The first clear theorem of this type was due to Mantel [5] in 1907 who proved that $t(n, K_3) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. This was rediscovered by Turán [7] in 1940 as a special case of his results on $t(n, K_k)$. Namely, for $n = (k-1)m + r$ for integer m, r, and with $0 \leq r < k - 1$,

$$t(n, K_k) = m^2 \binom{k-1}{2} + rm(k-2) + \binom{r}{2}.$$

One of the major open problems in Turán numbers concerns r-uniform hypergraphs (or r-graphs, for short). We denote by $t_r(n, H)$ the smallest integer m such that every r-graph on n vertices with $m + 1$ edges must contain H as a subgraph. When H is a complete graph on k vertices, we write $t_r(n, k) = t_r(n, H)$. In 1941, Turán [**?**] determined the Turán number $t_2(n, k)$ for 2-graphs and he asked the problem of determining the limit

$$\lim_{n\to\infty} \frac{t_r(n, k)}{\binom{n}{r}}$$

for $2 < r < k$. For this problem, Erdős offered \$1000 in honor of Paul Turán, (see [2] and [7]). Since 1941, the above problem has remained open, even for the first non-trivial case of $r = 3$ and $k = 4$. For small values of n, the conjectured values of 5/9 for $t_3(n, 4)$, $n \leq 13$, have been verified [6]. For the lower bound, Kostochka [4] gave several different constructions which achieve the conjectured value for $t_3(n, 4)$. For the upper bound for $t_3(n, 4)/\binom{n}{3}$, de Caen [3] gave an upper bound of $0.6213\cdots$ which is the real root of $9x^3 - 33x^2 + 46x - 18$. For the upper bound, in [1] we improve the previous bound of Giraud (unpublished, see [3]) by proving

$$\lim_{n\to\infty} \frac{t_3(n,4)}{\binom{n}{3}} \leq \frac{3+\sqrt{17}}{12} = .5936\ldots$$

The proof techniques involve using certain weight functions that are tight. There is still a considerable gap from the conjectured value.

References

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