Research on Isoperimetric inequalities

A summery of the paper:
Higher eigenvalues and isoperimetric inequalities on Riemannian manifolds and graphs,
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One of the earliest problems in geometry was the isoperimetric problem, which was considered by the ancient Greeks. The problem is to find, among all closed curves of a given length, the one which encloses the maximum area. The basic isoperimetric problem for graphs is essentially the same. Namely, remove as little of the graph as possible to separate out a subset of vertices of some desired "size". Here the size of a subset of vertices may mean the number of vertices, the number of edges, or some other appropriate measure defined on graphs. A classical isoperimetric inequality concerns the Cheeger constant, which is defined by

$$h_G \quad = \quad \min_S \frac{|E(S,\bar{S})|}{\min(\mathrm{vol}\, S\, , \mathrm{vol}\,)}$$

where the volume of S, denoted by vol S, is the sum of $\mu(v)$ for $v \in S$ for some measure μ, (e.g., for $\mu = 1$, vol S is the number of vertices in S, here we will use the measure $\mu(v) = degv$). The celebrated *Cheeger inequality* [2] states that

$$2h_G \geq \lambda_1 > \frac{h_G^2}{2}.$$

In general, isoperimetric inequalities are "global" properties that are hard to control by using "local" conditions. One recent research direction is precisely focusing on this aspect. In [4], we are able to give an isoperimetric inequality concerning subsets S that are not too large when the distance-type functions are locally "under control".

To be precise, we first give some definitions. A function $\gamma : V \times V \rightarrow \mathbb{R}^+ \cup \{0\}$ is said to be a *distance-type function* defined on a graph $G = (V, E)$ if

$$|\nabla_{x,y}\gamma_\xi| := |\gamma(\xi, x) - \gamma(\xi, y)| \leq 1$$

for any $\{x, y\} \in E$. Obviously, the graph distance satisfies the above inequality. A distance-type function is said to be *positive* if $\gamma(x, y) = 0$ if and

only if $x = y$. For a vertex $\xi \in V$, we write $\gamma_\xi(v) = \gamma(\xi, v)$. The γ-ball $B_\xi(R) = \{x \ : \ \gamma(x,\xi) \le R\}$.
Theorem [4]
We are given positive numbers, c_1, c_2 and R_0. Suppose a (weighted) graph G has a distance-type function γ and a positive distance-type function η satisfying, for any $\xi \in V$, $x, y \in \gamma$-ball $B_\xi(R_0)$.

$$\nabla_{x,y}\eta_\xi \ := \ \eta(y,\xi) - \eta(x,\xi) \le \gamma(x,\xi) + c_1$$

$$\Delta_{\eta_\xi}(x) \ := \ \frac{1}{d_x}\sum_{\substack{y \\ y\sim x}}(\eta(y,\xi) - \eta(x,\xi)) \ge c_2$$

Then, we have

$$|E(S,\bar{S})| \ge c \ (\text{vol}S)^{1-1/n}$$

for subsets S satisfying $|S| \le cR_0^n$, where $n \ge 1$ satisfies

$$n = c_2 \inf_{\substack{\xi\in V \\ x\in B_\xi(r)}} \frac{d_x}{|\{y \sim x : \gamma(y,\xi) < \gamma(x,\xi)\}|}.$$

Here, c is a constant depending only on c_1, c_2, n and R_0, and d_x denotes the degree of x.

The concept of such isoperimetric inequality orignated from the result on minimal surfaces by Bombieri, de Giorgi and Miranda [1]. Michael and Simon [5] proved a similar result for general subminfolds of $\mathbb{R}^n$. The discrete version above is more complicated (involving two distance-type functions instead of just one). It is desirable to find alternative conditions which can guarantee some range of isoperimetric conditions.

References

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