OPTIMAL SPREADING IN n-DIMENSIONAL RECTILINEAR GRID

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ABSTRACT

A disturbance spreads in a rectilinear n-dimensional grid moving from each affected point to at most one neighbor in each unit of time. The question we consider: How many points can be affected in N units of time? In this note we give the two highest degree terms in N of the answer.

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INTRODUCTION

The following question has been raised by S. T. Hedetniemi et al. [1]: A disturbance starts at the origin in an n-dimensional rectangular grid. In each unit of time it can spread from any grid point it has already reached to <u>one</u> adjacent grid point. To how many points can it spread in N units of time, and how can the spreading be arranged to maximize the extent of the disturbance?

It is the intent of this note to point out that the two terms of highest degree in N in the answer to this question can easily be deduced, and the arrangements that give them found. These do not include spreading schemes of the form suggested by Cockayne and Hedetniemi in [2] for this problem. The Cockayne and Hedetniemi arrangement is seen to be suboptimal in its second term in n dimensions for $n \geq 3$.

It is apparent that in N units of time only grid points a distance of N or less from the origin (in the "Manhattan" or L_1 metric) could ever be reached. It is easy to verify that there are $V(N,n) = \sum_{i=0}^{n} \binom{n}{i}\binom{N}{i} 2^i$ points a distance N or less from the origin in n dimensional grid. Let $S(N,n)$ denote the maximum number of points that can be reached in N units of time in n dimensional grid. The terms in $S(N,n)$, which is the answer to our problem, of highest degree in N will agree with $V(N,n)$.

RESULTS

We obtain our results through the following sequence of observations.

Remark 1: Only one grid point a distance of N from the origin can be reached in N units of time.

Proof: Let the path leading to such a grid point be $p_0, p_1, \ldots, p_N$. For each j, the spreading p_j to p_{j+1} must be the unique first spreading from p_j after its infection. The path therefore is unique.

Remark 2: The set of grid points reached in N units of time a distance at least N-k from the origin is at most $\sum_{j=0}^{k} \binom{N}{j}$ in number.

Proof: Such a point must be reached by a path experiencing at most k delays. These delays can occur in at most N time intervals. If the delay times are specified, the path is uniquely determined by the spreading rules if any path with such delays exists.

Remark 3: The number of grid points in the half space oppositely directed to the first at a distance of at least N-k from the origin which can be reached in N time units are at most $\sum_{j=0}^{k-1} \binom{N-1}{j} + \sum_{j=0}^{k-2} \binom{N-2}{j}$.

Proof: This follows as Remark 2 does. The first term represents spreading along the paths that do not use the first link. The second reflects that fact that paths using it that get into the opposite half space can experience only k-2 delays if they reach a Manhattan distance of N-k from the origin, as they must run forward and then back in the first direction.

Remark 4: The number of points a distance of k from the origin within any orthant (with all coordinates non-zero and having certain fixed signs) is $\binom{k-1}{n-1}$. There are $2^n\binom{k-1}{n-1}$ such points altogether. This is well known and will not be proven here. (It follows easily by induction on dimension.)

Remark 5: For N large, we can reach a negligible proportion of the points a distance N-n+2 or further from the origin and a proportion at most approaching 2^{-n} of those a distance N-n+1 from the origin.

Proof: The binomial coefficients $\binom{N}{k}$ for large N and fixed k are strongly increasing in k and only slightly increasing in N. In particular we have

$$\binom{N}{k} = \frac{N-k}{k}\binom{N}{k-1} = \frac{N}{k}\binom{N}{k-1}\left(1+0(N^{-1})\right)$$

$$\binom{N}{k} = \frac{N}{N-k}\binom{N-1}{k} = \binom{N-1}{k}\left(1+0(N^{-1})\right).$$

Thus, the number of reached grid points a distance at least N-X from the origin is by Remark 2 at most $\binom{N}{X}$ $(1+0(N^{-1}))$. This will be smaller than $\binom{N-X}{n-1}$ by a factor of the order of N unless X = n-1, in which case we have $\binom{N-n+1}{n-1} = \binom{N}{X}$ $(1+0(N^{-1}))$ and we can reach on the order of $\binom{N}{n-1}$ points at a distance N-n+1 from the origin.

By Remark 4, we know that this is a proportion essentially 2^{-n} of all points a distance N-n+1 from the origin, for large N.

Remark 6: Among points in the "back" half space, directed opposite to the initial spread, we can reach a negligible portion of points a distance N - n+1 from the origin, and asymptotically $\binom{N-1}{n-1}$ points a distance N-n from the origin.

Proof: This follows from Remarks 3 and 4 exactly as the last result follows from Remarks 2 and 4.

Theorem 1: An upper bound to the number of points accessible in N time units from a point is, to the first two leading orders in N,

$$\frac{1}{2} V(N-n-1,n) + \frac{1}{2} V(N-n,n) + 2 \binom{N}{n-1}(1+0(N^{-1}))$$

$$= \frac{2^n}{n!} N^n - \frac{n \cdot 2^n - 2}{(n-1)!} N^{n-1} + 0(N^{n-2}).$$

Proof: By Remarks 5 and 6, we can reach only points a distance up to N-n-1 from the origin in the "back" halfspace and a distance N-n in the "front" half space except $2\binom{N}{n-1}(1+0(N^{-1}))$ points. (We are ignoring corrections of order N^{-1} with respect to these. The bound above includes all points with any zero coordinates a distance up to N-n-1 from the origin. The number of such points that are between N-n-1 and N from the origin is of the order N^{-1} compared to the "surface" term considered, and we shall make no effort to estimate these.) These give the desired bound.

Theorem 2: We can achieve the bound just given, to leading two orders of N.

Proof: We use the following construction. A spreading can be described by giving the rules for the spread at each point reached by the disturbance. We do this after ordering the axes of the grid from 1 to n, by describing the non-delay and first delay paths from each point by the following four rules (0. - 3.). We describe the first three rules.

0. The spread from the origin is first to (1,0,...,0), then to (-1,0,...,0).

1. The non-delay spread from a point with components $(a_1,a_2,\ldots,a_{j-1},a_j,0,0,\ldots,0)$ for $j \leq n$ with all $a_i \neq 0$ is to $(a_1,a_2,\ldots,a_{j-1},a_j+(-1)^{a_{j-1}},0,\ldots,0)$ with $a_0 = 0$ when $a_1 > 0$, $a_0 = 1$ when $a_1 < 0$.
2. The single delay spread from such point for $j < n$ is $(a_1,a_2,\ldots,a_j,(-1)^{a_j},0,\ldots,0)$.

These rules have the following meaning. The first merely tells us how to start - go first positively then negatively in the first direction. The second tells us that when we reach any new point we move first so as to increase the magnitude of the last non-zero component. The third tells us that the first delayed spread from points whose last component is zero is to move in the first direction whose component was zero in a direction determined by the parity of the previous component.

These rules so far are incomplete in that they leave second delayed spreads unspecified, and do not specify even first delayed spreads when the last component is non-zero. Our final rule will specify some of the latter. Before describing it we notice what is accomplished by the rules so far described.

We will concentrate our attention on points with all $|a_i|$ at least two, since: (a) the number of other points within a distance N of the origin is only of order

N^{-1} in comparison with $V(N,n)$; and (b) by obvious second delay spreadings we can include all but a proportion $c\ N^{-1}$ of the points in our spreading. These two facts imply that, as far as the first two orders in N are concerned we need not worry about points with any $|a_j|$ one or less. We could attempt to treat these points optimally by defining explicit second delay spreads or by altering the scheme just described. It is not clear, however, that even the rules given so far are optimal to within the next order.

Within any orthant, the rules above provide spreading to only a proportion of roughly $2^{-(n-1)}$ of the points. This is so since any points reached whose parity in any of the first n-1 components differ will have been reached by paths that turned differently and so will be in different orthants. (For example in the "positive" orthant with all $a_j > 0$, only those points whose first n-1 components are all even will be reached.) We note that $2\binom{N}{n-1}$ points a distance N-n+1 from the origin are reached.

3. The final spreading rule is as follows. Construct $2^{n-1} - 1$ paths in the grid hyperplane with $a_n = 0$, leading from the origin to the $2^{n-1} - 1$ points having components 0 or 1 with at least one 1. Each of these paths should be increasing in each component, and should change one component at each step; (the magnitude of the Manhattan distance to the origin must

increase by one at each step). It is very easy to find such paths and we have no need of specifying any particular set of them. We order them so that the larger ones are first. Let them be $p_1, \ldots, p_{2^{n-1}-1}$, in some such order.

We now represent the single delay spreads for $1 \leq |a_n| \leq 2^{n-1} - 1$.

If $|a_n| = j$, the single delay spread is to proceed "along the first arc of p_j" (defined below) keeping a_n fixed. From the point thus reached one continues without further delay to the end of p_j and then turns again without delay to the direction of increasing $|a_n|$. Moving along p_j here means changing the same component that changes in an arc of p_j, but always moving in a direction away from the origin.

To see what is accomplished by this rule, recall that when $a_n \neq 0$, the spreading according to the first rules moves along some but not all lines of increasing $|a_n|$ in each orthant. The effect of the last step is (with one more delay) first to spread the disturbance along the $\{p_j\}$ to all values of magnitude two or more of the first n-1 components and then to spread in the nth direction (parallel to the first mentioned spread).

It is evident that for $a_n \geq 2^{n-1} - 1$ we will reach all points with $a_1 > 0$, $|a_j| \geq 2$ and $\Sigma|a_j| \leq N-n$ by these rules.

It is straightforward to define second delay rules, etc., to pick up those points with $a_n \geq 2^{n-1} - 1$ and those with $|a_j| \leq 1$ that are omitted here, except for those having $\Sigma|a_j|$ near N-n. The latter represent a second order set (of order N^{-2} with respect to the leading term) which we will ignore here.

The "back" halfspace spreading is identical to that in front with one additional delay unit.

It is therefore evident that this construction achieves the bound in its first two orders.

It does not seem impossible to extend this result to the next order. As already noted, the spreading rules above (particularly 2) may not be optimal for that order. We chose them here to simplify description of the rule 3.

FURTHER COMMENTS

In two dimensions we can evaluate both the upper and lower bounds leading to Theorems 1 and 2 explicitly, and can extend them without much difficulty to obtain the exact result,

$$S(N,2) = 2N^2 - 6N+8 \text{ for } N \geq 2.$$

The exact form for S(N,3) can surely be obtained by considering the various corrections to the higher two orders. It is questionable that the exact result is of sufficient interest to justify the effort required to get it.

It is also of interest to know whether we can reach S(N,n) points in N units of time by a "greedy algorithm", i.e., whether it exists a scheme such that we can reach S(M,n) points in M units of time for any $M \leq N$.

We suspect that for sufficiently large N, S(N,n) is a polynomial in N for any given n. We have not, however, been able to prove this.

REFERENCES

1. A. Farley, S. Hedetniemi, and E. J. Cockayne, Broadcasting in grid graphs, Proc. of Ninth Southeast Conference on Combinatorics, Graph Theory and Computing 1978.

2. E. J. Cockayne and S. Hedetniemi, A conjecture concerning broadcasting in m-dimensional grid graphs, preprint 1978.