ON TRIANGULAR AND CYCLIC RAMSEY NUMBERS WITH k COLORS

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ABSTRACT

Define $r(G;k)$ to be the smallest integer with the following property: For any $n \geq r(G;k)$, color the edges of K_n in k colors, then there exists a monochromatic graph isomorphic to G. In this paper, we discussed the bounds for $r(K_3;k)$ and $r(C_4;k)$.

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Let G be a finite graph and k be a positive integer. Define $r(G;k)$ to be the smallest integer with the following property: For any $n \geq r(G;k)$, color the edges of K_n in k colors; then there exists a monochromatic graph isomorphic to G. The existence of $r(G;k)$ is assured by Ramsey's theorem [1,2].

In the case of $G = K_3$ and $k = 2$, $r(K_3;2) = 6$. This is one of the most interesting fundamental problems that appeared in Putnam Mathematics Competition [3] in 1953. The problem can be stated as follows: Color the edges of K_6 in red or blue; then either a red triangle or a blue triangle exists.

In 1955 Greenwood and Gleason [4] proved that $r(K_3;3) = 17$ and $r(K_3;4) > 41$. The value of $r(K_3;4)$ is still unknown. Whitehead and Taylor [5] proved that $r(K_3;4) > 49$ in 1971. In 1972, G. J. Porter (unpublished) and the author [6] proved independently that $r(K_3;4) > 50$ and a lower bound for $r(K_3;k)$ was obtained. A simpler proof will now be presented.

<u>Theorem 1</u>. Let $f(k) = r(K_3;k) - 1$ and let $t = 0.103 \ldots$ be the only positive root of $x^3 + 6x^2 + 9x - 1 = 0$ and $C = 50t^2 = 0.5454 \ldots$. Then $f(k+1) \geq 3f(k) + f(k-2)$ for $k > 3$ and $f(k) \geq (3+t)^k C$.

We need the following lemma.

<u>Lemma 1</u>: The edges of K_n can be colored in k colors without any monochromatic triangle if and only if its adjacency matrix A_n is the sum of k symmetric binary matrices $M_1, M_2, \ldots, M_k$ where the componentwise product of M_i and M_i^2 is zero for $i = 1,2,\ldots,k$, i.e., $A_n = M_1 + M_2 + \ldots + M_k$ and $(M_1 * M_i^2)_{jm} = (M_i)_{jm}(M_i^2)_{jm} = 0$ for $i = 1,2,\ldots,k$.

<u>Proof</u>: If the edges of K_n are colored in k colors without monochromatic triangles, then define

$$(M_i)_{jm} = \begin{cases} 1 & \text{if the edge } (jm) \text{ has color } i \\ 0 & \text{otherwise} \end{cases}$$

Obviously $(M_i^2)_{jm}$ is the number of paths of length 2 joining points j and m. But $(M_i)_{jm}$ should be zero when (M_i^2) is non-zero. Hence $A_n = M_1 + M_2 + \dots + M_k$ and $M_i * M_i^2 = 0$ for all i.

Conversely, given k symmetric binary matrices $M_1, M_2, \dots, M_k$ with their sum A_n and $M_i * M_i^2 = 0$ for $i = 1,2,\dots,k$, we have a k-coloring of K_n without any monochromatic triangles.

Proof of theorem. The edges of $K_{f(k)}$ can be colored in k colors withoug monochromatic triangles. By Lemma 1 there exist $M_1, M_2, \dots, M_k$ and $N_1, N_2, \dots, N_{k-2}$ such that

$$A_{f(k)} = M_1 + M_2 + \dots + M_k \quad \text{and} \quad M_i * M_i^2 = 0 \quad \text{for} \quad i = 1,2,\dots,k$$

$$A_{f(k-2)} = N_1 + N_2 + \dots + N_{k-2} \quad \text{and} \quad N_j * N_j^2 = 0 \quad \text{for} \quad j = 1,2,\dots,k-2$$

and let J be the f(k-2) x f(k) matrix with all entries 1.

Let $L_1, L_2, \dots, L_{k+1}$ be square matrices of order $(3f(k) + f(k-2))$; then symmetric matrices are defined as follows:

$$L_1 = \begin{bmatrix} 0 & & & \\ M_2 & M_2 & & \\ M_1 & I & M_1 & \\ J & 0 & 0 & 0 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} M_1 & & & \\ M_1 & 0 & & \\ I & M_2 & M_2 & \\ 0 & J & 0 & 0 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} M_2 & & & \\ I & M_1 & & \\ M_2 & M_1 & 0 & \\ 0 & 0 & J & 0 \end{bmatrix}$$

$$L_i = \begin{bmatrix} M_i & & & \\ M_i & M_i & & \\ M_i & M_i & M_i & \\ 0 & 0 & 0 & N_{i-3} \end{bmatrix} \quad \text{for } i = 4,5,\dots,k+1.$$

It is clear that $L_1 + L_2 + \dots + L_{k+1} = A_{3f(k) + f(k-2)}$ and $L_i * L_i^2 = 0$ for $i = 1,2,\dots,k+1$.

Since the complete graph $K_{3f(k)+f(k-2)}$ can be colored in k+1 colors without any monochromatic triangle,

$$f(k+1) \geq 3f(k) + f(k-2) \quad \text{for} \quad k \geq 3 .$$

From the above inequality we can get $f(k) \geq (3+t)^k C$ where $t = 0.103...$ is the only positive root of $x^3 + 6x^2 + 9x - 1 = 0$ and $C = 50t^2 = 0.5454...$

The classical upper bound [4] for $r(K_3;k)$ is $[k!e] + 1$. Whitehead [5] proved $r(K_3;4) \leq 65$ $[4!e] + 1$. Combining these, we get the next inequality.

Theorem 2. $r(K_3;k) \leq [k!(e-1/24)] + 1.$

From Theorems 1 and 2, we know that the limit of k'th root of $f(k)$ will be between $3+t$ and ∞ if it exists.

Lemma 2. $f(jk) \geq (f(k))^j$.

Proof. Let $f(k) = n$ so that the edges of K_n can be k-colored without any monochromatic triangles.

Define K_{nj} with vertices the vectors $(i_1,i_2,\ldots,i_j)$, $i_s = 1,2,\ldots,n$. Let c_s, $s = 1,\ldots,j, m=1,\ldots,k$, be the jk colors available.

The edge joining $(i_1,i_2,\ldots,i_j)$ and $(i_1', i_2', \ldots, i_j')$ is colored in the color c_{jm} if and only if $i_1 = i_1', \ldots, i_{j-1} = i_j', i_j \neq i_j'$ and the edge joining i_j and i_j' has color m.

It is clear that this gives a coloring of edges of K_{nj} without any monochromatic triangle in kj colors.

Therefore $$f(jk) \geq (f(k))^j.$$

Theorem 3. $\lim_{k\to\infty} (f(k))^{1/k}$ exists.

Proof: Let $x = \limsup (f(k))^{1/k}$

There exists an integer m such that $f(m)^{1/m} > x-\varepsilon$

For any $n \geq m/\varepsilon$, $f(n)^{1/n} \geq f(m[n/m])^{1/n}$

$$\geq f(m)^{[n/m]/n}$$

$$\geq (x-\varepsilon)^{(1-\varepsilon)}$$

Hence $\liminf f(k)^{1/k} = \limsup f(k)^{1/k} \geq 3.103...$

<u>Theorem 4</u>. Let $r(K_m;k)$ be the classical Ramsey number $N(\underbrace{m,m,\ldots,m}_{k\text{ times}};2)$. Then $\lim r(K_m;k)^{1/k}$ exists for any m and is greater than $m-1$.

<u>Proof</u>: By a similar method we can prove $r(K_m;kj)^{1/kj} \geq r(K_m;k)^{1/k}$ and the limit exists.

Let $\xi_m = \lim r(K_m;k)^{1/k}$. Then $\xi_3 = 3.103\ldots$ It is not known that ξ_3 is finite or infinite. It was shown in [7] that $\xi_4 \geq \sqrt{17}$, $\xi_5 \geq \sqrt{37}$, $\xi_6 \geq \sqrt{101}$, $\xi_7 \geq \sqrt{109}$, $\xi_8 \geq \sqrt{281}$, $\xi_9 \geq \sqrt{373}$ and ξ_m is strictly increasing.

Some upper and lower bounds for $r(C_4,k)$ have been obtained.

<u>Lemma 3</u>. The edges of K_n can be colored in k colors without any monochromatic triangle if and only if the matrix A_n is the sum of k symmetric binary matrices $M_1,M_2,\ldots,M_k$ where $(M_i^{\,2})_{jm} \leq 1$ for $j \neq m$ $i = 1,2,\ldots,k$.

The proof of Lemma 3 is clear.

<u>Lemma 4</u>. Let M be an $n \times n$ symmetric binary matrix and $(M^2)_{jm} \leq 1$ for $j \neq m$. Then $S = \sum_{i,j=1}^{n} M_{ij} \leq n\sqrt{n-3/4} + n/2$.

<u>Proof</u>: $\sum_{j=1}^{n} M_{ij}M_{jk} \leq 1$ $(i \neq k)$

Sum over $k = 1,\ldots,n$, $k \neq j$, to get

$$\sum_{j=1}^{n} M_{ij} \sum_{\substack{k=1\\k\neq 0}}^{n} M_{jk} \leq n-1 ,$$

or

$$\sum_{j=1}^{n} M_{ij}(r(i)-M_{ji}) \leq n-1,$$

where $r(i)$ is the i'th column sum or row sum.

Then sum over j, to get $\sum_{j=1}^{n} r(i)^2 - \sum_{i,j=1}^{n} M_{ij} \leq n(n-1)$,

$$\sum_{i=1}^{n} r(i)^2 \leq n(n-1) + S .$$

For any positive numbers $r(1), r(2), \ldots, r(n)$,

$$\sum_{i=1}^{n} r(i)^2 \geq \left(\sum_{i=1}^{n} r(i)\right)^{2/n} .$$

So $$S^2/n \leq n(n-1) + S$$

and $$S \leq n/2 + n\sqrt{n-3/4}\,.$$

The equality holds when all the $r(i)$ are equal.

Theorem 5. $k^2 + k + 1 \geq r(C_4;k)$.

Proof. Let $r(C_4;k) - 1 = n$. By Lemma 3 we know that $A_n = \sum_{i=1}^{n} M_i$ and $(M_i)_{jm} \leq 1$ for $j \neq m$, $i = 1,2,\ldots, k$. There is some M_i with the property that

$$\sum_{j,m=1}^{n} (M_i)_{jm} \geq n(n-1)/k\,.$$

By Lemma 4, we have $n/2 + n\sqrt{n-3/4} \geq n(n-1)/k$.

Then $k^2 + k + 1 \geq n$.

The equality holds when the row sums of M_i are all equal to $k+1$. In this case M_i is the adjacency matrix of a projective plane. But there does not exist [8] an adjacency matrix of a projective plane of trace 0.

Hence $$k^2 + k + 1 > n$$

and $$k^2 + k + 1 \geq r(C_4;k).$$

Theorem 6. $r(C_4;k) \geq k^2/16$ for infinitely many k's.

The proof is established by an explicit construction.

After the conference the author proved that $r(C_4;(1+\varepsilon)k \geq k^2$ for any small ε and large k and that $r(C_4;k)$ is asymptotically equal to k^2.

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