ROTATABLE GRACEFUL GRAPHS

by

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Abstract

Consider a graph with n vertices and m edges and a labeling of the vertices by a set N of n non-negative integers. Suppose the number a_i labels the vertex v_i, $i = 1,\ldots,n$. Let $|a_i-a_j|$ be the *edge number* of the edge $\{v_i,v_j\}$ and M the collection of the m edge numbers. If M is the set $\{1,\ldots,m\}$ and N is a subset of $\{0,1,\ldots,m\}$, the labeling is called a *graceful* labeling by Golomb. He also defines a graceful graph to be a graph with a graceful labeling.

Methods have been given which construct graceful graphs from certain smaller graceful graphs. In this type of construction, the vertex which is labeled by the smallest number (we might as well assume it to be zero) in a graceful labeling is often of special interest. A graceful graph will be called a rotatable graceful graph if for every vertex v there exists a graceful labeling in which v is labeled by the number zero.

In this paper, we give a construction of rotatable graceful graphs from smaller rotatable graceful graphs. We also prove that caterpillars, a type of graph which have been well studied in the literature, are rotatable graceful graphs provided that the caterpillar has the same number of toes at each foot.

1. *Introduction.*

Consider a graph with n vertices and m edges and a labeling of the vertices by a set N of n non-negative integers. Suppose the number a_i labels the vertex v_i, $i = 1,\ldots,n$. Let $|a_i-a_j|$ be the *edge number* of the edge $\{v_i,v_j\}$ and M the collection of the m edge numbers. If M is the set $\{1,\ldots,m\}$ and N is a subset of $\{0,1,\ldots,m\}$, the labeling is called a *graceful* labeling by Golomb [4]. He also defines a graceful graph to be a graph with a graceful labeling. A graceful labeling is also called a β-valuation of the graph by Rosa [7].

When the graph is a tree with n vertices, the set of labels N may be assumed to be the set $\{0,1,\ldots,n-1\}$. A conjecture attributed to Ringel [6] says that all trees are graceful. This conjecture is still open though many special cases have been investigated [1,2,3,5,9].

Methods have been given which construct graceful graphs from certain smaller graceful graphs. In this type of construction, the vertex which is labeled by the smallest (or the largest) number in a graceful labeling is often of special interest. For example, Stanton and Zarnke [9] prove the following two theorems.

THEOREM 1. *Let* S *be a graceful graph and let* $L^S:\{v_i\} \to \{a_i\}$ *be a graceful labeling of* S. *Then the graph obtained from* S *by attaching* k *edges at the vertex* p *with* $L^S(p) = 0$ *is also a graceful graph (assume without loss of generality that zero is a label).*

THEOREM 2. *Let* S *and* T *be two graceful trees and let* $\{L^S(a)\}$ *and* $\{L^T(a)\}$ *be the graceful labelings of* S *and* T. *Then the graph obtained by attaching a copy of* T *to every vertex of* S *except the one labeled by the largest value in* $\{L^S(a)\}$ *is also a graceful graph. However the vertex of* T *at which the attachment is made has to be the vertex* p *with* $L^T(p) = 0$.

To take full advantage of this kind of theorem, it is desirable to know all the vertices on a graph which can be labeled by the number zero, hence also by the largest number, in a graceful labeling. In this repect, a graceful graph will be called a *rotatable graceful graph* if for every vertex v there exists a graceful labeling in which v is labeled by the number zero.

In this note we first prove some useful properties of rotatable graceful graphs by generalizing some results of Stanton and Zarnke. Then we prove by construction that "caterpillar" graphs, a type of graph which has been studied previously in graceful labeling, are rotatable graceful graphs provided the caterpillar has the same number of toes at each foot.

2. *Some Useful Properties of Rotatable Graceful Graphs.*

Let T be a graceful graph with n vertices. Consider a graceful labeling: $a \to L^T(a)$. Let $d(a,b)$ denote the length of the shortest path between two vertices a and b in T. Let $T_1,\ldots,T_r$ be r copies of T with the following labeling:

(i) Select a fixed vertex z in T.

(ii) Use $L_i(a)$ to designate the label of the copy in T_i of the vertex a of T. Define

$$L_i(a) = (r+1-i)n-1 - L^T(a) \quad \text{if } d(a,z) \text{ is odd,}$$

$$L_i(a) = in - 1 - L^T(a) \quad \text{if } d(a,z) \text{ is even.}$$

Stanton and Zarnke [9] give the following two construction for graceful graphs.

Construction 1. Let S be a graceful tree with r vertices and let the vertex a be labeled by $L^S(a)$. Relabel S by the rule

$$L^S(a) \leftarrow nL^S(a) + m \quad \text{where } 0 \le m \le n-1.$$

Attach to vertex a of S the tree containing vertex b of T_i, where $L^T(b) = L^S(a)$, and the attachment is at vertex b. Then the resultant tree is gracefully labeled.

Construction 2. Let S be a graceful tree with $r+1$ vertices and let the vertex a be labeled by $L^S(a)$. Relabel S by the rule

$$L^S(a) \leftarrow nL^S(a).$$

Attach T_i to S just as in Construction 1. Then the resultant tree is gracefully labeled.

Construction 2 is particularly useful since repeated applications of it yield a graceful labeling for any rooted tree whose nodes at the same level have the same outdegree.

Note that zero is a label of the tree T_1 or T_r. Furthermore, for every vertex a in T, either

$$L_1(a) > L_i(a) > L_r(a)$$

or

$$L_r(a) > L_i(a) > L_1(a)$$

for every $i = 2,\ldots,r-1$. This means that when we attach r copies of T to the r vertices of S, with the attachment made at the vertex a of T, then $L_1(a)$ and $L_r(a)$ are the smallest and the largest labels in S. This observation is crucial to the proof of Theorem 3.

THEOREM 3. *Let* S *be a rotatable graceful tree with* r *vertices and* T *a rotatable graceful tree with* n *vertices. Then the tree* R *obtained by attaching a copy of* T, *at a fixed vertex of* T, *to every vertex of* S *is also a rotatable graceful graph.*

Proof. Let q be an arbitrary vertex of R. We show that a graceful labeling exists in which q is labeled by the value zero.

Since every vertex of R is on a copy of T, we assume q is on the copy of T which is attached to the vertex p of S (p can be q). Furthermore assume that the positions of p and q correspond to the vertices P and Q in T. Let $\{L^T(a)\}$ be a graceful labeling on T such that $L^T(q) = 0$. Let $\{L^S(a)\}$ be a graceful labeling on S such that

$$L^S(p) = 0 \quad \text{if} \quad d(P,Q) \text{ is even in } T,$$

$$L^S(p) = r-1 \quad \text{if} \quad d(P,Q) \text{ is odd in } T.$$

Such $\{L^T(a)\}$ and $\{L^S(a)\}$ exist since T and S are rotatable graceful graphs. Using Construction 1 of Stanton and Zarnke, we obtain a graceful labeling for R.

3. *Some Background on Labeling Caterpillars.*

A *caterpillar* is a graph where all vertices of degree greater that one lie on a chain. Let $\{U_0, U_1, \ldots, U_{f+1}\}$ be a longest chain of the caterpillar. Then U_0 and U_{f+1} are of degree one and $U_1, \ldots, U_f$ are of degree greater than one. U_0 and U_{f+1} will be referred to as the *head* and *tail* of the caterpillar and $U_1, \ldots, U_f$ as the f *feet*. All other vertices, which will be referred to as *toes*, are of degree one and are connected to feet. It is well known [7] that a caterpillar is a graceful graph (it also follows from Theroem 1).

In the case that every foot has the same number t of toes, the caterpillar will be called a *t-toe caterpillar*. Note that every foot is a star graph which can be easily shown to be a rotatable graceful graph. Furthermore, Rosa [8] proves that a chain of any length is a rotatable graceful graph. Therefore from Theorem 3, we know a t-toe caterpillar without its head and tail is a rotatable graceful graph. But how about the t-toe caterpillar with both its head and tail intact? We now show that this species also belongs to the genus of rotatable graceful graphs.

For a $t \ge 1$, consider a t-toe caterpillar with f feet. Suppose the number zero is assigned to the head of the caterpillar. Then a graceful labeling readily exists by Rosa's construction which is valid for general caterpillars: Let F_i denote the set of toes con-

nected to U_i and $L(F_i)$ the set of labels on F_i. Then Rosa's construction can be described by

$$L(U_j) = \begin{cases} \frac{j}{2}(t+1) & \text{for } j \text{ even,} \\ (f - \frac{j-1}{2})(t+1) + 1 & \text{for } j \text{ odd;} \end{cases}$$

$$L(F_j) = \begin{cases} \{(f - \frac{j}{2})(t+1)+2, \ldots, (f - \frac{j}{2})(t+1)+t+1\} & \text{for } j \text{ even,} \\ \{\frac{j-1}{2}(t+1)+1, \ldots, \frac{j-1}{2}(t+1)+t\} & \text{for } j \text{ odd.} \end{cases}$$

For example, for $t = 2$ and $f = 6$, Rosa's construction yields the labeling in Figure 1.

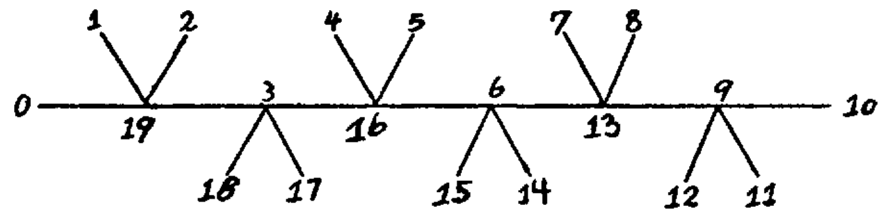

Figure 1

Now suppose we want to assign the number zero to the vertex U_2. We use the same principle shown in Figure 1 for labeling except we start at U_2 and proceed towards U_0. After we finish with the left end we pick up U_3 and proceed toward U_{f+1}. The resultant labeling is shown in Figure 2.

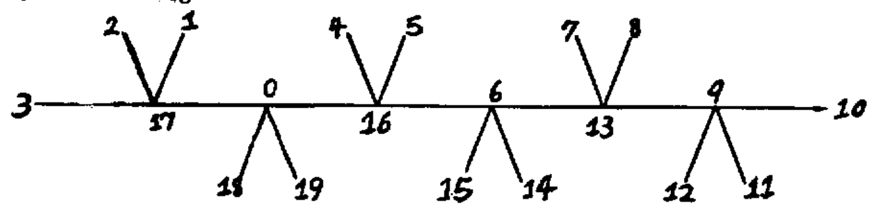

Figure 2

Note that the labeling in Figure 2 is not really graceful since both the edges {17,1} and {16,0} yield the number 16. Furthermore, the edge {17,3} yields the number 14 and the edge {16,4} yields the number 12 with the number 13 being skipped. However, a simple modification of the above scheme will remedy both discrepancies simultaneously. This is done by leaving out the number 1 during the labeling process so to avoid the duplication and then insert it somewhere else to produce the number being skipped. (See Figure 3).

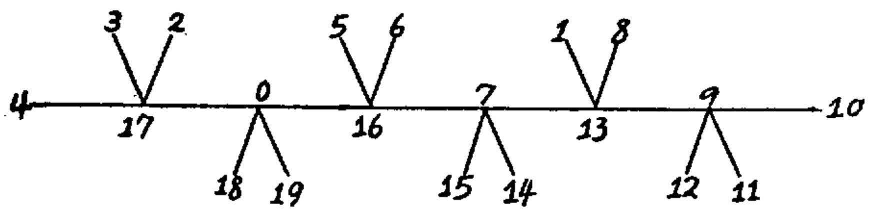

Figure 3

For another example, suppose the number zero is assigned to U_3. Figure 4 gives a graceful labeling.

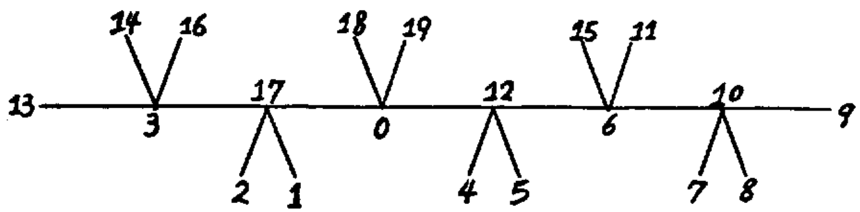

Figure 4

Again the graceful labeling is obtained by taking the number 15 out of its order and inserting it somewhere else. Also note that U_{i+1} (assuming U_i is labeled by zero) is always labeled by the largest unassigned number (excluding the number taken out of order).

In the following section, we show that such a simple scheme works for all f, all t and all i. Note that in this scheme, if the number zero is assigned to U_i, then the largest number, i.e., $n - 1$,

is always assigned to a toe in the set F_i. Therefore by taking a complimentary labeling, i.e., replacing $L(v_i)$ by $n - 1 - L(v_i)$, we can obtain a graceful labeling with the number zero assigned to any toe in F_i for any i.

4. A formal prescription for construction

We give a graceful labeling for a t-toe (f-feet) caterpillar with the number zero assigned to the vertex U_i. Define $k = t+1$. Then the caterpillar has $n = fk+2$ vertices. (We use the convention that the set of consecutive integers from a to b, denoted by $\{a,\ldots,b\}$, is the empty set if $a > b$). Without loss of generality we may assume $i \le f/2$.

Case (i). $i = 4W < n$, $W = 1,2,\ldots$.

For j odd:

$$L(U_j) = \begin{cases} (f - 2W + \frac{j-1}{2})k + 1, & j < 2W, \\ (f - 2W + \frac{j-1}{2})k + 2, & 2W < j < 4W, \\ (f - \frac{j-1}{2})k, & 4W < j < 6W, \\ (f - \frac{j-1}{2})k + 1, & 6W < j; \end{cases}$$

$$L(F_j) = \begin{cases} \{(2W - \frac{j-1}{2} - 1)k + 1,\ldots,(2W - \frac{j-1}{2})k - 1\}, & j < 4W, \\ \{\frac{j-1}{2}k + 1,\ldots,\frac{j+1}{2}k - 1\}, & 4W < j. \end{cases}$$

For j even:

$$L(U_j) = \begin{cases} (2W - \frac{j}{2})k, & j \le 4W, \\ \frac{j}{2}k, & 4W < j; \end{cases}$$

$$L(F_j) = \begin{cases} \{(f-2W-1+\frac{j}{2})k + 2,\dots,(f-2W+\frac{j}{2})k\}, & j < 2W, \\ \{(f-W-1)k + 2,\dots,(f-W)k + 1\} - \{(f-W)k\}, & j = 2W, \\ \{(f-2W-1+\frac{j}{2})k + 3,\dots,(f-2W+\frac{j}{2})k + 1\}, & 2W < j \le 4W, \\ \{(f - \frac{j}{2})k + 1,\dots,(f - \frac{j}{2} + 1)k - 1\}, & 4W < j < 6W, \\ \{(f-W)k\} \cup \{(f-3W)k + 2,\dots,(f-3W+1)k - 1\}, & j = 6W, \\ \{(f - \frac{j}{2})k + 2,\dots,(f - \frac{j}{2} + 1)k\}, & 6W < j. \end{cases}$$

Case (ii) $i = 4W + 1 < n$, $W = 0,1,\dots$

For j odd:

$$L(U_j) = \begin{cases} (2W - \frac{j-1}{2})k + 1, & j < 2W, \\ (2W - \frac{j-1}{2})k, & 2W < j \le 4W + 1, \\ \frac{j-1}{2}k + 1, & 4W + 1 < j < 6W + 2, \\ \frac{j-1}{2}k, & 6W + 2 < j; \end{cases}$$

$$L(F_j) = \begin{cases} \{(f-2W-1+\frac{j-1}{2})k + 3,\dots,(f-2W + \frac{j-1}{2})k + 1\}, & j \le 4W + 1, \\ \{(f - \frac{j-1}{2})k,\dots,(f - \frac{j-1}{2} - 1)k + 2\}, & 4W + 1 < j. \end{cases}$$

For j even:

$$L(U_j) = \begin{cases} (f - 2W - 1 + \frac{j}{2})k + 2, & j < 4W + 1, \\ (f - \frac{j}{2})k + 1, & 4W + 1 < j; \end{cases}$$

$$L(F_j) = \begin{cases} \{(2W - \frac{j}{2})k + 2,\dots,(2W + 1 - \frac{j}{2})k\}, & j \le 2W, \\ \{(2W - \frac{j}{2})k + 1,\dots,(2W + 1 - \frac{j}{2})k - 1\}, & 2W < j < 4W + 1, \\ \{(\frac{j}{2} - 1)k + 2,\dots,\frac{j}{2} k\}, & 4W + 1 < j < 6W + 2, \\ \{(Wk+1)\} \cup \{3Wk + 2,\dots,(3W+1)k - 1\}, & j = 6W + 2, \\ \{(\frac{j}{2} - 1)k + 1,\dots,\frac{j}{2}k - 1\}, & 6W + 2 < j. \end{cases}$$

Case (iii) $i = 4W + 2 < n$, $W = 0,1,\dots$.

For j odd:

$$L(U_j) = \begin{cases} (f - 2W - 1 + \frac{j-1}{2})\ k + 2, & j < 4W + 2, \\ (f - \frac{j-1}{2})\ k + 1, & 4W + 2 < j; \end{cases}$$

$$L(F_j) = \begin{cases} \{(2W - \frac{j-1}{2})k+2,\dots,(2W - \frac{j-1}{2} + 1)k\}, & j \le 2W + 1, \\ \{(2W - \frac{j-1}{2})k + 1,\dots,(2W - \frac{j-1}{2} + 1)k - 1\}, & 2W + 1 < j < 4W + 2, \\ \{\frac{j-1}{2}k + 2,\dots,(\frac{j-1}{2} + 1)k\}, & 4W + 2 < j < 6W + 5, \\ \{Wk+1\} \cup \{(3W+2)k + 2,\dots,(3W+3)k - 1\}, & j = 6W + 5, \\ \{\frac{j-1}{2}k + 1,\dots,(\frac{j-1}{2} + 1)k - 1\}, & 6W + 5 < j. \end{cases}$$

For j even:

$$L(U_j) = \begin{cases} (2W + 1 - \frac{j}{2})k + 1, & j < 2W + 1, \\ (2W + 1 - \frac{j}{2})k, & 2W + 1 < j \le 4W + 2, \\ \frac{j}{2}k + 1, & 4W + 2 < j < 6W + 5, \\ \frac{j}{2}k + 2, & 6W + 5 < j; \end{cases}$$

$$L(F_j) = \begin{cases} \{(f-2W-2 + \frac{j}{2})k + 3,\dots,(f-2W-1+\frac{j}{2})k + 1\}, & j \le 4W + 2, \\ \{(f - \frac{j}{2})k + 2,\dots,(f - \frac{j}{2} + 1)k\}, & 4W + 2 < j. \end{cases}$$

Case (iv). $i = 4W + 3 < n, \quad W = 0,1,\ldots$

For j odd:

$$L(U_j) = \begin{cases} (2W + 1 - \frac{j-1}{2})k, & j \le 4W + 3, \\ \frac{j-1}{2}\,k, & 4W + 3 < j; \end{cases}$$

$$L(F_j) = \begin{cases} \{(f-2W-2+\frac{j-1}{2})k + 2,\ldots,(f-2W-1+\frac{j-1}{2})k\}, & j < 2W + 1 \\ \{(f-W-2)k + 2,\ldots,(f-W-1)k + 1\} - \{(f-W-1)k\}, & j = 2W + 1 \\ \{(f-2W-2+\frac{j-1}{2})k + 3,\ldots,(f-2W-1+\frac{j-1}{2})k + 1\}, & 2W + 1 < j \le 4W + 3 \\ \{(f-\frac{j-1}{2} - 1)k + 1,\ldots,(f - \frac{j-1}{2}\,k - 1\}, & 4W + 3 < j < 6W + 5, \\ \{(f-W-1)k\} \cup \{f-3W-3)k + 2,\ldots,(f-3W-2)k - 1\}, & j = 6W + 5 \\ \{(f - \frac{j-1}{2} - 1)k + 2,\ldots,(f - \frac{j-1}{2})k\}, & 6W + 5 < j \end{cases}$$

For j even:

$$L(U_j) = \begin{cases} (f-2W-2+\frac{j}{2})k + 1, & j < 2W + 1, \\ (f-2W-2+\frac{j}{2})k + 2, & 2W + 1 < j < 4W + 3, \\ (f - \frac{j}{2})k, & 4W + 3 < j < 6W + 5, \\ (f - \frac{j}{2})k + 1, & 6W + 5 < j; \end{cases}$$

$$L(F_j) = \begin{cases} \{(2W - \frac{j}{2} + 1)k + 1,\ldots,(2W - \frac{j}{2} + 2)k - 1\}, & j < 4W + 3, \\ \{(\frac{j}{2} - 1)k + 1,\ldots,\frac{j}{2}\,k - 1\}, & 4W + 3 < j. \end{cases}$$

The following is a summary table for the number taken out of order, the vertex it is taken from and the vertex it is given to for each case.

i	the number	taken out from	inserted into
$4W$	$(f-W)k$	F_{2W}	F_{6W}
$4W + 1$	$Wk + 1$	F_{2W}	F_{6W+2}
$4W + 2$	$Wk + 1$	F_{2W+1}	F_{6W+5}
$4W + 3$	$(f-W-1)k$	F_{2W+1}	F_{6W+5}

5. *General Caterpillars are not rotatable.*

Can we delete the t-toe condition and prove that any caterpillar is a rotatable graceful graph? No, since there is no way to complete the labeling in Figure 5 (this example is taken from [2]).

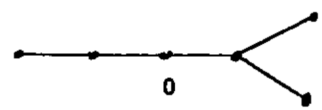

Figure 5

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