AN ALGEBRAIC APPROACH TO SWITCHING NETWORKS

by

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ABSTRACT

In telephone switching networks, a crossbar switch functions by allowing all permutations between its input lines and its output lines. Such a network can therefore be studied from an algebraic point of view. For example, Beneš pioneered the use of group theory to establish the rearrangeability of certain classes of switching networks. In this paper we introduce an alternative algebraic model for switching networks which has certain advantages over the earlier model of Beneš. In particular, the usual problems of realization and control for the network can be phrased as natural algebraic questions in the new model.

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I. INTRODUCTION

In telephone switching networks, a crossbar switch functions by allowing all permutations between its input lines and its output lines. Such a network can therefore be studied from an algebraic point of view. For example, Beneš [1] [3] [4] pioneered the use of group theory to establish the rearrangeability of certain classes of switching networks. In this paper we introduce an alternative algebraic model for switching networks which has certain advantages over the earlier model of Benes. In particular, the usual problems of realization and control for the network can be phrased as natural algebraic questions in the new model.

A switching network N can be specified by a set of switches, a set of links which connect switches, and two sets of inlet and outlet terminals, denoted by I and Ω respectively. A <u>request</u> is a pair consisting of an inlet terminal and an outlet terminal for which a connection is desired. An <u>assignment</u> is a set of requests in which each inlet or outlet occurs at most once. A <u>path</u> is a sequence of links such that any two consecutive links are an input and an output of some switch. Let P denote the set of all possible paths connecting inlet terminals to outlet terminals in N. A state is a set of link-disjoint paths in N. An assignment A is realizable in N if there exists a state S such that for each request in A there is a path in S connecting the inlet terminal of the request to the outlet terminal of the request. In this case we say A is realized by S. Finally, a network N is said to be rearrangeable if any assignment is realizable.

Most existing switching networks consist of several stages of switches. We say the switching network N is a k-stage network if the set of switches of N can be partitioned into a sequence of k subsets, called stages, such that links exist only between two switches in consecutive stages.

In this paper, we restrict ourselves to multi-stage networks with the property that the set of inlet terminals and the set of outlet terminals are disjoint and of the same size, say, $|I| = |\Omega| = n$. Moreover, we shall assume all switches are square switches, i.e., the number of inputs of a switch is the same as the number of outputs of the switch. Since every assignment is a subset of a maximal assignment (i.e., one consisting of n requests), all assignments will hereafter be assumed to be maximal. An assignment can then be interpreted as a permutation of $\{1,2,\ldots,n\}$. Thus, a rearrangeable network can realize any permutation of $\{1,2,\ldots,n\}$. Let A(N) denote the set of permutations realizable in N. For example, N is rearrangeable if and only if $A(N) = S_n$, where S_n denotes the symmetric group on n elements (see [7] for terminology).

II. An Algebraic Model for a k-Stage Switching Network of 2x2 Switches

In this section, we assume all switches are of size 2x2, (i.e., each switch has two input lines and two output lines). These simplified networks still contain much of the complexity of the general case (which will be treated in Section IV). Furthermore, we assume the number $n = |I| = |\Omega|$ is a power of 2.

In a given state, there are at most two paths passing through a 2x2 switch. Also, there are just two different ways for connections using a 2x2 switch (shown in Figure 1).

FIGURE 1

In Figure 1(a), the switch provides a "through" connection and in Figure 1(b), the switch provides a "cross" connection. The through connection corresponds to the identity permutation. The cross connection functions as a transposition (ab), (i.e., a cyclic permutation of length 2). Two transpositions (ab), (cd) are said to be disjoint if $\{a,b\} \cap \{c,d\} = \emptyset$. We note that (ab) is the same as (ba).

An algebraic model of a k-stage network N can be described as follows (see Figure 2).

(1) The inlet terminals are labeled by 1,2,...,n.

(2) The set of n/2 switches of size 2x2 in the first stage can be described by 2-element subsets of {1,2,...,n}. The switch s(ab) is connected to the inlet terminals a and b.

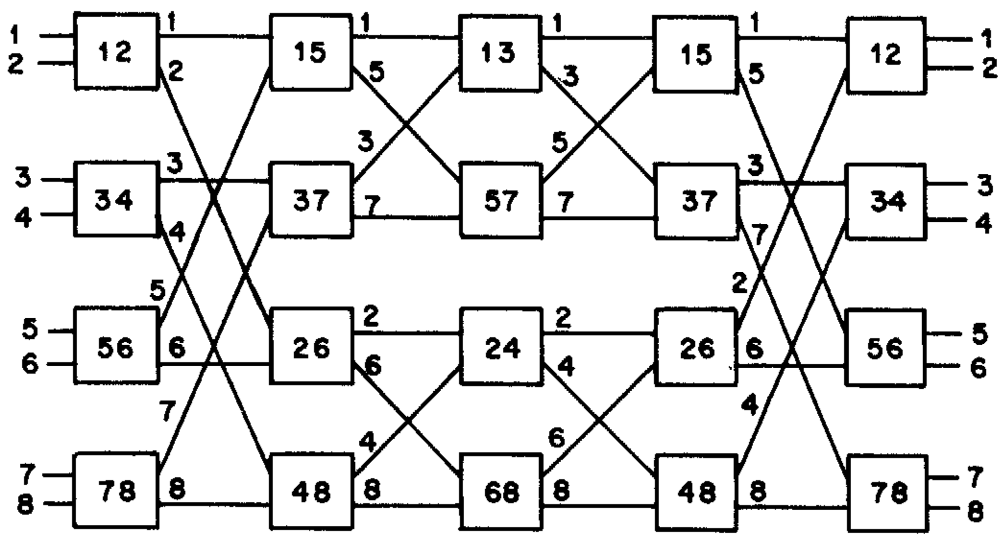


FIGURE 2

(3) The set of links between the i-th and (i+1)-th stage will be labeled as follows: Two links which go out of the switch s(ab) will be labeled by a and b, respectively, so that the input links and output links of the switch s(ab) are labeled in the same order (see Figure 3).

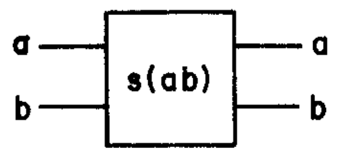


FIGURE 3

(4) The switch s in the i-th stage will be labeled by s(ab) if links a and b come into s. Thus, the switches in the i-th stage are labeled by a set of n/2 mutually disjoint 2-element subsets.

(5) The outlet terminals connected to the switch s(ab) are labeled by a and b, respectively, as in (2).

By combining (1), (2), (3) and (4), all links, inlet and outlet terminals are labeled by elements of $\{1,\ldots,n\}$ and all switches are labeled by 2-element subsets of $\{1,2,\ldots,n\}$. This gives an algebraic model of the network N which has many interesting properties.

Theorem 1: In the preceding model of a k-stage switching network N of 2x2 switches, let G_i denote the group generated by all the transpositions $\{(ab):s(ab)$ is a switch in the i-th stage$\}$. Then the set of assignments which can be realized in N is exactly the product $G_1G_2\ldots G_k$. In other words,

$$A(N) = G_1G_2\ldots G_k = \{g_1g_2\ldots g_k : g_i \varepsilon G_i\}.$$

(where the product of two permutations is defined by $(fg)(i)=g(f(i))$.)

Proof. Let p denote a permutation which is realizable in N. Thus, there exists a state S such that there are n link-disjoint paths, $P_1, P_2, \ldots, P_n$, connecting inlet terminal i to outlet terminal p(i) in S, $1 \leq i \leq n$. For any switch s(ab), there are two paths passing through it, say P_i and P_j. (See Figure 4(a), (b)).

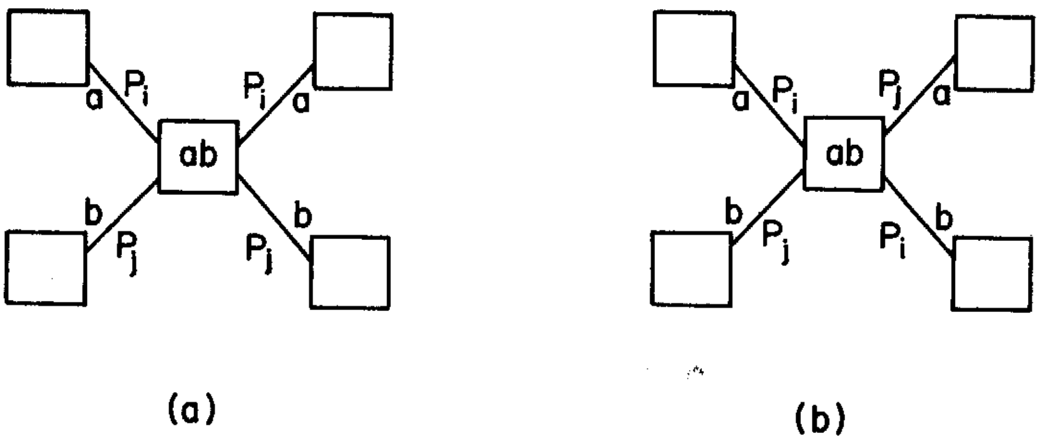


FIGURE 4

If the link of P_i which comes into s(ab) is labeled the same as the link of P_i which goes out of s(ab), the switch s(ab) provides a "through" connection in the state S, (see Figure 3(a)). If the link of P_i which comes into s(ab) is labeled differently from the link of P_i which goes out of s(ab), then s(ab) provides a "cross" connection in state S, (see Figure 3(b)). Let g_i be the element of G_i which is the product of the transpositions (ab) for all switches s(ab) which provide "cross" connections in the i-th stage. We note that P_i consists of links labeled by $g_1(i)$, $g_2(g_1(i))$,..., etc., and P_i passes through switches s(ab) in the j-th stage such that $(g_1 g_2 \ldots g_j)(i) \in \{a,b\}$. Thus, $p(i) = g_1 g_2 \ldots g_k(i)$ for all i. We have $p = g_1 g_2 \ldots g_k$. Therefore,

$$A(N) \supseteq G_1 \; G_2 \; \ldots \; G_k.$$

On the other hand, let $p = g_1 g_2 \ldots g_k$. It suffices to show there is a state S which realizes the assignment p. We let the switch s(ab) in i-th stage provide a "through" connection if and only if $g_i(a) = a$ and $g_i(b) = b$. This yields a state of n link-disjoint paths, connecting input terminals to output terminals. It is easily verified that S realizes A in N. Thus we have

$$G_1 \; G_2 \; \ldots \; G_k = A(N)$$

and Theorem 1 is proved. ∎

Let us define a <u>basic</u> subgroup of the symmetric group S_n, n even, to be a subgroup which is generated by n/2 mutually disjoint transpositions. It is easy to see that a basic subgroup is abelian. Also, any element a in a basic subgroup is of order 2, i.e., $a^2 = 1$. In our algebraic model of the switching network N, stage i correspond to a basic subgroup which is generated by all the transpositions $\{(ab): s(ab)$ is a switch in the i-th stage$\}$.

The following corollary follows immediately from the proof of Theorem 1.

<u>Corollary 1</u>: Let S be a state which realizes an assignment g. Then there are unique elements in basic subgroups corresponding to stages, say $g_i \in G_i$, $1 \leq i \leq k$, such that the following hold:

(i) $g = g_1 g_2 \cdots g_k$,

(ii) All the paths in S can be specified by g_i, $1 \leq i \leq k$. In other words, if P_i denotes the path in S connecting i and g(i), then P_i consists of links labeled by $g_1(i), g_2(g_1(i)), \ldots,$ etc.∎

We remark that Corollary 1 suggests a short expression for describing a state. A state can be viewed in form of a product of elements in basic subgroups. For example, the state illustrated in Figure 5 realizes the permutation g = (157)(683)(24). The state can be specified by writing g into a product as follows:

$$g = g_1 g_2 g_3 g_4 g_5$$

where
$$g_1 = (12),$$
$$g_2 = (37)(48),$$
$$g_3 = (13)(57)(68),$$
$$g_4 = (26),$$
$$g_5 = (34)(56) .$$

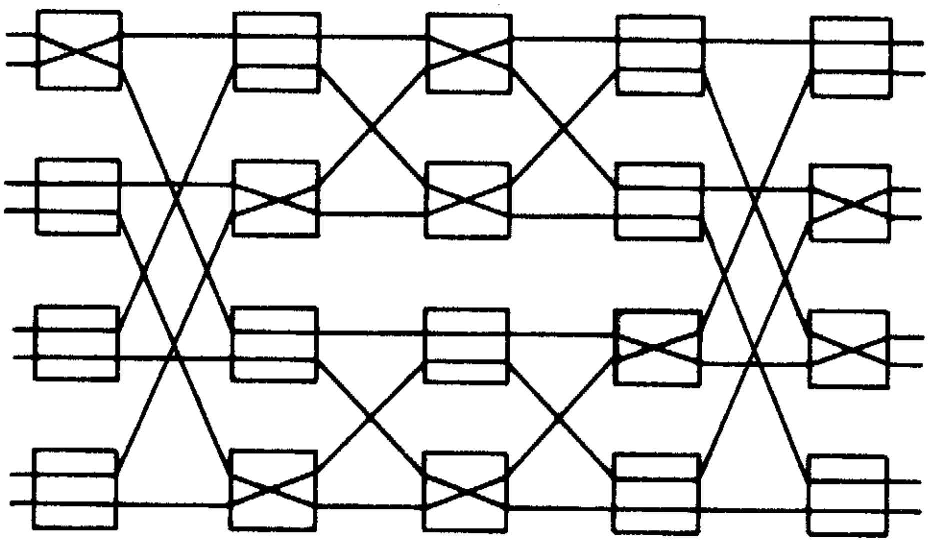


FIGURE 5

We note that for a given assignment A, there might exist more than one state which realizes A. Therefore a permutation can sometimes be written in the form $g_1 \ldots g_k$, $g_i \in G_i$ in more than one way.

III. Rearrangeable Networks

Our algebraic model is useful for studying rearrangeable switching networks, i.e., networks whose set of assignments is the full symmetric group S_n. We can study the structure of a multi-stage rearrangeable network of 2x2 switches by studying factorizations of S_n into a product of basic subgroups. From Theorem 1, we have the following.

Corollary 2: Let N denote a rearrangeable k-stage network of 2x2 switches with $|I| = |\Omega| = n$, n even. Then

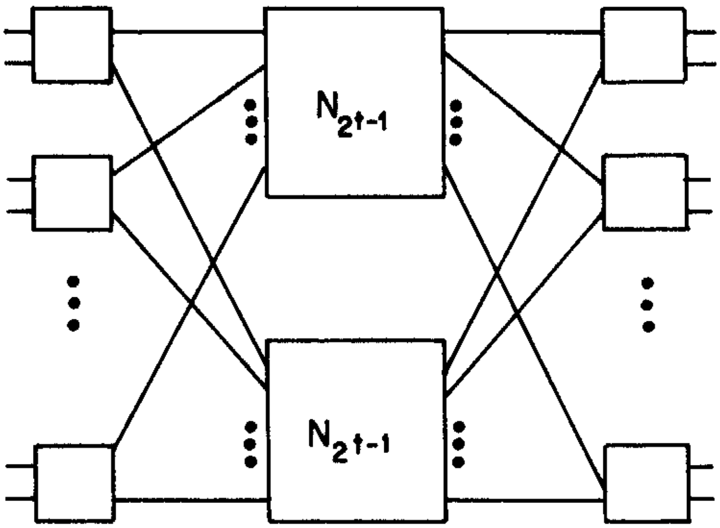
$$G_1 G_2 \ldots G_k = S_n$$

where G_i is the basic subgroup which corresponds to the i-th stage in N.

In order to minimize the number of crosspoints in a multi-stage rearrangeable network of 2x2 switches, we must minimize the number of stages. This can be phrased into the following problem:

What is the minimum number $\tau(n)$ of basic subgroups into which the symmetric group S_n can be factorized?

Based on the well-known Slepian-Duguid Rearrangeability Theorem [5] [8], the switching network which is built recursively from three-stage Clos networks is rearrangeable and has 2t-1 stages where $n = 2^t$ (see Figure 6). Thus $\tau(n)$ is bounded above by $2\lceil \log_2 n \rceil - 1$ where $\lceil x \rceil$ denotes the least integer greater than or equal to x.



N_{2^t}

FIGURE 6

The basic subgroups which correspond to stages in the network N_{2^t} (shown in Figure 6) can be described as follows:

G_1 or G_{2t-1} is the subgroup generated by transpositions $(2i+1, 2i+2)$, $0 \leq i < 2^{t-1}$, and we write

$$G_1 = G_{2t-1} = \langle (2i+1,\ 2i+2):\ 0 \leq i < 2^{t-1} \rangle$$

In general, we have

$$G_m = G_{2t-m}$$

$$= \langle (i+j\cdot 2^{t-m+1},\ i+j\cdot 2^{t-m+1} + 2^{t-m}):\ 1 \leq i \leq 2^{t-m},\ 0 \leq j < 2^{m-1} \rangle,$$

where $2 \leq m \leq t$.

For example, the basic subgroups for the rearrangeable network N_8 shown in Figure 2 are the following:

$$G_1 = G_5 = \langle (12), (34), (56), (78) \rangle,$$

$$G_2 = G_4 = \langle (15), (26), (37), (48) \rangle,$$

$$G_5 = \langle (13), (57), (24), (68) \rangle.$$

The proof that $G_1\ G_2\ \dots\ G_{2t-1} = S_{2^t}$ is established by using the fact that every permutation has a "Hall decomposition" (this is implied by Hall's Theorem on systems of distinct representatives; see [4], [6] for references).

A lower bound for $\tau(n)$ can be established by the following.

Corollary 3: In a k-stage rearrangeable network of 2x2 switches with $|I| = |\Omega| = n$, n even, we have

$$k \geq 2 \log_2 n - 2 \log_2 e$$

Proof: Let G_i be the basic subgroup corresponding to the i-th stage, $1 \leq i \leq k$. We have

$$G_1\ G_2\ \dots\ G_k = S_n.$$

Now, G_i is generated by n/2 generators and we have

$$|G_i| = 2^{n/2},\ 1 \leq i \leq k.$$

Therefore,

$$2^{nk/2} \geq n!$$

By using Stirling's approximation, we have

$$k \geq \frac{2}{n} (n \log_2 n - n \log_2 e)$$

$$= 2 \log_2 n - 2 \log_2 e$$

$$= 2 \log_2 n - 2.8853\ldots \quad \blacksquare$$

We note that Corollary 3 can also be obtained by information-theoretic arguments. When n is a power 2, we have the following.

Corollary 4: If $n = 2^t$, we have

$$2t-1 \geq \tau(n) \geq 2t-2.$$

It doesn't seem unreasonable to conjecture that $\tau(n) = 2\lceil \log n \rceil - 1$. Beneš [4] suggests that there is a strong relation between the number d of links one must use to go from any fixed input terminal to reach all switches of a stage and the number R of stages needed to guarantee that the network is rearrangeable. In particular Beneš conjectures that $R = 2d+1$ if the cross-field connection is fixed.

IV. An Algebraic Model for a k-Stage Switching Network of mxm Switches.

We now assume all switches in N are of size mxm where $|I| = |\Omega| = n = mr$. An algebraic model of a k-stage network

N can be described as follows.

(1) The inlet terminals are labeled by $1,2,\ldots,n$.

(2) The set of mxm switches in the first stage will be labeled by m-element subsets of $\{1,2,\ldots,n\}$. Let Q be the set of inlet terminals which are connected to s. Then s is labeled by s(Q). We note that if two switches in the first stage are labeled by $s(Q_1)$, $s(Q_2)$, respectively, then we have $Q_1 \cap Q_2 = \emptyset$. In a given state the switch s(Q) functions by allowing all m! permutations on elements of Q.

(3) The set of links between the i-th and (i+1)-st stage will be labeled as follows: The links which go out of the switch s(Q) will be labeled by elements in Q such that the input links and output links of the switch s(Q) are labeled in the same order.

(4) The switch s in the i-th stage will be labeled by a set Q such that $i \in Q$ implies link i comes into Q.

(5) The outlet terminals connected to the switch s(Q) are labeled by m-element subsets in Q as in (2).

As before, we have the following theorem.

Theorem 3: In the algebraic model of the k-stage switching network N of mxm switches, let G_i denote the group generated by all permutations $\{q: q$ is a permutation on elements of Q where s(Q) is one of the switches in the i-th stage$\}$. Then the set of all assignments which can be realized in N is exactly the product $G_1\ G_2 \ldots G_k$.

The proof is similar to the proof of Theorem 1. The G_i's are basic subgroups corresponding to stages in this network.

V. Concluding Remarks

The algebraic model for switching networks we have proposed is a little different from the model of Beneš. For example, Beneš' model for the rearrangeable networks in Figure 2 will be factorized as follows:

$$S_8 = (S_2)^4 \ \phi_1 \ (S_2)^4 \ \phi_2 \ (S_2)^4 \ \phi_2^{-1} \ (S_2)^4 \ \phi_1^{-1} \ (S_2)^4,$$

where ϕ_1, ϕ_2 denote the permutations for the cross-field connections. In our model, the structure of the cross-field connections is implicit in the factorization of basic subgroups. Moreover, any state can be explicitly specified by an element of S_n written in a certain form (see Corollary 1). This allows us to gain certain insights of the structure of these networks.

The problems of realization and control will be solved if we can understand more about the factorization of the corresponding groups. For example, for a given k-stage switching network N with basic subgroups $G_1, G_2, \ldots, G_k$ which correspond to stages in N, we can phrase the problems of realization and control for N as follows.

(1) The realization problem for a given assignment p is equivalent to the problem of determining if $p \in G_1 \, G_2 \, \ldots \, G_k$.

(2) The control problem for a given realizable assignment p is equivalent to the problem of finding an expression for p in the form

$$p = g_1 \cdot g_2 \cdot \ldots \cdot g_k \text{ where } g_i \in G_i.$$

Of course, much more work will be required in order to fully develop this algebraic approach to the analysis of switching networks. What we have tried to do is to point the way to what seems to the author to be an attractive and useful direction.

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