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Self-organizing Sequential Search and Hilbert's Inequalities

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In this paper we describe a general technique which can be used to solve an old problem in analyzing self-organizing sequential search. We prove that the average time required for the move-to-front heuristic is no more than $\pi/2$ times that of the optimal order and this bound is the best possible. Hilbert's inequalities will be used to derive large classes of inequalities some of which can be applied to obtain tight worst-case bounds for several self-organizing heuristics. © 1988 Academic Press, Inc.

1. INTRODUCTION

Sequential search, a very simple way to retrieve data, has long been studied in the literature. Various enhancements, especially the self-organizing heuristics, have been extensively investigated by many researchers [1–5, 7, 9–13]. In this paper we will examine a basic problem in self-organizing sequential search, which can be described as follows.

Suppose a set of n keys are stored in a linear list which will then be sequentially searched for some string of requests. The so called "optimal static ordering" method finds out the count or access probability for each key in the request string and then places the keys in decreasing order of probabilities. Suppose we do not want to use large memory in deriving such a count. An old memoryless scheme of ensuring that more frequently accessed keys are closer to the "top" of the list is the "move-to-front rule." Namely, each time a key is requested, it is moved to the front of the list and the order of the other keys remains unchanged. A natural question then arises: How good is the move-to-front rule in comparison with the optimal static ordering?

Suppose the request string has the probability distribution $p=(p_1, p_2, ..., p_n)$. The cost (the expected search cost for a single key) for the optimal static ordering (denoted by $\mathrm{Opt}(p)$) is just $\sum ip_i$, where $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$. The cost $M(p)$ for the move-to-front rule was derived by McCabe in 1965 and can be written as follows (also see [11, 13]):

$$M(p)=2 \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{p_i p_j}{p_i+p_j}.$$

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It can be easily checked [11, 12] that the ratio of $M(p)$ and $\mathrm{Opt}(p)=\sum_{i=1}^{n} i\, p_i$ is bounded above by 2. On the other hand Gonnet *et al.* [7] have shown that the value can be arbitrarily close to $\pi/2$ by considering the distribution p_i, $i=1, ..., n$ with $p_i = Ci^{-2}$. Thus the supremum of $M(p)/\mathrm{Opt}(p)$ over all possible distributions p is between $\pi/2$ and 2. The problem of determining the exact value of $\sup_p M(p)/\mathrm{Opt}(p)$ remained an open problem [2, 7, 13].

In this paper we will prove that for any distribution p we have $M(p)/\mathrm{Opt}(p) < \pi/2$ by using Hilbert's inequality. We will also demonstrate that this technique can produce a general class of combinatorial inequalities some of which can be used to generate tight bounds for other self-organizing heuristics.

Here we will give a brief history of other self-organizing heuristics. The transpose rule (a requested key is moved one closer to the front of the list) was proved by Rivest [13] to have lower cost than the move-to-front rule, and he conjectured that the transpose rule is optimal. This was further backed by the result of Yao and Bitner [4] that for some special distributions the transpose rule is optimal over all rules. However Anderson *et al.* [1] found a counterexample to this conjecture by deriving a rule that is better than the transpose rule for a specific distribution. Bitner later [3, 4] showed that while the transpose rule is asymptotically more efficient, the move-to-front rule converges more quickly and proposed a hybrid of these two rules with mixed performance. Rivest [12] introduced the "move-ahead-k" heuristics where a requested key is moved ahead k positions. Gonnet *et al.* [7] and Kan and Ross [10] proposed the "k-in-a-row" heuristics, where a key is moved only after it is requested k times in a row. Gonnet *et al.* [7] also considered the "k-in-a-batch" heuristics, where requests are grouped into batches of size k and a key will be moved if it is requested k times in a batch. They proved that the k-in-a-batch rule is better than the k-in-a-row rule when in combination with either the move-to-front rule or the transpose rule.

In Section 2 we will introduce Hilbert's inequalities and a few auxiliary facts. In Section 3 the main theorem on the worst-case behavior of $M(p)/\mathrm{Opt}(p)$ will be presented. Section 4 contains several general classes of combinatorial inequalities some of which are used to derive worst-case bounds for the k-in-a-batch move-to-front heuristics. Section 5 includes some concluding remarks.

2. Hilbert's Inequality

In this section we will illustrate several auxiliary tools in mathematical analysis which will later be used in our proofs:

(1) Hilbert's inequality (see Hardy *et al.* [8]). For $p, q > 1$ satisfying $1/p + 1/q = 1$, suppose that $K(x, y)$ is nonnegative and homogeneous of degree -1 and that

$$\int_0^\infty K(x, 1)\, x^{-1/p}\, dx = \int_0^\infty K(1, y)\, y^{-1/q}\, dy = C.$$

Then

$$\text{(a)} \quad \int_0^\infty \int_0^\infty K(x, y) f(x) g(y)\, dx\, dy \leqslant C\left(\int_0^\infty f^p\, dx\right)^{1/p} \left(\int_0^\infty g^q\, dy\right)^{1/q}$$

$$\text{(b)} \quad \int_0^\infty dy \left(\int_0^\infty K(x, y) f(x)\, dx\right)^p \leqslant C^p \int_0^\infty f^p\, dx$$

$$\text{(c)} \quad \int_0^\infty dx \left(\int_0^\infty K(x, y) g(y)\, dy\right)^q \leqslant C^q \int_0^\infty g^q\, dy.$$

A simple way to prove (a) is to use Hölder's inequality twice. For nonnegative functions f and q, Hölder's inequality is

$$\int f(x) g(x)\, dx \leqslant \left(\int f^p(x)\, dx\right)^{1/p} \left(\int g^q(x)\, dx\right)^{1/q}.$$

We have

$$\begin{aligned} &\iint K(x, y) f(x) g(y)\, dx\, dy \\ &\quad = \iint f(x) K^{1/p} \left(\frac{x}{y}\right)^{1/pq} g(y) K^{1/q} \left(\frac{y}{x}\right)^{1/pq} dx\, dy \\ &\quad \leqslant P^{1/p} Q^{1/q}, \end{aligned}$$

where $P = \int f^p(x)\, dx \int K(x, y)(x/y)^{1/q}\, dy = C \int f^p\, dx$ and $Q = C \int g^q\, dy$. Then (b) follows from (a) by taking $g = f^{p-1}$.

(2) A generalization of (1) can be described as follows [8]: For t numbers $p, q, ..., r$ satisfying $p > 1$, $q > 1, ..., r > 1$, $1/p + 1/q + \cdots + 1/r = 1$, and a positive function $K(x, y, ..., z)$ of t variables $x, y, ..., z$, homogeneous of degree $-t + 1$ with

$$\int_0^\infty \cdots \int_0^\infty K(1, y, ..., z)\, y^{-1/q} \cdots z^{-1/r}\, dy \cdots dz = C,$$

we have

$$\begin{aligned} &\int_0^\infty \cdots \int_0^\infty K(x, y, ..., z) f(x) g(y) \cdots h(z)\, dx\, dy \cdots dz \\ &\quad \leqslant C\left(\int_0^\infty f^p\, dx\right)^{1/p} \left(\int_0^\infty g^q\, dy\right)^{1/q} \cdots \left(\int_0^\infty h^r\, dz\right)^{1/r}. \end{aligned}$$

(3) The gamma function $\Gamma(z) = \lim_{n \to \infty} n!\, n^z / z(z+1) \cdots (z+n)$ $(z \neq 0, -1, -2, ...)$.

(4) The beta function

$$B(z, w)=\int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}}\,dt=\frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)},$$

where the real parts of z and w are both positive.

(5) The Dirichlet integral [6]

$$\int_0^\infty \cdots \int_0^\infty x_1^{i_1-1}\cdots x_n^{i_n-1} f(x_1+x_2+\cdots+x_n)\,dx_1\cdots dx_n$$
$$=\frac{\Gamma(i_1)\cdots\Gamma(i_n)}{\Gamma(i_1+\cdots+i_n)}\int_0^\infty v^{i_1+\cdots+i_n-1} f(v)\,dv$$

for $i_1, ..., i_n > 0$.

3. The Main Theorem

We want to show the following:

THEOREM 1. *For any probability distribution* $p=(p_1, p_2, ..., p_n)$, *we have*

$$\frac{M(p)}{\mathrm{Opt}(p)} \leqslant \frac{\pi}{2}.$$

This will be proved by using the inequality in Theorem 2, which will be proved later.

THEOREM 2. *If* $x_i > 0$ $(i=1, ..., n)$, *then*

$$\sum_{1 \leqslant i, j \leqslant n} \frac{2x_i x_j}{x_i+x_j} \Bigg/ \sum_{1 \leqslant i, j \leqslant n} \min(x_i, x_j) \leqslant \frac{\pi}{2}.$$

As an immediate consequence of Theorem 1 and Rivest's result [13] we have the following:

COROLLARY. *For any probability distribution of the request string, the cost for the transpose rule is no more than* $\pi/2$ *times that of the optimal static ordering.*

Proof of Theorem 1. Using the expressions for $M(p)$ and $\mathrm{Opt}(p)$ described in Section 1, we have

$$\frac{M(p)}{\mathrm{Opt}(p)}=2\sum_{i \leqslant j} \frac{p_i p_j}{p_i+p_j} \Bigg/ \sum_i i p_i$$

for $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$. We note that

$$\sum_{i \leqslant j} \frac{2p_i p_j}{p_i + p_j} = \sum_{i,j} \frac{p_i p_j}{p_i + p_j} + \frac{1}{2} \sum p_i$$

and

$$\sum_i i p_i = \frac{1}{2} \sum_{i,j} \min(p_i, p_j) + \frac{1}{2} \sum p_i.$$

Therefore, by using Theorem 2, we have

$$\frac{M(p)}{\mathrm{Opt}(p)} = \left(\left(\sum_{i,j} \frac{2p_i p_j}{p_i + p_j} \right) + 1 \right) \Bigg/ \left(\left(\sum_{i,j} \min(p_i, p_j) \right) + 1 \right)$$
$$< \sum_{i,j} \frac{2p_i p_j}{p_i + p_j} \Bigg/ \sum_{i,j} \min(p_i, p_j) \leqslant \frac{\pi}{2}.$$

To prove Theorem 2, we will first prove a continuous version from which Theorem 2 can then be derived.

LEMMA 2. *Suppose f is an integrable function on $(0, \infty)$ with $\int_0^\infty f\,dx = 0$. For any k satisfying $k < 0$,*

$$\frac{\int_0^\infty \int_0^\infty (x^k + y^k)^{1/k} f(x) f(y)\,dx\,dy}{\int_0^\infty \int_0^\infty \min(x, y) f(x) f(y)\,dx\,dy} \leqslant \frac{1-k}{k} B\left(1 - \frac{1}{2k}, 1 - \frac{1}{2k}\right),$$

where B is the beta function.

Proof. Set $F(x) = \int_\infty^x f(x)\,dx$. First,

$$\int_0^\infty \int_0^\infty \min(x, y) f(x) f(y)\,dx\,dy$$
$$= \int_0^\infty \int_0^x y f(x) f(y)\,dy\,dx + \int_0^\infty \int_0^y x f(x) f(y)\,dx\,dy$$
$$= 2\int_0^\infty f(x) \left(\int_0^x y f(y)\,dy \right) dx$$
$$= 2\int_0^\infty f(x) \left(yF(y)|_0^x - \int_0^x F(y)\,dy \right) dx$$
$$= 2\left(\int_0^\infty x f(x) F(x)\,dx - \int_0^\infty \int_0^x f(x) F(y)\,dy\,dx \right)$$

$$= 2\left(\int_0^\infty xf(x)\,F(x)\,dx - \int_0^\infty F(y)\int_y^\infty f(x)\,dx\,dy\right)$$

$$= 2\left(\int_0^\infty xf(x)\,F(x)\,dx + \int_0^\infty F^2(y)\,dy\right)$$

$$= 2\left(\frac{xF^2(x)|_0^\infty}{2} - \int_0^\infty \frac{F^2(x)}{2}\,dx + \int_0^\infty F^2(y)\,dy\right)$$

$$= \int_0^\infty F^2(x)\,dx.$$

Also,

$$\int_0^\infty \int_0^\infty (x^k + y^k)^{1/k} f(x)\,f(y)\,dx\,dy$$

$$= \int_0^\infty f(x)\,dx\left((x^k + y^k)^{1/k}\,F(y)|_0^\infty - \int_0^\infty (x^k + y^k)^{1/k-1}\,y^{k-1}F(y)\,dy\right)$$

$$= -\int_0^\infty \int_0^\infty (x^k + y^k)^{1/k-1}\,y^{k-1}f(x)\,F(y)\,dx\,dy$$

$$= -\int_0^\infty F(y)\,y^{k-1}\,dy \int_0^\infty (x^k + y^k)^{1/k-1}\,f(x)\,dx$$

$$= (1-k)\int_0^\infty (x^k + y^k)^{1/k-2}\,x^{k-1}y^{k-1}F(x)\,F(y)\,dx\,dy$$

$$\leqslant (1-k)\left(\int_0^\infty F^2(x)\,dx\right)\left(\int_0^\infty (x^k + 1)^{1/k-2}\,x^{k-1-1/2}\,dx\right).$$

This last inequality is derived from Hilbert's inequality (a) by setting

$$K(x, y) = (x^k + y^k)^{1/k-2}\,x^{k-1}y^{k-1}.$$

It is easy to verify that

$$\int_0^\infty \frac{x^{k-3/2}}{(1+x^k)^{2-1/k}}\,dx = \frac{1}{k}\int_0^\infty \frac{t^{-1/(2k)}}{(1+t)^{2-1/k}}\,dt$$

$$= \frac{1}{k}\,B\left(1 - \frac{1}{2k}, 1 - \frac{1}{2k}\right),$$

where B is the beta function.

Therefore we have

$$\frac{\int_0^\infty \int_0^\infty (x^k + y^k)^{1/k} f(x)\,f(y)\,dx\,dy}{\int_0^\infty \int_0^\infty \min(x, y)\,f(x)\,f(y)\,dx\,dy} \leqslant \frac{1-k}{k}\,B\left(1 - \frac{1}{2k}, 1 - \frac{1}{2k}\right).$$

The discrete version of Lemma 2 is as follows:

THEOREM 3. *If* $x_i > 0$ $(i = 1, ..., n)$ *and* $k < 0$ *then*

$$\frac{\sum_{1 \leqslant i, j \leqslant n} (x_i^k + x_j^k)^{1/k}}{\sum_{1 \leqslant i, j \leqslant n} \min(x_i, x_j)} \leqslant \frac{1-k}{k} B\left(1 - \frac{1}{2k}, 1 - \frac{1}{2k}\right).$$

Proof. Let $0 < x_1 < x_2 < \cdots < x_n$. Let $0 < \delta < \frac{1}{4} \min_{i \neq j} |x_i - x_j|$. Let f_δ denote the function with $f_\delta = 1$ in intervals of length δ centered at x_i, $i = 1, ..., n$ and zero elsewhere. By Lemma 2 we have

$$\frac{\int_0^\infty \int_0^\infty (x^k + y^k)^{1/k} f_\delta(x) f_\delta(y)\, dx\, dy}{\int_0^\infty \int_0^\infty \min(x, y) f_\delta(x) f_\delta(y)\, dx\, dy} \leqslant \frac{1-k}{k} B\left(1 - \frac{1}{2k}, 1 - \frac{1}{2k}\right).$$

Theorem 3 then follows, by letting δ approach 0.

We note that Theorem 2 is just a special case of Theorem 3 by taking $k = -1$, for $B(\frac{3}{2}, \frac{3}{2}) = \pi/8$.

4. GENERALIZATIONS

In this section, we will deduce several generalizations of Theorem 3. One of these combinatorial inequalities can be used in the worst-case analysis of k-in-a-batch heuristics.

THEOREM 4. *If* $x_i > 0$ $(i = 1, ..., n)$, $k < 0$, *and* $t \geqslant 0$ *is an integer, then*

$$\sum_{\substack{1 \leqslant i_j \leqslant n \\ 1 \leqslant j \leqslant t}} (x_{i_1}^k + x_{i_2}^k + \cdots + x_{i_t}^k)^{1/k} \Big/ \sum_{\substack{1 \leqslant i_j \leqslant n \\ 1 \leqslant j \leqslant t}} \min(x_{i_1}, ..., x_{i_t}) \leqslant C(k, t)$$

where

$$C(k, t) = \frac{(1-k)(1-2k) \cdots (1-(t-1)k)}{k^{t-1}} \frac{\Gamma(1 - 1/tk)^t}{\Gamma(t - 1/k)}.$$

As before, it suffices to show the following:

LEMMA 3. *Suppose* f *is an integrable function on* $(0, \infty)$ *and* $\int_0^\infty f\, dx = 0$. *For any* $k < 0$ *and fixed integer* t *we have*

$$\frac{\int_0^\infty \cdots \int_0^\infty (x_1^k + x_2^k + \cdots + x_t^k)^{1/k} f(x_1) \cdots f(x_t)\, dx_1 \cdots dx_t}{\int_0^\infty \cdots \int_0^\infty \min(x_1, ..., x_t) f(x_1) \cdots f(x_t)\, dx_1 \cdots dx_t} \leqslant C(k, t).$$

Proof. First using integration by parts, it can be easily checked that

$$\int_0^\infty \cdots \int_0^\infty \min(x_1, ..., x_t) f(x_t) \cdots f(x_t)\, dx_1 \cdots dx_t = (-1)^t \int_0^\infty F^t(x)\, dx.$$

Also we can get

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty (x_1^k + \cdots + x_t^k)^{1/k} f(x_1) \cdots f(x_t)\, dx_1 \cdots dx_t \\
&\quad = (-1)^t \int_0^\infty \cdots \int_0^\infty \frac{\partial^t}{\partial x_1 \cdots \partial x_t} (x_1^k + \cdots + x_t^k)^{1/k} \\
&\quad \times F(x_1) \cdots F(x_t)\, dx_1 \cdots dx_t \\
&\quad = (-1)^t (1-k) \cdots (1-(t-1)k) \int_0^\infty \cdots \int_0^\infty (x_1^k + \cdots + x_t^k)^{1/k - t} \\
&\quad \times x_1^{k-1} \cdots x_t^{k-1} F(x_1) \cdots F(x_t)\, dx_1 \cdots dx_t.
\end{aligned}$$

Now we use the generalized form of Hilbert's inequality by taking $p = q = \cdots = r = 1/t$ and the evaluation of the Dirichlet integral (see [5])

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \frac{x_2^{k-(1+(1/t))} \cdots x_t^{k-(1+(1/t))}}{(1 + x_2^k + \cdots + x_t^k)^{t-(1/k)}}\, dx_2 \cdots dx_t \\
&\quad = \frac{1}{k^{t-1}} \frac{\Gamma(1-(1/kt))^t}{\Gamma(t-(1/k))}.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty (x_1^k + \cdots + x_t^k)^{1/k} f(x_1) \cdots f(x_t)\, dx_1 \cdots dx_t \\
&\quad \leqslant \frac{(-1)^t (1-k) \cdots (1-(t-1)k)\, \Gamma(1-1/kt)^t}{k^{t-1} \Gamma(1-1/k)} \int_0^\infty F^t(x)\, dt.
\end{aligned}$$

The proof for Theorem 4 is then complete.

Another generalization of Theorem 3 which comes up in connection with the k-in-a-batch heuristics is the following:

THEOREM 5. *Suppose* $k > \frac{1}{2}$, $x_i > 0$, $i = 1, ..., n$. *Define* $H_k(x, y) = (x^k y + x y^k)/(x^k + y^k)$. *Then we have*

$$\frac{\sum_{1 \leqslant i, j \leqslant n} H_k(x_i, x_j)}{\sum_{1 \leqslant i, j \leqslant n} \min(x_i, x_j)} \leqslant \frac{\pi}{2k} \csc \frac{\pi}{2k}.$$

Again this follows from the following continuous version.

LEMMA 4. *Let f be an integrable function and suppose $\int_0^\infty f\,dx$ exists. Then*

$$\frac{\int_0^\infty \int_0^\infty H_k(x, y)\, f(x)\, f(y)\, dx\, dy}{\int_0^\infty \int_0^\infty \min(x, y)\, f(x)\, f(y)\, dx\, dy} \leqslant \frac{\pi}{2k} \csc \frac{\pi}{2k}.$$

Proof. By similar argument as in Lemma 2 we have

$$\int_0^\infty \int_0^\infty H_k(x, y)\, f(x)\, f(y)\, dx\, dy = \int_0^\infty \int_0^\infty K(x, y)\, F(x)\, F(y)\, dx\, dy$$

where

$$\begin{aligned} K(x, y) &= \frac{d^2}{dx\, dy} H_k(x, y) = \frac{d}{dx} \left(\frac{kx^{k+1}y^{k-1} + x^{2k} + x^k y^k - kx^k y^k}{(x^k + y^k)^2} \right) \\ &= [(x^k + y^k)(k(k+1)\, x^k y^{k-1} + 2kx^{k-1} + kx^{k-1}y^k - k^2 x^{k-1} y^k) \\ &\quad -2k(kx^{k+1}y^{k-1} + x^{2k} + x^k y^k - kx^k y^k)\, x^{k-1}](x^k + y^k)^{-3}. \end{aligned}$$

By Hilbert's inequality we have

$$\int_0^\infty \int_0^\infty H_k(x, y)\, f(x)\, f(y)\, dx\, dy \leqslant \left(\int_0^\infty K(1, y)\, y^{-1/2}\, dy \right) \left(\int_0^\infty F^2(x)\, dx \right).$$

Now

$$\begin{aligned} \int_0^\infty K(1, y)\, y^{-1/2}\, dy &= \int_0^\infty \frac{ky^{k-3/2}}{(1+y^k)^3} ((1-k)(1+y^{k+1}) + (1+k)\, y(1+y^{k-1}))\, dy \\ &= C(k) + C(-k), \end{aligned}$$

where $C(k) = (1-k)\, \Gamma(1 - 1/2k)\, \Gamma(2 + 1/2k)$. Since $\Gamma(1+z) = z\Gamma(z)$ and $\Gamma(z)\, \Gamma(1-z) = \pi \csc z\pi$ for $0 < z < 1$, we get $C(k) + C(-k) = (\pi/2k) \csc(\pi/2k)$.

As an immediate application of Theorem 5 we prove the following.

THEOREM 6. *Let $M_k(p)$ denote the cost for the k-in-a-batch move-to-front rule. Then we have*

$$\frac{M_k(p)}{\text{Opt}(p)} \leqslant \frac{\pi}{2k} \csc \frac{\pi}{2k}.$$

Proof. Gonnet *et al.* [7] showed that

$$M_k(p) = \sum_{1 \leqslant i \leqslant j \leqslant n} \frac{p_i p_j^k + p_j p_i^k}{p_i^k + p_j^k}.$$

Theorem 6 is an immediate consequence of Theorem 5.

It is easy to check that $M_2(p)/\mathrm{Opt}(p)=\sqrt{2}\,\pi/4\sim 1.110\ldots$, which is better than the bound 1.207 given in [7]. Also $M_3(p)/\mathrm{Opt}(p)\leqslant(\pi/6)\csc(\pi/6)=\pi/3\sim 1.04$ which improves upon the bound 1.08 given in [7].

5. Some Remarks

We want to point out that Hilbert's inequalities can be used to derive large classes of inequalities because the functions involved are very general. Although in the statement of Hilbert's inequalities the homogeneous function K is required to have degree -1 (see Section 2), we can use integration by parts first and then use Hilbert's inequalities (as demonstrated in Sections 2 and 3) to obtain inequalities for sums of higher degree homogeneous functions.

References

1. E. J. Andersen, P. Nash, and R. R. Weber, A counterexample to a conjecture on optimal list orderings, *J. Appl. Probab.* **19** (1982), 730–732.
2. J. L. Bentley and C. C. McGeoch, Worst-case analyses of self-organizing sequential search heuristics, *Comm. ACM* **28**, No. 4 (1985), 404–411.
3. J. R. Bitner, "Heuristics That Dynamically Alter Data Structure to Reduce Their Access Time," Ph.D. thesis, Univ. of Illinois, 1976.
4. J. R. Bitner, Heuristics that dynamically organize data structures, *SIAM J. Comput.* **8** (1979), 82–110.
5. P. J. Burville and J. F. C. Kingman, On a model for storage and search, *J. Appl. Probab.* **10** (1973), 697–701.
6. J. Edward, "Treatise on Integral Calculus," Vol. 2, pp. 160–161, Chelsea, New York, 1922.
7. G. Gonnet, J. I. Munro, and H. Suwanda, Exegesis of self-organizing sequential search heuristics, *SIAM J. Comput.* **10** (1981), 613–637.
8. G. H. Hardy, J. E. Littlewood, and G. Polya, "Inequalities," pp. 229–232, Cambridge Univ. Press, Cambridge, 1952.
9. W. J. Hendrick, An account of self-organizing systems, *SIAM J. Comput.* **5** (1976), 715–723.
10. Y. C. Kan and S. M. Ross, Optimal list order under partial memory constraints, *J. Appl. Probab.* **17** (1980), 1004–1015.
11. D. E. Knuth, "The Art of Computer Programming," Sorting and Searching, Vol. 3, pp. 398–399, Addison–Wesley, Reading, MA, 1973.
12. J. McCabe, On serial file with relocatable records, *Oper. Res.* **12** (1965), 609–618.
13. R. Rivest, On self-organizing sequential search heuristics, *Comm. ACM* **19** (1976), 63–67.

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