On Unavoidable Hypergraphs

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ABSTRACT

A r-uniform hypergraph H (or a r-graph, for short) is a collection $E = E(H)$ of r-element subsets (called edges) of a set $V = V(H)$ (called vertices). We say a r-graph H is (n, e)-unavoidable if every r-graph with n vertices and e edges must contain H. In this paper we investigate the largest possible number of edges in an (n, e)-unavoidable 3-graph for fixed n and e. We also study the structure of such unavoidable 3-graphs.

1. INTRODUCTION

By r-uniform hypergraphs H (or r-graphs, for short) we mean a collection $E = E(H)$ of r-element subsets, called edges, of a set $V = V(H)$, called the vertices of H. A r-graph H is said to be (n, e)-unavoidable if H is contained in every r-graph with n vertices and e edges. Let $f_r(n, e)$ denote the largest integer m with the property that there exists an (n, e)-unavoidable r-graph having m edges. In this paper we study the case of $r = 3$ and we prove the following:

(1) For $e \le (n^2/6) - 2n$, we have

$$f_3(n, e) = \sqrt{\frac{e}{n}} + O(1).$$

(2) For $(n^2/6) - 2n < e < n^{15/7}$ we have

$$c_1 e^{3/2}/n^{5/2} \le f_3(n, e) \le c_2 e^{3/2}/n^{5/2}.$$

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(3) For $n^{15/7} < e$

$$c_3 \frac{e^{1/3} \log n}{\log\left(\binom{n}{3} \Big/ e\right)} < f_3(n, e) < c_4 \frac{e^{1/3} \log n}{\log\left(\binom{n}{3} \Big/ e\right)}.$$

where c_i's are suitable constants.

Unavoidable 3-graphs often consist of disjoint unions of "stars" or "books" or some combinations or modifications of these structures (which will be defined in the later sections.)

2. PRELIMINARIES

Unavoidable 2-graphs (or unavoidable graphs) have been studied by the authors in [2]. First we will state some known theorems on unavoidable graphs that will later be used in deriving results for unavoidable hypergraphs. Note that we define $f(n, e) = f_2(n, e)$.

Facts on Unavoidable Graphs

$$f(n, e) = 1 \qquad \text{if } e \leq \left\lfloor \frac{n}{2} \right\rfloor. \tag{1}$$

$$f(n, e) = 2 \qquad \text{if } \left\lfloor \frac{n}{2} \right\rfloor < e \leq n. \tag{2}$$

$$f(n, e) = \left(\frac{e}{n}\right)^2 + O\left(\frac{e}{n}\right) \qquad \text{if } n < e \leq n^{4/3}. \tag{3}$$

($O(X)$ denotes a quantity within a constant ratio of X).

$$c_5 \frac{\sqrt{e} \log n}{\log\left(\binom{n}{2} \Big/ e\right)} < f(n, e) < c_6 \frac{\sqrt{e} \log n}{\log\left(\binom{n}{2} \Big/ e\right)} \tag{4}$$

for some constants c_5 and c_6 where $cn^{4/3} < e < \binom{n}{2} - n^{1+c'}$ and c' is between 0 and 1.

In particular, we have the following:

$$f(n, e) > (1 + o(1)) \sqrt{2e} \qquad \text{if } e \gg n^{4/3}\left(or \frac{n^{4/3}}{e} = o(1)\right). \tag{5}$$

In particular, $f(n, e) = (1 + o(1))\sqrt{2e}$ if $n^{4/3} < e < n^{3/2}$.

$$f(n,e) = (1 + o(1))\sqrt{2e}\ \log n/\log\left(\binom{n}{2}\Big/e\right) + o(\sqrt{e}) \qquad (6)$$

$$\text{if } n^{4/3} \ll e = o(n^2).$$

The unavoidable graphs in proving (1), (2), and (3) are forests that are disjoint unions of stars. In proving (4), (5), and (6), we use unavoidable graphs that are disjoint unions of bipartite subgraphs. [We note that (3) is better than that in [2], and in (5) we have $f(n, e) = (1 + o(1))\sqrt{2e}$ for $n^{4/3} < e < n^{3/2}$ by considering the common subgraph of three graphs, a clique on $\sqrt{2e}$ vertices, a bipartite graph on e/n and n vertices, and the graph formed by using projective plane [6]].

We will introduce some definitions, useful facts, and related theorems that will eventually be used in the proofs of our main theorems.

A r-graph H is said to be a star if there is a vertex $u \in V(H)$ such that for any two edges E_i, E_j in $E(H)$ we have $E_i \cap E_j = \{u\}$. A r-graph H' is said to be a book if there are $r - 1$ vertices contained in every edge of H'. A *r-graph* is said to be i-intersective if the intersection of any two edges has at most i vertices. By using results on Steiner triple systems [11] it is not too difficult to prove the following (also see [12]).

Lemma 1. If $e \leq (n - 3)(n - 1)/6$, then there exists a 3-graph on e edges which is 1-intersective.

In the other direction, one of the authors proved [1] the following:

Lemma 2. For fixed odd k, any 3-graph with $k(k - 1)n + O(k^3)$ edges must contain a star with k edges. For fixed even k, any 3-graph with $k(k - 3/2)n + O(n + k^3)$ must contain a star with k edges.

Lemma 2 implies that any 3-graph with n vertices and e edges contains a star of $\sqrt{e/n} - 1$ edges. This is almost best possible because of the following result in [4,10].

Lemma 3. There exists a 3-graph on e edges which does not contain a star of at least $\sqrt{e/n} + 2$ edges.

Proof: Consider a 3-graph G with vertex set $X \cup Y \cup Z$ where $|X| = |Y| = k = \lfloor e/n \rfloor + 1$ and $|Z| = n - 2k + 2$ and G has edge set $\{\{u, v, w\}: \{u, v\} \subseteq X \text{ or } \{u, v\} \subseteq Y \text{ and } w \in Z\}$. ∎

It is easy to see that G has at least e edges and G does not contain a star of $k + 1$ edges.

The following two lemmas are obtained by combination probabilistic methods (see [9]).

Lemma 4. If H is a r-graph on p vertices and q edges with the property that H is contained in every graph on n vertices and e edges, then we have

$$q < \frac{p \log n}{\log\left(\binom{n}{r}\Big/ e\right)}.$$

Proof. There are at most n^p ways to map $V(H)$ into $\{1, 2, \ldots, n\}$. Therefore there are at most

$$n^p \binom{\binom{n}{r} - q}{e - q}$$

r-graphs on n vertices and e edges which contain H. Since there are

$$\binom{\binom{n}{r}}{e}$$

r-graphs containing H, we have

$$n^p \binom{\binom{n}{r} - q}{e - q} \geq \binom{\binom{n}{r}}{e}$$

This implies

$$n^p > \left(\frac{\binom{n}{r}}{e}\right)^q$$

and

$$\frac{p \log n}{\log\left(\binom{n}{r}\Big/ e\right)} > q.$$

Lemma 5. Suppose G is a bipartite graph on e edges with $V(G) = V_1 \cup V_2$ where $|V_1| = m$, $|V_2| = n$, and $E(G) \subseteq V_1 \times V_2$. Then G contains a complete bipartite subgraph $K_{a,b}$ with vertex set $U_1 \cup U_2, |U_1| = a, |U_2| = b$, $U_i \subseteq V_i, i = 1, 2$ if

$$m\binom{e/m}{b} \geq a\binom{n}{b}.$$

The proof will be omitted (see [9]).

3. ON $f_3(n, e)$ for $e < n^2/6 - 2n$

Throughout Sections III and IV we deal only with 3-graphs. For small value of e, the values for f_3 (n, e) can be easily determined. For example, for $e \leq n - 2$, it is not difficult to see that $f_3(n, e) \leq 1$, since the common subgraph of a book and an 1-intersective graph has just one edge. Thus, $f_3(n, e) = 1$ for $e \leq n - 2$.

Theorem 1. For $e \leq n^2/6 - 2n$

$$f_3(n, e) = \sqrt{\frac{e}{n}} + O(1).$$

Proof. The lower bound for $f_3(n, e)$ is an immediate consequence of Lemma 2. We only have to consider the upper bound.

Suppose H is an (n, e)-unavoidable 3-graph. Let G_1 denote the 3-graph on $\binom{n-1}{2}$ edges with the property that there is a vertex u being contained in every edge of G_1. Let G_2 denote an 1-intersective graph on e edges. Since H is contained in both G_1 and G_2, H must be a star. Now consider G_3 to be the graph on e edges not containing a star of $\sqrt{e/n} + 2$ edge. This implies that

$$f(n, e) \leq \sqrt{\frac{e}{n}} + 2.$$

This completes the proof of Theorem 2. ∎

We note that for $n < e \leq (n^2/6) - 2n$, the value of $f_3(n, e)$ is just the maximum number of edges in a star that is contained in all 3-graphs on n vertices and e edges. This problem of determining the largest star has been studied in the past [4,10]. It is of interest to determine the exact value of $f_3(n, e)$ for e in this range.

4. ON $f_3(n, e)$ for $cn^2 < e < n^{15/7}$

First we will establish the following upper bound for $f_3(n, e)$.

Theorem 2. If $cn^2 < e < n^{15/7}$, we have

$$f_3(n, e) \leq c' e^{3/2}/n^{5/2}$$

for some constants c and c'.

Proof. Suppose H is an (n, e)-unavoidable graph. We consider the graph G_1 that is the disjoint union of $n^{3/2}/(6e)^{1/2}$ copies* of complete 3-graphs on $(6e/n)^{1/2}$ vertices. Since H is a subgraph on G_1, every connected component of H has at most $(6e/n)^{1/2}$ vertices.

Now we consider the graph G_2 which has $2e/n^2 + 8$ special vertices such that $E(G_2)$ consists of all 3-sets of $V(G_2)$, each of which contains exactly one special vertex. It is easy to check that G_2 has at least e edges. Thus G_2 contains H. Suppose H has t connected components. Since each component contains at least one special vertex, we have $t \leq 2e/n^2 + 8$. Therefore H has at most $(2e/n^2 + 8)(6e/n)^{1/2}$ vertices. From Lemma 4 we know that the number of edges in H is bounded above by $(7/6)|V(H)|$. We conclude that $|E(H)|$ is at most $6e^{3/2}/n^{5/2}$. ∎

Before we go on to establish the lower bound for $f_3(n, e)$, we need some auxiliary facts about a special kind of subgraphs, called book-stars. We say a graph T is a book-star of type (r, s) if T has vertices $u_0, u_1, \ldots, u_r$ such that for each i there are exactly s edges containing $\{u_0, u_i\}$ and any vertex in $V(H) - \{u_0, u_1, \ldots, u_r\}$ is in exactly one such edge. A book-star of type (r, s) can be viewed as the union of r books of size s intersecting at one vertex. We call u_0 to be the center of T and u_i to be a spine of T. A vertex of T which is not the center or a spine is called a leaf.

Lemma 6. Any 3-graph on n vertices and e edges contains a book-star of type $(n^{3/2}e^{-1/2}/10, en^{-2}/10)$ if $e \leq n^{7/3}$.

Proof. Let H denote a 3-graph on n vertices and e edges that does not contain a book-star T of type $(n^{(1-\alpha)/2}/10, n^{\alpha/10})$ where $e = n^{2+\alpha}$ [note that $(1 - \alpha)/2) > \alpha$ for $\alpha < (1/7)$]. Suppose v is a vertex of H having degree at least $n^{1+\alpha}$. We consider the neighborhood graph H_v whose edge set is $\{\{u_1, u_2\} = \{u_1, u_2, v\} \in E(H)\}$ (note that H_v is a 2-graph with at least $n^{1+\alpha}$ edges). Since H does not contain T, H_v does not contain a disjoint union of $n^{(1-\alpha)/2}/10$ copies of (2-) stars each with $n^{\alpha/10}$ edges. Let d_v denote $|E(H_v)|$.

*Strictly speaking, we should use $\lfloor n^{3/2}/(6e)^{1/2} \rfloor$ instead of $n^{3/2}/(6e)^{1/2}$. However we will usually not bother with this type of detail since it has no significant effect on the argument or results.

Here we use Theorem 2 in [2], which states that any (2-) graph on n vertices and m edges contains a disjoint union of (2-) stars; $S_t, S_{t-2}, \ldots$, where $t = [(1-\epsilon)(2m/n)]$ if $n < \epsilon n^{4/3}$. This leads to a contradiction, and Lemma 6 is proved.

Theorem 3. Any 3-graph on n vertices and e edges with $n^2 \le e \le n^{15/7}$ contains the vertex disjoint union of $e/20n^2$ copies of book-stars of type $(n^{3/2}e^{-1/2}/20, en^{-2}/20)$.

Proof. Suppose $e = n^{2+\alpha}$ and $\alpha \le 1/7$. Let H denote a 3-graph on n vertices and e edges that does not contain the vertex disjoint union of $n^{\alpha}/20$ copies of book-stars T of type $(n^{(1-\alpha)/2}/20, n^{\alpha}/20)$. Now we partition the vertex set $V(H)$ into three parts. A vertex of degree at least $10n^{(3+\alpha)/2}$ is said to be an A-vertex. A vertex is said to be a B-vertex if it has degree at least $10n^{(3-\alpha)/2}$ and at most $10n^{(3+\alpha)/2}$. The rest of the vertices are called C-vertices. Note that there are at most $n^{(1+\alpha)/2}/3$ A-vertices and there are at most $n^{(1+3\alpha)/2}/3$ B-vertices. Let t denote the maximum number such that t copies of book-star T can be embedded into H such that the leaves of T are embedded into C-vertices and the spines of T are embedded into vertices that are not A-vertices. We have $t < n^{\alpha}/20$. Let R denote the set of vertices of H onto which t copies of T are embedded. Then $|R| = t \cdot n^{(1+\alpha)/2}/400$. We make the following observations:

(1) The number of edges containing some vertices in R is at most

$$t \cdot n^2 + t \cdot n^{(1-\alpha)/2} n^{(3+\alpha)/2} + tn^{(1+\alpha)/2} n^{(3-\alpha)/2}/10 \le 3n^{2+\alpha}/20$$

since there are at most tn^2 edges containing the t centers, there are at most $tn^{(1-\alpha)/2}n^{(3+\alpha)/2}$ edges containing the spines, and there are at most $tn^{(1+\alpha)/2}n^{(3-\alpha)/2}/10$ edges containing the leaves.

(2) The number of edges containing one A-vertex and two B-vertices is at most $n^{1+3\alpha+(1+\alpha)/2}/10 \le n^{2+\alpha}/10$. The number of edges containing three B-vertices is at most $n^{(1+3\alpha)3/2}/27 \le n^{2+\alpha}/20$ since $\alpha \le 1/7$.

(3) The number of edges that contain at least two A-vertices is at most $n^{2+\alpha}/10$ since there are at most $n^{(1+\alpha)/2}/3$ A-vertices.

(4) Let W denote the set of A-vertices or B-vertices, each of which is contained in at most $n^{(3-\alpha)/2}/10$ edges in the induced subgraph of H on $S = V(H) - R$, denoted by H'. The number of edges in H' containing some vertex in W is at most $n^{(1+3\alpha)/2}n^{(3-\alpha)/2}/10 \le n^{2+\alpha}/10$ since $\alpha \le 1/7$.

(5) Let W' denote the set of pair $\{u, v\}$ where u is an A-vertex or B-vertex, v is a B-vertex, and $\{u, v\}$ is contained in at most $n^{(1+\alpha)/2}/10$ edges. The number of edges in H' containing a pair in W' is at most $n^{1+3\alpha+(1+\alpha)/2}/90 \le n^{2+\alpha}/90$.

Therefore by removing all edges mentioned in (1)–(5) we obtain a subgraph H'' of H satisfying the following properties:

(i) H'' has at least $n^{2+\alpha}/2$ edges and $V(H'') \cap R = \phi$.

(ii) An A-vertex in H'' has degree at least $n^{(3-\alpha)/2}/10$ and is not adjacent to other A-vertices.

(iii) A B-vertex in H'' has degree at least $n^{(3-\alpha)/2}/10$. There is no edge containing one A-vertex and two B-vertices. There is no edge containing three B-vertices.

(iv) Any pair of vertices $\{u, v\}$ where u is an A-vertex or a B-vertex and v is a B-vertex contained in at least $n^{(1+\alpha)/2}/10$ edges in H''.

Suppose H'' contains an A-vertex or B-vertex u. If u is adjacent to $n^{(1-\alpha)/2}/20$ B-vertices, then by (iv) we can embed a copy of T into H'' by choosing u to be the center and the $n^{(1-\alpha)/2}/20$ B-vertices to be the spine. By (iii) the leaves then are all C-vertices. This is impossible. We may assume u is adjacent to at most $n^{(1-\alpha)/2}/10$ B-vertices. In the neighborhood graph H''_u of u in H'', there are at least $n^{(3-\alpha)/2}/10$ edges. It can be easily proved that H''_u contains the forest F consisting of $n^{(1-\alpha)/2}/10$ copies of stars with $n^{\alpha}/10$ edges such that all leaves are in C. (Otherwise, all vertices in $V(H''_u) - V(F)$ have degree at most $n^{\alpha}/10$. There are at most one vertex in each star of F adjacent to more than $n^{\alpha}/5$ vertices in $V(H''_u) - V(F)$. H''_u can have at most $n^{1+\alpha} + n^{(3-\alpha)/2}/20$ edges that is less than $n^{(3-\alpha)/2}/10$.) This contradicts the assumption that t is maximum.

We may assume H'' does not contain A vertices or B vertices. Now from Lemma 6 we know there is a copy of book-star of type $(n^{(1-\alpha)/2}/20, n^{\alpha}/20)$ being contained in H'' since H'' has $e/2$ edges. This leads to a contradiction. Theorem 3 is proved. ∎

From Theorem 2 and 3, we conclude:

Theorem 4. If $n^2 \leq e < n^{15/7}$, we have

$$c_1 e^{3/2} n^{-5/2} \leq f(n, e) \leq c_2 e^{3/2} n^{-5/2}$$

for some constant c_1 and c_2.

We remark that an unavoidable graph H that is the vertex-disjoint union of stars cannot contain more than $10e^{1/2}n^{-1/2}$ edges. This can be proved as follows. We consider the graph containing $(2e/n)^{1/2}$ special vertices with the property that any edge of H contains exactly two special vertices. It is easy to see that H can contain at most $2(2e/n)^{1/2}$ vertices. We note that $e^{1/2}n^{-1/2}$ is smaller than $e^{3/2}n^{-5/2}$ for $n^2 \leq e$. Also an unavoidable graphs $\overline{H}$, which is the vertex-disjoint union of books, can contain at most $n^{2\alpha}$ edges (by considering two graphs of e edges: G_1 with every pair in $\leq 10n^{\alpha}$ edges, G_2 consisting of n^{α} special vertices and all edge containing one special vertex). The book-stars, a modified combination of books and stars, are the largest unavoidable graphs we can find so far for this range of e. It would be of interest to prove that the maximum unavoidable graphs have the structure of book-stars or similar types.

5. ON $f_3(n, e)$ for $n^{15/7} < e < n^{3-\epsilon}$

Using Lemma 4 we can easily obtain the following upper bound for $f_3(n, e)$.

Theorem 5. If $n^{15/7} < e \leq n^{3-\epsilon}$, we have

$$f_3(n, e) \leq ce^{1/3}$$

where c is a constant depending on ϵ.

Proof. An (n, e)-unavoidable graph H must be a subgraph of a graph on $(6e)^{1/3}$ vertices. H has at most $(6e)^{1/3}$ vertices. Thus from Lemma 4 H has at most $(6e)^{1/3}/\epsilon$ edges. ∎

We will prove the above upper bound for $f_3(n, e)$ is asymptotically best possible within a constant factor for this range of e.

The proof is very similar to that of Theorem 3, although the values of various parameters are different.

Theorem 6. Any 3-graph on n vertices and e edges with $e > n^{15/7}$ contains the vertex-disjoint union of $n^{1/2}/20e^{1/6}$ copies of book-stars of type $(n^{3/2}e^{-1/2}/20, en^{-2}/20)$.

Proof. Suppose $e = n^{2+\alpha}$ where $1/7 \leq \alpha$. Let H denote a 3-graph on n vertices and e edges that does not contain the vertex-disjoint union of $n^{(1-\alpha)/6}/20$ copies of book-stars T of type $(n^{(1-\alpha)/2}/20, n^{\alpha}/20)$. We first partition the vertex set $V(H)$ into three parts. A vertex of degree at least $10n^{(4+5\alpha)/3}$ is said to be an A-vertex. A vertex is called to be a B-vertex if it has degree at least $10n^{(4+2\alpha)/3}$ and at most $10n^{(4+5\alpha)/3}$. The rest of the vertices of H are called C-vertices. Note that there are at most $n^{(2-2\alpha)/3}/3$ A-vertices. There are at most $n^{(2+\alpha)/3}/3$ B-vertices. Let t denote the maximum number such that t copies of book-stars T can be embedded in H in a way that the leaves of T are embedded into C-vertices and the spines of T are embedded into vertices that are not A-vertices. We have $t < n^{(1-\alpha)/6}/20$. Let R denote the set of vertices of H onto which t copies of T are embedded. Then $|R| = tn^{(1+\alpha)/2}/400$. We first make the following observations:

(1) The number of edges containing some vertices in R is at most

$$tn^2 + tn^{(1-\alpha)/2}n^{(4+5\alpha)/3} + tn^{(1-\alpha)/2}n^{\alpha}n^{(4+2\alpha)/3}/20 < 3n^{2+\alpha}/20 \quad \text{since} \quad \alpha \geq 1/7.$$

(2) The number of edges that does not contain any C-vertex is at most $n^{2+\alpha}/10$.

(3) The number of edges which contain at least two A-vertices is at most

$$n^{1+(4-4\alpha)/3}/10 < n^{2+\alpha}/10.$$

(4) Let W denote the set of vertices v such that v is an A-vertex or a B-vertex and v is contained in at most $n^{(4+2\alpha)/3}/10$ edges in the induced subgraph of H on $S = V(H) - R$, denoted by H'. The number of edges in H' containing some vertex in W is at most $n^{2+\alpha}/10$.

(5) Let W' denote the set of pairs $\{u, v\}$ where u is an A-vertex or a B-vertex, v is a B-vertex and $\{u, v\}$ is contained in at most $n^{(1+\alpha)/2}/10$ edges. The number of edges in H' containing a pair in W' is at most $n^{(4+2\alpha)/3}n^{(1+\alpha)/2}/90 \leq n^{2+\alpha}/90$.

Therefore by removing all edges mentioned in (1)–(5), we obtain a subgraph H'' of H satisfying the following properties:

(i) H'' has $n^{2+\alpha}/2$ edges and $V(H'') \cap R = \phi$.
(ii) An A-vertex in H'' has degree at least $n^{(4+2\alpha)/3}/10$ and is not adjacent to other A-vertices.
(iii) A B-vertex in H'' has degree at least $n^{(4+2\alpha)/3}/10$. Every edge contains at least one C-vertex.
(iv) Any pair of vertices $\{u, v\}$ where u is an A-vertex or a B-vertex and v is a B-vertex is contained in at least $n^{(1+\alpha)/2}/10$ edges in H''.

Suppose H'' contains an A-vertex or a B-vertex u. If u is adjacent to $n^{(1-\alpha)/2}/20$ B-vertices, then we can embed a copy of T into H'' by choosing u to be the center. This is impossible. We may assume u is adjacent to at most $n^{(1-\alpha)/2}/20$ B-vertices. In the neighborhood H_u of u in H, there are $n^{(4+2\alpha)/3}/10$ edges. From the theorem on 2-graphs we know that H_u contains $n^{(1-\alpha)/2}/10$ copies of stars with $n^{\alpha}/10$ edges since $\alpha > 1/7$, and $(1 + 2\alpha)/3 \geq (1 - \alpha)/2$. (We can easily choose the leaves of the stars to be C-vertices). This again contradicts the assumption that t is maximum. We may assume H'' does not contain any A-vertex or B-vertex. From Lemma 6 we know that H'' contains a book-star of type $(n^{(1-\alpha)/2}/20, n^{\alpha}/20)$. This contradicts the maximality of t. Theorem 6 is proved. ∎

As an immediate consequence of Theorem 5 and Theorem 6, we obtain the following:

Theorem 7. If $n^{15/7} \leq e < n^{3-\epsilon}$, we have

$$c'e^{1/3} < f(n, e) < ce^{1/3}.$$

6. ON $f_3(n, e)$ for $n^{3-\epsilon} < e < cn^3$

The value of $f_3(n, e)/e^{1/3}$ is unbounded when e is close to $\binom{n}{3}$. To see this, we need the following fact, which can be proved in a similar way as that in [5].

Lemma 7. Any 3-graph H on n vertices and e edges contains the complete 3-partite subgraph $K(s, s, t)$ with $s = (\log n/(2 \log(\binom{n}{3}/e)))^{1/2}$ and $t = n^{1/3}$ where $K(n_1, n_2, n_3)$ has vertex set $V_1 \cup V_2 \cup V_3$, $|V_i| = n_i$ and edge set $\{\{v_1, v_2, v_3\}: v_i \in v_i\}$.

Proof. First we use Lemma 5 repeatedly. Consider a bipartite graph G with vertex set $V_1 \cup V_2$ where V_1 consists of pairs of vertices in H and $V_2 = V(H)$ and $\{w_1, w_2\} \in E(G)$ if $w_1 \cup w_2$ is an edge of H. Then G contains a complete bipartite graph $K_{s,t'}$ with $t' = n^{2-2/s}$ since

$$\frac{n^2}{2}\binom{\frac{6e}{n^2}}{s} \geq n^{2-2/s}\binom{n}{s}.$$

This means there is a set A of s in $V(H)$ vertices with the property that there are t' pairs of vertices $\{u, v\}$ of $V(H)$ such that $\{u, v, w\} \in E(H)$ for any $w \in A$. Let G'' denote the 2-graph on these t' edges. From the theorems on 2-graphs (see [2]), it follows that G'' contains a bipartite subgraph $K_{s',t}$ where $s' = \log n/(2 \log(\binom{n}{2}/t')) = s$ and $t = n^{1/3}$. This completes the proof of Lemma 7. ∎

Theorem 8. Any 3-graphs on n-vertices and e-edges with $e > n^{8/3}$ contains the vertex-disjoint union of $e^{1/3}n^{-1/3}$ copies of $K(s, s, t)$ where $s = \log n/(3 \log(\binom{n}{3}/e))$ and $t = n^{1/3}/10$.

Proof. Suppose H is a 3-graph on n vertices and e edges that does not contain $e^{1/3}n^{-1/3}/10$ copies of $K(s, s, t)$. A vertex is H is called a C-vertex if its degree is at most $10e^{2/3}$. There are at most $e^{1/3}/3$ vertices that are not C-vertices. Let w denote the maximum number of vertex disjoint copies of $K(s, s, t)$ that can be embedded in H, whereas in each copy the third vertex set (of the t vertices) of $K(s, s, t)$ are embedded into C-vertices. Suppose $w < e^{1/3}n^{-1/3}$. Let R denote the set of vertices of H onto which copies of $K(s, s, t)$ are embedded and S denotes $V(H) - R$. We make the following observations:

(1) The number of edges containing some vertices in R is at most

$$2w \cdot sn^2 + e^{1/3} \cdot e^{2/3} \leq e/3 \quad \text{since} \quad e > n^{8/3}.$$

(2) The number of edges that do not contain any C vertex is at most $e/10$.

Therefore by removing all edges mentioned in (1) and (2), we obtain a subgraph H'' of H satisfying the following properties:

(i) H'' has $e/2$ edges and $V(H'') \cap R = \phi$.
(ii) Any edges of H'' contains at least one C-vertex.

From Lemma 7, H'' contains a $K(s,s,t')$ where $t' = n^{1/3}$. Since there are at most $e^{1/3}/3$ vertices which are not C-vertices, we can choose $n^{1/3}/10$ out of t' vertices to be C-vertices. This contradicts the assumption that w is maximum. Theorem 8 is proved. ∎

Theorem 9. If $n^{15/7} < e$ we have

$$c_1 \frac{e^{1/3}\log n}{\log\left(\binom{n}{3}\Big/e\right)} < f_3(n,e) < c_2 \frac{e^{1/3}\log n}{\log\left(\binom{n}{3}\Big/e\right)}$$

for some constants c_1 and c_2.

Proof. From Theorems 7 and 8 we have an unavoidable graph on $ce^{1/3}$ vertices and edges for $n^{15/7} < e < n^{8/3}$ and (n,e)-unavoidable subgraphs on $(c'e^{1/3}\log n/\log(\binom{n}{3})/e)$ edges for $n^{8/3} < e$. From Lemma 4 we have an upper bound $((6e)^{1/3}\log n/\log(\binom{n}{3})/e))$ since an (n,e)-unavoidable subgraph can have at most $(6e)^{1/3}$ (nontrivial) vertices. This completes the proof of Theorem 9. ∎

7. CONCLUDING REMARKS

In this paper we deal with unavoidable 3-graphs. A natural direction is to investigate unavoidable r-graphs for general r. As we can see the structures of unavoidable 3-graphs are more complex than unavoidable 2-graphs, and proofs for theorems on 3-graphs are more complicated. For the case of $r = 4$, the authors have obtained some partial results. We note that the unavoidable hypergraphs often turn out to be combinations and modifications of strong Δ-systems, which can be viewed as generalizations of stars. [A strong Δ-system with parameter (k,t) consists of a collection of k r-sets such that the intersection of any pair of them is a fixed t-set (see [7,8]).] P. Erdös and R. A. Duke [4] first investigated maximum unavoidable strong Δ-systems in large families of r-sets. Namely, they ask the question of determining the smallest integer $M = f(n,r,k,t)$ with the property that if any r-graph on n vertices and m edges must contain a strong Δ-system of type (k,t). P. Frankl, Z. Furedi, and the authors have studied the asymptotical estimates or bounds for such unavoidable strong Δ-systems. Except for the case of $r = 3$, the results for higher ($r \geq 4$) strong Δ-systems are far from satisfactory. This explains the difficulty in determining $f_r(n,e)$.

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