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Diameters Of Communication Networks

F. R. K. CHUNG*

ABSTRACT. When graphs are used to model the linkage structure of communication networks, the diameter of the graph corresponds to the maximum number of links over which a message between two nodes must travel. In cases where the number of links in a path is roughly proportional to the time delay or signal degradation encountered by messages sent along the path, the diameter is then involved in the complexity analysis for the performance of the networks. A variety of interrelated diameter problems will be discussed here, including: determining extremal graphs of bounded degrees and small diameters, finding orientations for undirected or mixed graphs to minimize diameters; investigating diameter bounds for networks with possible node and link failures, and algorithms aspects for determining the diameters of graphs.

1. Introduction.

Modern communication networks typically are highly complex structures formed by various interconnected components. Many principal characteristics of communication networks often result from the topology of the underlying connection patterns of the network. Graph theory can then be used to study the linkage structure of the network and to model problems arising in the optimization and analysis of the networks, (see [11]).

A graph G consists of a finite set $V(G)$ of vertices (or nodes) together with a prescribed set $E(G)$ of unordered pairs of vertices of $V(G)$. The vertices represent objects in a network and the pairs, called edges (or links) represent the interconnections between objects. We note that the exact geometric positions of the vertices or the lengths of the edges are not important unless specified.

For two vertices u and v in a graph G, a path P of length t from u to v is a sequence of distinct vertices $u = a_0, a_1, \ldots, a_t = v$, together with edges $\{a_i, a_i+1\}, i = 0, \ldots, t-1$, *in* G. A graph is said to be *connected* if every pair of vertices are joined by a path. In a connected graph G, the distance $d_G(u,v)$ between two vertices u and v is the length of a shortest path joining u and v in G. The diameter $D(G)$ *of* G is the maximum value of $d_G(u,v)$, taken over all pairs of vertices $u,v \epsilon V(G)$ (see Fig. 1).

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Bell Communications Research, Morristown, New Jersey 07960.

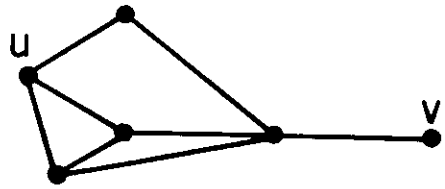

FIGURE 1

In the graph model for communication networks, the diameter of the graph corresponds to the maximum number of edges over which a message between two nodes must travel. In cases where the number of edges in a path is roughly proportional to the time delay or signal degradation encountered by messages sent along the path, the diameter is directly involved in the analysis and the optimization of the networks. In particular, diameter-related problems often arise in connection with analyzing the computational complexity of routing, distributing and scheduling algorithms.

Before we proceed to several interrelated diameter problems, we will first introduce some definitions.*

For a given graph G, a subgraph G' *of* G is given by taking $V(G') \subsetneq V(G)$ and $E(G') \subsetneq E(G)$. A maximum connected subgraph of G is called a *connected component of* G. A *bridge* of a connected graph G is an edge whose removal disconnects G. If G has no bridge, it is called bridgeless.

In a graph G, a path is formed by a sequence of distinct vertices $a_0, a_1, \ldots, a_t$ together with edges

* For undefined graph-theoretic terminology, the reader is referred to [2, 15, 33].

$\{a_i, a_i+1\}$, $i = 0,...,t-1$. A cycle is formed by a path $a_0, \ldots, a_t$ together with the edge $\{a_t, a_0\}$. An acyclic connected graph is called a *tree*. For a vertex v in G the degree of v, denoted by $deg_G(v)$, is the number of edges $\{u,v\}$ in G which contain v. These u's are called the neighbors of v. We note that the maximum degree of a graph is also a useful parameter in network optimization since practical networks commonly satisfy some degree constraints. We will discuss extremal graphs with small diameter and bounded degrees in Sections II and III.

In cases that some links only allow one-way traffic in communication networks we will then consider directed graphs or mixed graphs. A directed graph G is formed by a vertex set together with a prescribed set of ordered pairs of vertices. The edge set of a mixed graph contains ordered and unordered pairs of vertices. Note that both undirected and directed graphs are special cases of mixed graphs. We shall denote an edge with end vertices u and v,, by $[u,v]$ if it is either undirected or directed from u to v. Similarly we define paths and connectivity in mixed graphs. A path P in G from u to v is a sequence of distinct vertices $u = a_0, a_1, \ldots, a_t = v$ so that $[a_i, a_{i+1}], i = 0,...,t-1$, are edges in G, and P is called a walk if a_i's are not necessarily distinct. G is strongly connected if every pair of vertices is joined by a path. An orientation of a mixed graph G is any directed graph obtained by directing every undirected edge in G. G is orientable if there is an orientation of G which is strongly connected. In Section IV we will discuss the problem of choosing orientations of a mixed graph subject to minimizing the diameter in the resulting directed graphs.

In Section V we will consider diameter problems associated with networks having possible link or node failures. We will study extremal problems for fault-tolerant graphs with small diameters and investigate the diameter bounds in general graphs while a small number of vertices or edges are deleted. In Section VI we will discuss fast algorithms for finding the diameter of a graph. Many unresolved problems will be mentioned.

2. Large graphs with bounded degree and diameter.

The following problem has been studied by many researchers in the past (see [3, 4, 5, 6]):

How many vertices can a graph have which has diameter D and degree at most k?

This problem can be viewed as a network optimization problem of connecting as many processors as possible while each processor has a bounded number of ports (degree constraint) and the delay in data transmission is small (diameter constraint). The maximum number $n(k,D)$ of vertices in a graph with diameter D and maximum degree k is bounded above by

$$n(k,D) \leqslant 1+k+\cdots+k(k-1)^{D-1} = n_0(k,D)$$

since there are at most $k(k-1)^{i-1}$ vertices at a distance $i \geqslant 1$ from a vertex.

The upper bound $n_0(k,D)$, called Moore bound, is provably unreachable [34] for almost all nontrivial values of k and D. The only graphs, called Moore graphs, which achieve the Moore bounds must be one of the following [34]:

(i) $D = 1$, $(k+1)$-cliques

(ii) $k = 2$, $(2D+1)$-cycles

(iii) $D = 2$ and $k = 3$, the Petersen graph

(iv) $D = 2$ and $k = 7$, the Hoffman-Singleton graph

(v) (possibly) $D = 2$ and $k = 57$.

Problem 1 [34]: Is there a Moore graph of diameter 2 and degree 57?

Best known lower bound for $n(k,D), k,D \leqslant 10$ can be found in Table 1 (see [7]).

D / k	2	3	4	5	6	7	8	9	10
3	10	20	38	70	128	180	286	462	708
4	15	40	95	364	731	856	1872	3708	7000
5	24	66	174	532	2734	2988	7000	11340	30240
6	32	105	317	820	7817	10920	19138	43744	131232
7	50	122	420	1550	8998	31248	62536	156340	562824
8	57	200	807	2550	39223	40593	154800	327689	1310729
9	74	585	1178	5050	74906	156864	480250	1176690	5883450
10	91	650	1755	7550	132869	380835	1117550	2696616	14981200
11	94	715	2925	11388	142494	723060	1990050	5580498	33217250
12	133	780	4680	17563	354323	1065285	3778261	11757325	85887453
13	136	345	5265	25844	394616	1414440	7211386	24340680	130631232
14	183	910	5850	37107	804481	2130310	12694773	46243080	322828871
15	186	1215	7605	54796	892062	5133375	22303302	68145480	550731776

TABLE 1

For the upper bound the only result beyond $n(k,D)<n_0(k,D)$ (except for (i)-(v)) was obtained by P. Erdös, S. Fajtlowicz and A. J. Hoffman [27] who proved that

$$n(2k,2) \leqslant n_0(2k,2)-2 \text{ for } k > 1.$$

Problem 2 [12, 26]: Is it true that for every integer c there exist k and D such that

$$n(k,D) \leqslant n_0(k,D)-c \ ?$$

There are two different approaches for establishing the lower bounds for $n(k,D)$: explicitly construct such a good graph or prove by probabilistic methods the existence of a good graph. For practical concern the first approach is much more desirable although the second often gives better bounds (so far).

Many of the explicit constructions are extensions or modifications of the de Bruijn graphs. Here we describe briefly the structure of de Bruijn graphs.

For given integers r and s, the de Bruijn graph $B(r,s)$ has s^r vertices represented by r-tuples $(a_1,a_2,\ldots,a_r)$, where $a_i \epsilon \{1,\ldots,s\}$ and $(a_1,a_2,\ldots,a_r)$ is adjacent to $(a_2,\ldots,a_r,b)$ and $(b,a_1,\ldots,a_{r-1})$ for any $b \epsilon \{1,\ldots,s\}$. It is easily seen that $B(r,s)$ has diameter r and degree $2s$.

This gives

$$\mu_D = \liminf_{k \to \infty} \frac{n(k,D)}{n_0(k,D)} \geqslant 2^{-D}.$$

For some small fixed values of D, the ratio of $n(k,D)$ and $n_0(k,D)$ can be arbitrarily close to one for sufficiently large k due to the explicit construction of large classes of graphs using combinational structures such as generalized n-gons and product constructions (see [2,5]), which we will describe.

A generalized polygon can be viewed as a bipartite graph G with vertex set $P \cup L$ having the property that for any two vertices x,y of distance $d(x,y) <$ diameter (G), there is a unique path of length $d(x,y)$ joining x and y. Elements in P and L will be called points and lines, respectively, and we say $x \epsilon P$ belongs to (or is incident with) $l \epsilon L$ if and only if (x,l) is an edge in G.

If the degree of each vertex of G is at least 3 and the diameter is n, G is called a thick generalized n-gon. It turns out that any two vertices in P have the same degree and any two vertices in L have the same degree. We say the generalized n-gon has parameter (s,t) if every line contains $s+1$ points and every point is contained in $t+1$ lines. A theorem of Feit and Higman [29] states that thick generalized n-gons only exist for $n = 2,3,4,6$ and 8.

For example, the graphs of generalized 2-gons are precisely the complete bipartite graphs. The thick generalized 3-gons are the nondegenerate projective planes with q^2+q+1 points and lines where q is a prime power (see [37]).

Generalized 4-, 6-, and 8-gons are known as generalized quadrangles, hexagons, and octagons, respectively. There are several types of generalized quadrangles, namely, the classical polar spaces Sp (4,q), $0^-(6,q)$ and $U(5,q^2)$ where q is a prime power. These constructions are rather complicated (the reader is referred to [37,48,49]). There are two known types of generalized hexagons $G_2(q)$ and ${}^3D_4(q^3)$ [see 37,48] and one generalized octagon ${}^2F_4(q)$ which requires q to be an odd power of 2 (see [37,52]).

Now for a bipartite graph G with vertex set $P \cup L$, the self product G^2 has vertex set $P \times L = \{(p,l): p \epsilon P, l \epsilon L\}$ while (p,ℓ) is adjacent to (p',ℓ') if p is contained in ℓ' and p' is contained in ℓ. It is not difficult to check that G^2 has diameter one less than the diameter of G and G is regular of degree k_2k_2 if vertices in P has degree k_1 and vertrices in L has degree k_2 in G (see [5]).

Now suppose we consider the generalized octagon ${}^2F_4(q)$ which has diameter 8 with parameter (q,q^2) and contains $(1+o(1))q^{10}$ points and $(1+o(1))q^{11}$ lines. The self product of ${}^2F_4(q)$ has then $(1+o(1))q^{21}$ vertices with diameter 7 and maximum degree $(1+o(1))q^3$. As an immediate consequence, we have

$$\mu_7 = \liminf_{k\to\infty} \frac{n(k,7)}{n_0(k,7)} = 1.$$

Although these methods were known in the proofs of $\mu_3 = \mu_5 = 1$. Bert Wells first proved this fact about μ_7. Also, he pointed out that by using the following inequality derived by Delorme in [24]

$$\mu_{D+1} \geqslant 2\mu_D D^D (D+1)^{-(D+1)},$$

one gets $\mu_8 \geqslant 2\cdot 7^7 8^{-8}, \mu_9 \geqslant 4\cdot 7^7 9^{-9}$ and $\mu_{10} \geqslant 8\cdot 7^7 10^{-10.}$

Here are best known lower bounds for μ_D, for $D \leqslant 10$,

D	1	2	3	4	5	6	7	8	9	10
μ_D	1	1	1	$3^3\, 2^{-7}$	1	$2\cdot 5^5\, 6^{-6}$	1	$2\cdot 7^7\, 8^{-8}$	$4\cdot 7^7\, 9^{-9}$	$5\cdot 2^{10}$

Using probabilistic approaches B. Bollobas and W. F. de la Vega [14] obtained a much stronger lower bound: $n(k,D) \geqslant \dfrac{c}{k^3 D \log(k-1)} n_0(k,D)$ for some constant c.

The directed analogue of this problem turns out to be much more tractable. We can define the maximum number $\bar{n}(k,D)$ of vertices in a directed graph of outdegree $\leqslant k$ and diameter D. Then $\bar{n}(k,D)$ must satisfy

$$\bar{n}(k,D) < 1+k+k^2+ \ldots +k^D = \bar{n}_0(k,D).$$

W. G. Bridges and S. Toueg [15] showed that the directed Moore bound $\bar{n}_0(k,D)$ is not achievable for $k,D > 1$. M. Imase and M. Itoh [35] construct a regular directed graph with n vertices, outdegree k and diameter $\lceil \log_k n \rceil$. This implies $\bar{n}(k,D) \geqslant k^D$. A later result in [44] gives a construction of a directed graph on $k^D + k^{D-1}$ vertices with outdegree k and diameter D.

Problem 3 [16,35]: Determine the exact values for $\bar{n}(k,D)$.

3. Sparse graphs with bounded degree and diameter.

P. Erdös and A. Rényi [25] asked the following question in 1962:

Suppose there are n cities such that the airport of each city can handle at most k flights. It is desirable to schedule the flights in such a way that from each city it is possible to fly to another city with at most t stops along the way. What is the minimum number of flights which must be set up to satisfy the stated requirements? In other words, what is the least umber $e(n,k,D)$ of edges in a graph with n vertices having degree at most k and diameter at most D where $D = t+1$?

The above simple-looking problem turns out to be unexpectly difficult. Relatively few exact values for $e(n,k,D)$ are known so far except for some of the cases with $D \leqslant 3$. Some partial results and estimates were obtained in the past [12, 25, 28] and numerous questions still remain unresolved.

In Fig. 2 we illustrate the extremal graph G with n vertices having diameter 2, degree $\leqslant k, (2n-2)/3 \leqslant k \leqslant n-5$, and $e(n,k,2) = 2n-4$ edges (see [12,25]).

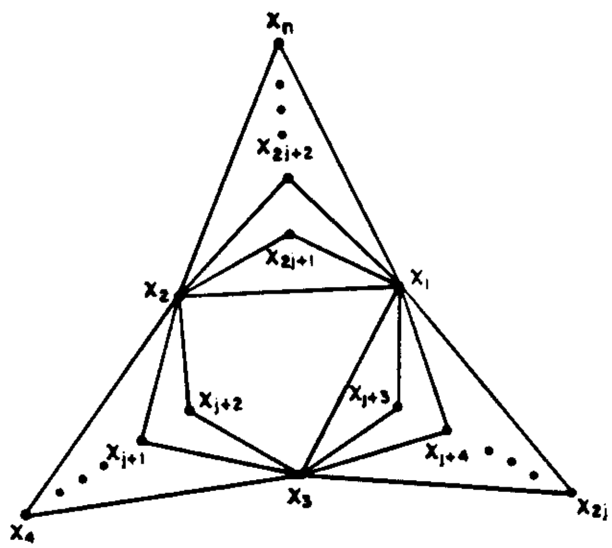

FIGURE 2

Known exact values for $e(n,k,d)$ are (see [25, 26], also see [12]):

$$e(n,k,2) = \begin{cases} n-1 & \text{if } k = n-1 \\ n+k-2 & \text{if } k = n-2 \text{ or } n-3 \\ 2n-5 & \text{if } k = n-4 \\ 2n-4 & \text{if } (2n-2)/3 \leqslant k \leqslant n-5 \\ 3n-k-6 & \text{if } \frac{3n-3}{5} \leqslant k < \frac{2n-2}{3} \\ 5n-4k-10 & \text{if } \frac{5n-3}{9} \leqslant k \leqslant \frac{3n-3}{5} \\ 4n-2k-11 & \text{if } \frac{n+1}{2} \leqslant k < \frac{5n-3}{9} \end{cases}$$

$e(n,k,3) = n + \binom{s}{2} - 1$ if $[n/(s+1)]+s-1 \leqslant k < [n/s]+s-2$ and $1 \leqslant s \leqslant [(n/2)^{1/3}]$.

J. Pach and L. Suranyi [41] proved that $e(n,cn,2)/n$ tends to a limit $g(c)$ for n large and any fixed c between 0 and 1. The function $g(c)$ is a piecewise linear function except at a sequence of "turning" points. The values of $g(c)$ can be determined using linear programming for any fixed c.

For $D \geqslant 3$, a lower bound for $e(n,k,D)$ is obtained in [26]:

$$e(n,k,D) \geqslant \frac{n^2}{k^{D-1}}(1-(n/k^D)^{1/3}).$$

It is also proved in [22] that

$$e(n+p,k,D+2) \leqslant e(n,k,D)+p$$

for $0 \leqslant p \leqslant kn-2e(n,k,D)$.

Problem 4: What is the least number $\bar{e}(n,k,D)$ of edges in a directed graph with n vertices having diameter D and degree (the sum of indegree and outdegree) $\leqslant k$?

Clearly, $\bar{e}(n,k,D) \leqslant 2e(n,[k/2],D)$.

G. Katona and E. Szemeredi [38] proved that any directed graph with n vertices, which does not contain any cycle of length 2 and has diameter 2, must have at least $n \log_2 n$ edges and that is the best possible. We can ask analogous questions for directed graphs with degree constraints and other constraints as well, for example, having no small cycles.

4. Orientations of mixed graphs with small diameters.

In 1939 H. Robbins [45] asked the following question: "When is it possible to find an assignment of one-way directions for all the streets in a town while preserving the property that it is possible to reach any point in town from any other point?".

He solved [45] this problem by proving an (undirected) graph is orientable if and only if it is connected and has no bridge. F. Boesch and R. Tindell [8] considered the more general case for a town in which some, but not all, of the streets are already one-way streets. They proved [8] that a mixed graph is orientable if and only if it is strongly connected and has no bridge.

J. A. Bondy and U. S. R. Murty raised the problem of determining how much the diameter can increase in the process of orienting edges while preserving strong connectivity. V. Chvátal and C. Thomassen [22] subsequently proved that every bridgeless (undirected) graph of diameter D admits an orientation of diameter $2D^2+2D$. On the other hand, they show there is a bridgeless graph G of diameter D with the property that any orientation of G has diameter at least $D^2/4+D$ (see Fig. 3 for the case of $D=4$).

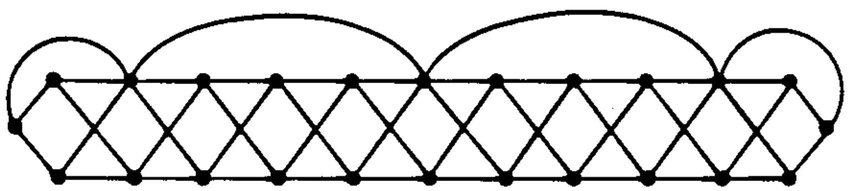

Figure 3

Let $\mathfrak{f}(D)$ denote the least number such that any bridgeless graph of diameter D admits an orientation of diameter $\leqslant \mathfrak{f}(D)$. Therefore we have

$$D^2/4+D \leqslant \mathfrak{f}(D) \leqslant 2D^2+2D.$$

Question 5 [21]: Tighten the bounds for $\mathfrak{f}(D)$.

In [22] Chvátal and Thomassen proved that the problem of deciding whether an undirected graph admits an orientation of diameter 2 is NP-complete. (The reader is referred to [32] for a discussion in NP-completeness.)

Recently, M. R. Garey, R. T. Tarjan and the author studied the problem of orienting all undirected edges in a mixed graph so as to minimize the diameter. It can be shown [21] that if a mixed graph of diameter D has any strongly connected orientation, then there is an orientation of diameter at most $8D^2+4D$. The proof gives a polynomial algorithm for constructing such an orientation. Suppose we define $\bar{\mathfrak{f}}(d)$ similarly for the case of mixed graphs. Then we have

$$D^2/4+D \leqslant \bar{\mathfrak{f}}(D) \leqslant 8D^2+4D$$

Question 6: Improve bounds for $\bar{\mathfrak{f}}(D)$.

5. Diameter bounds for altered graphs.

K. Vijayan and U. S. R. Murty [50] first investigated the following extremal problem which is motivated by constructing optimal networks with diameter and reliability considerations:

Determine the least number of edges for a graph on n vertices and diameter D having the property that, if any t edges (vertices) are deleted, the remaining graph has diameter no more than D'. Although this problem has received much attention in the past (see the surveys [3, 6, 12, 19]), it seems to be quite difficult in general, and relatively little is known beyond cases with small values of t and D (primarily $t = 1$ and $D \leqslant 5$). We will just mention the following intriguing problem:

Question 7 [12]: How many edges must a graph have so that after removing any t edges is still has diameter $\leqslant D$?

Motivated by the following examples B. Bollobás [12] conjectured that such graph G must have $(1+o(1))(n + \frac{n}{\lfloor 2D/3 \rfloor})$ edges for the case $t = 1$. In particular he verified this conjecture for the case that G has diameter $< \frac{2}{3}D$, before removing edges.

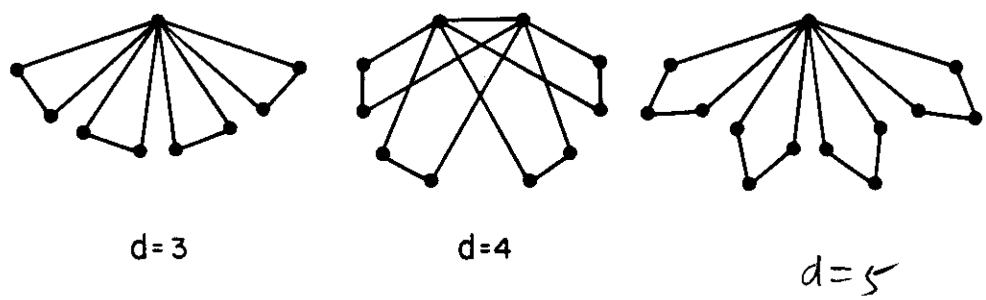

FIGURE 4

For the analogue of vertex deletion, B. Bollobás conjectured that any graph with n vertices must have at least $(1+o(1))(n + \frac{n}{\lfloor d/2 \rfloor})$ edges if it satisfies the property that after removing any edge the remaining graph has diameter $\leqslant D$ (see Fig. 5).

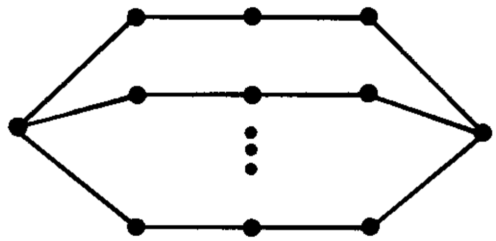

FIGURE 5

Again he verified [12] the conjecture for the case that the graph starts with diameter $< D/2$.

M. Garey and the author [20] recently studied the effect of edge or vertex deletion on the diameter bounds for general graphs. Suppose we delete an edge from a 2-connected graph, the new diameter can be twice as large as the old diameter, (deleting an edge from a cycle, for example). It is not difficult to prove that the maximum new diameter after deletion of an edge is $2D$ where D denotes the old diameter [20,43]. In general, it can be shown that when t edges are removed from a Graph G of diameter D, the resulting graph, if it is still connected, has maximum possible diameter approximately $(t+1)\cdot D$.

The corresponding vertex-deletion case is also studied in [20]. It is proved that if t vertices are deleted from a λ vertex-connected graph G with n vertices and diameter D, the resulting graph has diameter at most $(\frac{n-t-2}{\lambda-t}+1)D/2$ and this bound is best possible.

These problems are related to the following augmentation problems which is interesting in its own right. If t edges are added to a path P_n or a cycle C_n, how small can the diameter be? Let $P(n,t)$ denote this least possible diameter for the case of paths, and let $C(n,t)$ denote the corresponding value for the case of cycles, we have the following [20]:

$$\frac{n}{t+1}-1 \leqslant P(n,t) < \frac{n}{t+1}+3$$

and

$$\frac{n}{t+2} - 1 \leqslant C(n,t) < \frac{n}{t+2} + 3 \quad \text{if } t \text{ is even}$$

$$\frac{n}{t+1} - 1 \leqslant C(n,t) < \frac{n}{t+1} + 3 \quad \text{if } t \text{ is odd}$$

Problem 8: If t edges are removed from a strongly $(t+1)$-edge-connected directed graph, how large can the diameter of the resulting graph be?

Problem 9: Find a fast algorithm in determining the maximum diameter of a graph after any choice of t edges are removed.

We will also mention the following interesting problem which is relevant in the sense that many extremal graphs with diameter constraints contain relatively few small cycles [46].

Problem 10 [1,18]: For given integers r and s, suppose G is a directed graph with n vertices and outdegree at least r. Is it true that if $n \leqslant rs$ then G contains a (directed) cycle of length $\leqslant s$?

This problem is unanswered even for the case of $s = 3$!

6. The complexity of determining the diameter.

For a given graph G on n vertices a staighforward way to determine the diameter $D(G)$ is as follows:

(1) Find the breadth-first search tree [28, 46] for each vertex v of G. Thereby determine the maximum distance $d_v = \max_u d(u,v)$.

(2) Compare d_v and determine $D(G) = \max_v d_v$.

Since the time requirement [46] for finding a breadth-first search tree is $O(n+e)$, the preceding algorithm has complexity $O(n^2+ne)$ where $e = |E(G)|$. This algorithm has in fact calculating the distances among all pairs of vertices. The problem of finding all distances is a well-studied problem in graph algorithms (see [46,47] in a somewhat general setting.) Here we will mention the fact that

M. L. Fredman [30] has on $O(n^3(\log\log n/\log n)^{1/3})$ algorithm for finding distances of all pairs, which is faster than the straightforward algorithm for high density graphs.

The problem of finding the diameter of a graph is just to find the fartherest pair of vertices while pairs with short distances can be ignored. To take advantage of this, we can use the matrix multiplication algorithm to reduce the running time, which will be described in the following:

Let A denote the adjacency matrix of G, i.e., $A = (a_{ij})$ is a $n \times n$ and $a_{ij} = 1$ if and only if $[v_i, v_j]$ is an edge. (Note that G can be a mixed graph.) It is not difficult to see that in the k-th matrix product $\overline{A}^k = (A+I)^k$ of A the (i, j)-entry is nonzero if and only if there is a walk of length nor more than k from v_i to v_j. Therefore the diameter $D(G)$ is the last integer D with the property that $\overline{A}^D$ has all entries nonzero. The best known matrix multiplication algorithm due to D. Coppersmith and S. Winograd [23] requires running time $O(n^{2.496})$. The time required in computing $D(G)$ for G is then no more than $O(n^{2.496}\log D)$, since we can first find the least integer k with the property that A^{2^k} has only nonzero entries (by $n^{2.496}$ log D steps) and then use binary approximation to determine D (by another $n^{2.496}$ log D steps).

For the complexity lower bound, since every vertex and edge must be examined to determine the diameter, the obvious lower bound is $n+e$. There is of course the problem of further narrowing the gap between the upper and lower bounds on the complexity of determining $D(G)$.

Problem 11: Find a fast algorithm for determining the diameter of a graph.

Instead of finding the diameter of G, George and Lin [31] asked the question of finding a pair of pseudoperipheral vertices, i.e., a pair $\{x, y\}$ of vertices such that $d(x, y) \geqslant d(v, y)$ and $d(x, y) \geqslant d(x, v)$ for any vertex v in G.

A greedy algorithm for this problem can be described as follows:

Step (1) Start from any vertex v and find a farthest vertex u from v.

Step (2) If v is also furthest from u, $\{u, v\}$ is a solution. Stop. If there is a vertex w with $d(u, w) > d(u, v)$, go to (1) and replace v by w.

One interesting question is as follows:

Problem 12: Is it true that this greedy algorithm must stop before $c\sqrt{n}$ iterations of Step (1)?

There exists a graph together with a starting point such that the greedy algorithm takes $c\sqrt{n}$ iterations (see [42]). We remark that J. K. Pachl has another algorithm for solving this problem with worse case time $\sqrt{ne}$. (Note that each iteration in the greedy algorithm takes e steps). However, the complexity of the greedy algorithm still remains open.

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