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Monotone Subsequences in (0, 1)-Matrices

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Abstract. A (0, 1)-matrix contains an $S_0(k)$ if it has 0-cells $(i,j_1), (i+1,j_2), \ldots, (i+k-1,j_k)$ for some i and $j_1 < \ldots < j_k$, and it contains an $S_1(k)$ if it has 1-cells $(i_1,j), (i_2,j+1), \ldots, (i_k,j+k-1)$ for some j and $i_1 < \ldots < i_k$. We prove that if M is an $m \times n$ rectangular (0, 1)-matrix with $1 \le m \le n$ whose largest k for an $S_0(k)$ is $k_0 \le m$, then M must have an $S_1(k)$ with $k \ge \lfloor m/(k_0+1) \rfloor$. Similarly, if M is an $m \times m$ lower-triangular matrix whose largest k for an $S_0(k)$ (in the cells on or below the main diagonal) is $k_0 \le m$, then M has an $S_1(k)$ with $k \ge \lfloor m/(k_0+1) \rfloor$. Moreover, these results are best-possible.

1. Introduction

A basic result [2, 3, 4] in the theory of order within irregular patterns says that every linear arrangement of the first N positive integers has either an increasing or a decreasing subsequence of at least $n+1$ terms if, and only if, $N \ge n^2 + 1$. For example ($n = 3$), 7 8 9 4 5 6 1 2 3 has no 4-term monotone subsequence, but every list of $\{1, 2, \ldots, 10\}$ has such a subsequence.

The aim of this paper is to prove related but independent results for regular patterns of 0's or of 1's in rectangular (0, 1)-matrices and in triangular (0, 1)-matrices. We shall consider so-called monotone subsequences of 0's and of 1's in matrices that place no restrictions on the relative proportions of 0's and 1's apart from those dictated by the size of the matrix.

A *length-k monotone subsequence of* 0's in a (0, 1)-matrix, denoted by $S_0(k)$, is a sequence of k 0's in cells $(i,j_1), (i+1,j_2), \ldots, (i+k-1,j_k)$ for some i and $j_1 < j_2 < \cdots < j_k$. A *length-k monotone subsequence of* 1's in a (0, 1)-matrix, denoted by $S_1(k)$, is a subsequence of k 1's in cells $(i_1,j), (i_2,j+1), \ldots, (i_k,j+k-1)$ for some j and $i_1 < i_2 < \ldots < i_k$. These are illustrated in Fig. 1. When the matrix is lower-triangular, it is to be understood that all cells used for an $S_i(k)$ lie within the matrix, i.e., on or below the main diagonal.

Our two main results are similar to the basic result for linear arrangements of $\{1, \ldots, N\}$ in that they force the existence of suitably large monotone subsequences in (0, 1)-matrices. For any such matrix let

$$k_0 = \max k \text{ for which there is an } S_0(k)$$

$$k_1 = \max k \text{ for which there is an } S_1(k).$$

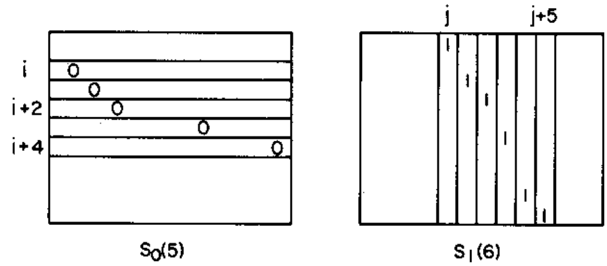

Fig. 1. Monotone subsequences

Clearly, neither k_0 nor k_1 can exceed the smaller dimension of the matrix.

Theorem 1. *For every $m \times n$ rectangular $(0, 1)$-matrix with $1 \le m \le n$, and for all $0 \le k_0 \le m$,*

$$k_1 \ge \left\lfloor \frac{m}{k_0 + 1} \right\rfloor,$$

with equality holding for some matrix for each (k_0, m, n).

Theorem 2. *For every $m \times m$ lower-triangular $(0, 1)$-matrix, and for all $0 \le k_0 \le m$,*

$$k_1 \ge \left\lfloor \frac{m}{k_0 + 1} \right\rfloor,$$

with equality holding for some matrix for each (k_0, m).

These theorems generalize the following corollary from [1].

Corollary 1. *For every $m \ge 1$ and every $m^2 \times m^2$ lower-triangular $(0, 1)$-matrix, $\max\{k_0, k_1\} \ge m$. Moreover, for every $m \ge 2$, there is an $(m^2 - 1) \times (m^2 - 1)$ rectangular $(0, 1)$-matrix with $\max\{k_0, k_1\} < m$.*

This implies that, when $n = m$ in Theorem 1, its conclusions hold even when we ignore all cells above the main diagonal. This will be reflected in our proof of Theorem 2.

The equality assertions at the ends of the theorems follow from very simple patterns of 0's and 1's. Given $1 \le m \le n$ and $0 \le p \le m$, let M be the $m \times n$ rectangular matrix with all 1's in row $t(p + 1)$ for $t = 1, \ldots, \lfloor m/(p + 1) \rfloor$, and 0's everywhere else. Because an $S_0(k)$ cannot jump over a row of 1's, $k_0 = p$. And, because there are $\lfloor m/(p + 1) \rfloor$ rows of 1's and $m \le n$, $k_1 = \lfloor m/(k_0 + 1) \rfloor$. The same construction applies to Theorem 2.

The proofs of the main inequalities are considerably more delicate, especially for Theorem 2. Although the inequality of Theorem 2 clearly implies that for Theorem 1 (consider any $m \times m$ lower-triangular submatrix of the matrix in Theorem 1), we shall prove both since the proof for Theorem 2 makes refinements on the simpler and more easily visualized proof of Theorem 1.

These proofs generalize the proof [1] of Corollary 1 in fairly straightforward ways. Because we believe that monotone subsequences, which were a secondary theme in [1], are interesting in their own right, it is hoped that their primacy in the present work will encourage further consideration of regularities in binary matrices.

2. Proof of Theorem 1

Given $1 \le m \le n$, let M denote any $m \times m$ submatrix of a rectangular $m \times n$ (0, 1)-matrix. The columns of M are labeled 1 to m.

Let $M(i,j)$ denote entry (i,j) in M. A *horizontal path* in M is a left-to-right sequence

$$M(i_1, 1), M(i_2, 2), \ldots, M(i_m, m)$$

with

$$i_j \le i_{j+1} \le i_j + 1 \quad \text{for } j = 1, \ldots, m-1.$$

Such a path uses $l = i_m - i_1 + 1$ contiguous rows; two such paths are *row-disjoint* if they have no row in common.

We say that a horizontal path is a *leading monotone sequence* of 0's of length l, abbreviated $LS_0(l)$, if $l = i_m - i_1 + 1$ and, for all $j > 1$,

$$i_j = i_1 \Rightarrow M(i_j, j) = 1,$$
$$i_j > i_1 \Rightarrow \{M(i_j, j) = 0 \Leftrightarrow i_j > i_{j-1}\}.$$

Each $LS_0(l)$ contains an $S_0(l)$ if $M(i_1, 1) = 0$, and an $S_0(l-1)$ if $M(i_1, 1) = 1$. A row of 1's is an $LS_0(1)$ and contains an $S_0(0)$.

Suppose M contains an $S_0(k)$ but no $S_0(k+1)$, so $k_0 = k$. Let $y = \lfloor m/(k+1) \rfloor$. We show that M has an $S_1(y)$, which completes the proof of Theorem 1.

With $k_0 = k$, construct a family $\mathscr{L}$ of y row-disjoint LS_0's in M, beginning in the upper left corner. Since each $LS_0(l)$ in $\mathscr{L}$ has $l \le k + 1$, the construction requires at most $y(k + 1)$ rows, so such a construction exists.

For each t with $0 \le t \le m - y$, let U_t be the y-term sequence

$$M(x(1, t), t + 1), M(x(2, t), t + 2), \ldots, M(x(y, t), t + y),$$

where $x(i, t)$ denotes the row corresponding to column $t + i$ in the ith LS_0 (top-to-bottom) in $\mathscr{L}$. If U_t consists entirely of 1's, then it is an $S_1(y)$ and we are done. Indeed, this must be true for some U_t, else there would be at least $(m - y + 1)$ 0's in the U_t's but no more than yk 0's in all of $\mathscr{L}$, hence $m - y + 1 \le yk$, or $m + 1 \le y(k + 1)$, which contradicts $y = \lfloor m/(k + 1) \rfloor$. □

3. Proof of Theorem 2

To prove the inequality in Theorem 2, assume that $m \ge 1$ with no loss in generality, and let M denote a rectangular $m \times m$ (0, 1)-matrix with 1 in every cell above the main diagonal. Given $k_0 = k$, let $y = \lfloor m/(k + 1) \rfloor$. We wish to show that there is an $S_1(y)$ composed solely of entries on or below the main diagonal.

Several definitions in addition to those in the preceding section are needed. Let

$\mathscr{L}_0$ be the family of all LS_0's in M. Define $\mathbf{s} \in \mathscr{L}_0$ as *canonical* if it has the greatest final row of all leading monotone sequences in $\mathscr{L}_0$ which begin in the same row as $\mathbf{s}$. Let $\mathscr{C}\mathscr{L}_0$ be the family of canonical sequences in $\mathscr{L}_0$. It is easily seen that different members of $\mathscr{C}\mathscr{L}_0$ nowhere coincide or cross.

Let $\mathbf{s} = [M(i_1, 1), \ldots, M(i_m, m)]$ and $\mathbf{s}^* = [M(i_1^*, 1), \ldots, M(i_m^*, m)]$ be adjacent members of $\mathscr{C}\mathscr{L}_0$ with $\mathbf{s}^*$ above $\mathbf{s}$, so $i_j^* < i_j$ for all j. We define the *distance* between the two as

$$d(\mathbf{s}, \mathbf{s}^*) = i_m - i_m^* - 1.$$

A 0 in $\mathbf{s} \in \mathscr{C}\mathscr{L}_0$ is *soft* if $\mathbf{s}$ is not the top-most member of $\mathscr{C}\mathscr{L}_0$ and the column cells strictly between this 0 and the next $\mathbf{s}^* \in \mathscr{C}\mathscr{L}_0$ above $\mathbf{s}$ contain at least one 1; otherwise, a 0 in $\mathbf{s}$ is *hard*: see Fig. 2.

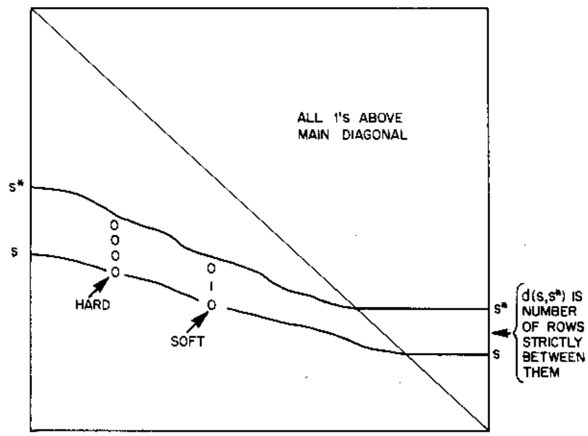

Fig. 2. Adjacent canonical sequences

The following lemma, which is proved at the end of the section, is needed for our proof that M has an $S_1(y)$ on or below the main diagonal.

Lemma 1. *If* $\mathbf{s}$ *and* $\mathbf{s}^*$ *are adjacent in* $\mathscr{C}\mathscr{L}_0$ *with* $\mathbf{s}^*$ *above* $\mathbf{s}$, *then* $\mathbf{s}$ *has at least* $d(\mathbf{s}, \mathbf{s}^*)$ *soft* 0's.

Given M as before, with $k_0 = k$ and 1's above the main diagonal, let $\mathbf{s}_1, \ldots, \mathbf{s}_y$ be the bottom-most members of $\mathscr{C}\mathscr{L}_0$ in sequence from $\mathbf{s}_1$ on up. Let

$$w_i = \text{number of 0's in } \mathbf{s}_i \qquad i = 1, \ldots, y$$

$$d_i = d(\mathbf{s}_i, \mathbf{s}_{i+1}) \qquad i = 1, \ldots, y-1.$$

By hypothesis, $w_i \leq k$ for all i. Moreover, since it is easily checked that $\mathbf{s}_1$ begins (from the right) at cell (m, m), there are at least y members of $\mathscr{C}\mathscr{L}_0$: $y(k+1) \leq m$.

The lowest row for $\mathbf{s}_y$ is $m + 1 - (y + \sum_1^{y-1} d_i) = r_y$, and (r_y, r_y) is the first cell (from the right) for $\mathbf{s}_y$ that is on (or below) the main diagonal. The number of columns from (r_y, r_y) to the left boundary of M, including column r_y, is simply r_y.

For each $0 \le t < r_y$, let V_t be the y-term sequence

$$M(x(1,t), t+1), M(x(2,t), t+2), \ldots, M(x(y,t), t+y),$$

where $x(i,t)$ is the row corresponding to column $t+i$ in $\mathbf{s}_{y+1-i}$. The r_y V_t are totally disjoint and lie on or below the diagonal: see Fig. 3. If any V_t consists entirely of 1's and soft 0's, then a modified construction from the bottom up for this V_t shows that it yields an $S_1(y)$ on or below the main diagonal. In particular, whenever a soft 0 is encountered, it is replaced by the 1 next above it in its column.

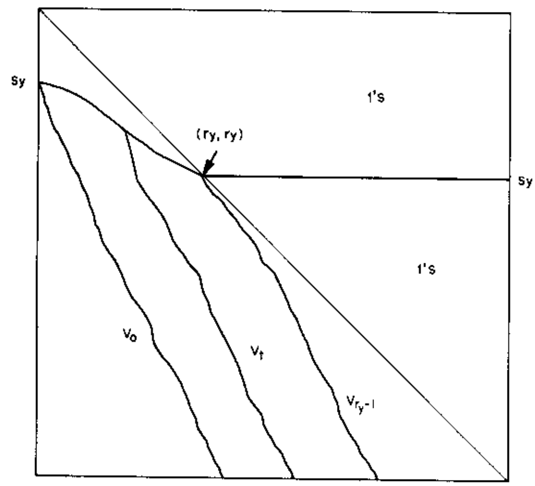

Fig. 3.

It follows that the desired result holds unless every one of the r_y V_t has at least one hard 0. In view of Lemma 1, the number of hard 0's from $\mathbf{s}_1$ through $\mathbf{s}_y$ that can be in the V_t does not exceed

$$\sum_{i=1}^{y-1} (w_i - d_i) + w_y,$$

which is bounded above by $y(k+1) - (m+1) + r_y$ since $w_i \le k$ for all i and $-\sum d_i = r_y + y - (m+1)$. If every V_t had a hard 0, then $r_y \le y(k+1) - (m+1) + r_y$, or $(m+1) \le y(k+1)$, and this contradicts the definition of y. □

Proof of Lemma 1. Suppose $\mathbf{s}^*$ is above and adjacent to $\mathbf{s}$ in $\mathscr{CL}_0$. Let $\mathbf{z}$ be the LS_0 constructed right-to-left, beginning in the row immediately below the bottom row of $\mathbf{s}^*$ (whenever a 0 is encountered, jump northwest into the next row). Since $\mathbf{s}^*$ is canonical, $\mathbf{z}$ does not meet $\mathbf{s}^*$. Since for any LS_0 there is a unique canonical LS_0 which shares its left-most cell, $\mathbf{z}$ must meet $\mathbf{s}$, for $\mathbf{s}$ and $\mathbf{s}^*$ are adjacent. Thus, going right-to-left, there are exactly $d(\mathbf{s}, \mathbf{s}^*)$ columns at which $\mathbf{s}$ jumps northwest but $\mathbf{z}$ stays horizontal. In each such column, $\mathbf{s}$ has a 0 and $\mathbf{z}$ has a 1, so the 0 for $\mathbf{s}$ is soft. This means that $\mathbf{s}$ has at least $d(\mathbf{s}, \mathbf{s}^*)$ soft 0's. □

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