The Forwarding Index of Communication Networks

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***Abstract*—A network is defined as an undirected graph and a *routing* which consists of a collection of simple paths connecting every pair of vertices in the graph. The forwarding index of a network is the maximum number of paths passing through any vertex in the graph. Thus it corresponds to the maximum amount of forwarding done by any node in a communication network with a fixed routing. For a given number of vertices, each having a given degree constraint, we consider the problem of finding networks that minimize the forwarding index. Forwarding indexes are calculated for cube networks and generalized de Bruijn networks. General bounds are derived which show that de Bruijn networks are asymptotically optimal. Finally, efficient techniques for building large networks with small forwarding indexes out of given component networks are presented and analyzed.**

I. Introduction

We CONSIDER a collection of n nodes that are to be interconnected for purposes of communicating data, messages, etc. Generally, the nodes are to be interpreted as computer/communication devices. In practice, the networks to be constructed may range from arrays of microcomputers to systems of large geographically remote communication centers.

The network connecting the n nodes is designed by specifying first the bidirectional communication lines or channels, i.e., those pairs of nodes having direct communication. Interconnection is limited by a port constraint $d \geq 2$ common to each node; i.e., at most d communication lines can be attached to any node. Since it follows in general that not every pair of nodes will have direct communication, the network design must also specify a set of $n(n-1)$ paths called a *routing*, indicating for each i and $j \neq i$ the path or fixed sequence of lines which carries the data transmitted from node i to node j. Implicit here is that in addition to being data sources and sinks, the nodes can serve a forwarding function for the data being communicated between other nodes. Note that, generally, the path from node i to node j need not be the reverse of the path from node j to node i.

In what follows the objective function for network design is the network *forwarding index* ξ, defined as the maximum number of paths passing *through* any node, i.e., ξ is the maximum forwarding being done in the network. With n and d given we consider the specific problem of finding networks that minimize the forwarding index; we call this the *forwarding index problem*.

Fig. 1 shows an example for $n = 6$ and $d = 3$. According to the routing indicated, nodes 1 and 4 forward the traffic on one path each, nodes 5 and 6 forward the traffic on two paths each, and nodes 2 and 3 forward traffic on a total of four paths each. Thus $\xi = 4$ for this network.

Concrete applications of the forwarding index problem can be found in problems of maximizing network capacity. (Indeed, such applications provided the original motivation for the research reported here.) For example, assume symmetric transmission requirements in the sense that the transmission rate, say λ, is the same from each node to every other node. The total rate at which data originates and terminates at each node is, therefore, $2(n-1)\lambda$, and the total transmission rate among the nodes is $n(n-1)\lambda$.

The amount of forwarding at a node is assumed to be limited by a capacity c common to all nodes. Specifically, the local transmission rate at a node $2(n-1)\lambda$, plus the rate at which it forwards data for other nodes cannot exceed c. In Fig. 1, for example, since nodes 2 and 3 forward the most traffic and since the traffic at these nodes is $2(n-1)\lambda + 4\lambda = 14\lambda$, we must have $c \geq 14\lambda$.

The constraint on node capacity requires that $(2n-1)\lambda + \xi\lambda \leq c$. The local traffic originating or terminating at each node must, therefore, satisfy

$$2(n-1)\lambda \leq \frac{2(n-1)c}{\xi + 2(n-1)} \tag{1.1}$$

thus defining an *effective* node capacity $c/(1 + \xi/2(n-1))$. The corresponding bound on the total data transmission rate defines the network capacity

$$n(n-1)\lambda \leq \frac{nc/2}{1 + \xi/2(n-1)}. \tag{1.2}$$

In Fig. 1 the effective node capacity is $5c/7$ and the network capacity is $15c/7$. From (1.1) and (1.2) the problem of maximizing capacity for given n and d clearly reduces to the forwarding index problem.

In the next section we formalize the problem using standard concepts from graph theory. In Section III we introduce basic network designs and determine the

Manuscript received May 1984; revised June 19, 1986.

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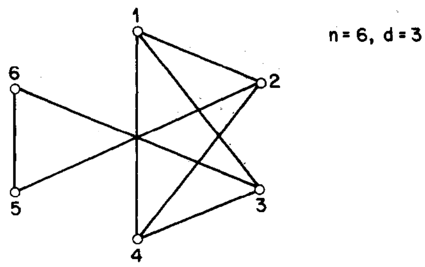
IEEE Log Number 8611659.

0018-9448/87/0300-0224$01.00



ROUTING:

1. BOTH PATHS BETWEEN

1	AND 5	PASS THROUGH	2
4	5		2
1	6		3
4	6		3
2	6		5
3	5		6

2. PATH FROM

2	TO 3	PASSES THROUGH	4
3	TO 2		1

3. ALL OTHER COMMUNICATIONS ARE DIRECT.

Fig. 1. Example network.

asymptotic behavior of their forwarding indexes. The asymptotic bounds on forwarding indexes are derived in Section IV. In Section V we illustrate methods for constructing larger networks out of smaller ones so that the growth of the forwarding index is kept small. We conclude with remarks on open problems and possible extensions in Section VI.

II. Mathematical Preliminaries

The reader is assumed to be familiar with the basic terminology of graphs. Any standard text on graph theory will supply definitions for the terms we leave undefined. We consider only undirected graphs G. $V(G)$ denotes the vertex set of G, and if $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent vertices in G, we write $\boldsymbol{u} \sim \boldsymbol{v}$. This symbol is also used to denote the edge between $\boldsymbol{u}$ and $\boldsymbol{v}$. The length of a shortest path between vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ in G defines the *distance* $\delta(\boldsymbol{u}, \boldsymbol{v})$ between $\boldsymbol{u}$ and $\boldsymbol{v}$. The maximum distance over all pairs of vertices in G is known as the *diameter* of G.

The degree of a vertex is the number of edges incident to it, and $\deg G$ is the maximum degree of the vertices in G. The correspondence between the terms in this and the previous section is the obvious one: instead of speaking of nodes, communication lines, and port constraints, we speak of vertices, edges, and degree constraints.

Consistent with the previous section we define a *routing* P for a given graph G as a set of $n(n-1)$ simple paths $p(\boldsymbol{u}, \boldsymbol{v})$ $(\boldsymbol{u} \neq \boldsymbol{v})$, specifying for each (ordered) pair of vertices $\boldsymbol{u}, \boldsymbol{v}$ in G a sequence of edges that begins with an edge incident to $\boldsymbol{u}$ and ends with an edge incident to $\boldsymbol{v}$. A *network* is defined by a pair (G, P) where P is a routing for G. We let $\xi_v(G, P) \geq 0$ denote the number of paths in P that pass through vertex $\boldsymbol{v}$ in G, i.e., the number of paths $\boldsymbol{u}_1 \sim \boldsymbol{u}_2, \boldsymbol{u}_2 \sim \boldsymbol{u}_3, \cdots, \boldsymbol{u}_{r-1} \sim \boldsymbol{u}_r$ in P such that $\boldsymbol{v} = \boldsymbol{u}_i$ for some $1 < i < r$. Thus the forwarding index introduced in the previous section is defined formally as

$$\xi(G, P) = \max_{v \in V(G)} \xi_v(G, P).$$

In this notation the dependence on G and P may be suppressed when the network is clear by context.

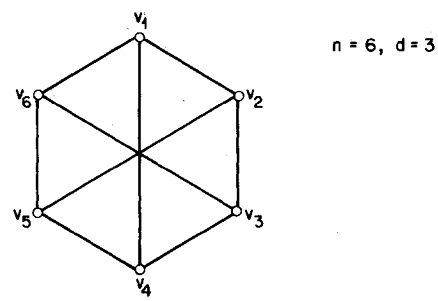
Our analysis also extends to the function

$$\xi(G) = \min_P \xi(G, P)$$

defining the minimum achievable forwarding index for networks restricted to G. In addition, for a given $d \geq 2$ we define

$$\xi_{d,n} = \min_{\substack{|V(G)|=n \\ \deg G \leq d}} \xi(G)$$

which in turn defines the forwarding index problem: for given n and d find a network (G, P) such that $n = |V(G)|$, $\deg G \leq d$, and $\xi(G, P) = \xi_{d,n}$. To illustrate these definitions, it is not difficult to verify that $\xi(G) = 4$ for the graph in Fig. 1. On the other hand, a solution to the forwarding index problem for $d = 3$ and $n = 6$ is given by the graph in Fig. 2 where $\xi(G) = \xi_{3,6} = 2$ is easily proved.



ROUTING:

PATH FROM v_i TO v_j IS THE REVERSE OF THE PATH FROM v_j TO v_i.

1. PATH BETWEEN

v_1	AND v_3	PASS THROUGH	v_2
v_1	v_5		v_6
v_2	v_4		v_3
v_2	v_6		v_1
v_3	v_5		v_4
v_4	v_6		v_5

ALL OTHER PATHS SINGLE EDGES.

Fig. 2. Graph for which $\xi(G) = \xi_{3,6}$.

Except for certain examples in the next section our analysis concentrates on asymptotics. As shown in Section IV, $\xi_{d,n} = \Omega(n \log n)$. It follows from the bounds in (1.1) and (1.2) for our network capacity example that if $\xi_{d,n} = \Theta(f(n))$, then the effective node capacity is $\Theta(n/f(n))$ and the network capacity is $\Theta(n^2/f(n))$. In a similar way asymptotic results on capacity can be obtained quite simply from our other forwarding index results.

Diameter problems have also been important in the design of communication networks. An up-to-date treatment can be found in [3]; for earlier surveys see [1] and [2]. A problem of particular interest to us is this. Given n and d, find a graph on n vertices with degree constraint d

whose diameter is minimum over all such graphs. We show in Section III-C that appropriate routings for certain small-diameter graphs lead to relatively small forwarding indexes as well. Before getting into the forwarding indexes of specific networks we prove the following simple bounds on $\xi(G)$.

Theorem 1: Let δ denote the distance function for a given graph G on n vertices. Then

$$\frac{1}{n}\sum_{u}\sum_{v\neq u}(\delta(u,v)-1)\leq \xi(G)\leq (n-1)(n-2).$$

Moreover, graphs exist for which these bounds are achieved.

Proof: The maximum number of paths passing through a vertex v is at most the total number of paths minus the number having v as a terminal vertex. Thus

$$\xi(G)\leq n(n-1)-2(n-1)=(n-1)(n-2).$$

A star graph on n vertices with edge-set $\{(v_i,v_n), 1\leq i\leq n-1\}$ achieves the bound since all but $2(n-1)$ paths must pass through the central vertex v_n. Indeed, it is easy to show that stars are the *only* graphs achieving the upper bound.

For the lower bound observe first that $\delta(u,v)-1$ is the minimum number of vertices through which $p(u,v)$ can possibly pass. Thus summing over all $v\neq u$ and then over u gives us a lower bound on the total number of instances in which paths pass through vertices. Finally, the lower bound on $\xi(G)$ follows from the observation that the maximum number of paths passing through vertices in G cannot be less than the average number. The graph of Fig. 2 achieves the lower bound. Further examples appear in the next section.

III. Forwarding Indexes of Some Basic Networks

Clearly, we need consider only those graphs to which we cannot add edges without violating the degree constraint; the forwarding index of the network corresponding to any other graph cannot be increased by the addition of one or more edges. Since the degree constraint is identical for each vertex, it is also natural to consider *symmetric* graphs, i.e., graphs in which each vertex has the same view of the remainder of the graph. Formally, for each vertex pair u, v in a symmetric graph an automorphism mapping vertex u into vertex v exists.

For $d\geq n-1$ we can fully connect a network, i.e., G is a complete graph. In this case P can be composed only of single-edge paths so that the minimum $\xi=0$ is achieved. Thus we now turn to graphs for which $n\geq 4$ and $d<n-1$, and hence forwarding is required.

A. Rings

For $d=2$ the only connected network fully utilizing the degree constraint is easily seen to be a ring; each vertex is adjacent to exactly two others. Because of the simplicity of rings, the forwarding index problem can be solved completely for $d=2$.

Theorem 2: For all $n\geq 2$

$$\xi_{2,n}=\left(\left\lceil\frac{n}{2}\right\rceil-1\right)\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right).$$

Proof: It is easy to see that a ring is symmetric. For any symmetric graph the sum $\sum_{v\neq u}\delta(u,v)$ is the same for each node u. Thus when G is symmetric, the lower bound in Theorem 1 simplifies to

$$\xi(G)\geq \sum_{v\neq u}[\delta(u,v)-1]=\sum_{v}\delta(u,v)-(n-1). \tag{3.1}$$

Now in a ring we note that two vertices exist at distance i from any vertex v for each $i=1,2,\cdots,\lfloor (n-1)/2\rfloor$, plus one vertex at distance $n/2$ if n is even. Thus from (3.1) we have

$$\xi(G)\geq\sum_{i=1}^{(n-1)/2}2(i-1)=\frac{(n-1)(n-3)}{4},\qquad n \text{ odd}$$

and

$$\xi(G)\geq\sum_{i=1}^{(n/2)-1}2(i-1)+\left(\frac{n}{2}-1\right)=\frac{(n-2)^2}{4},\quad n\text{ even}.$$

Combining these inequalities yields

$$\xi(G)\geq\left(\left\lceil\frac{n}{2}\right\rceil-1\right)\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right). \tag{3.2}$$

To show that this lower bound is achievable, we let P be a set of shortest paths. If n is odd, this set is clearly unique. Moreover, it is easily seen to be symmetric in the sense that it imposes the same forwarding index on each vertex. In this case equality in (3.1) and (3.2) follows directly from the fact that the average forwarding index is equal to the maximum.

For n even a set of shortest paths is not unique; two choices exist for the path from u to v when $\delta(u,v)=n/2$. In these cases we take one choice for $p(u,v)$ and the reverse of the other choice for $p(v,u)$ in the definition of P. It is readily verified that P then imposes the same forwarding index on each vertex. Equality in (3.2) follows as before. It remains only to observe that rings are the only graphs that need to be considered for $d=2$, i.e., $\xi_{2,n}=\xi(G)$ where G is an n-vertex ring.

B. Cubes

Rings can be generalized to higher dimensions in a natural way. Consider a graph with an even degree constraint d and $k^{d/2}$ vertices for some $k\geq 2$. Let the vertices be represented by vectors of $d/2$ elements $(a_1,a_2,\cdots,a_{d/2})$, $0\leq a_i\leq k-1$, $1\leq i\leq d/2$. The edges defining the *k-cube* $K_{d,n}$ are given by

$$(a_1,\cdots,a_i,\cdots,a_{d/2}) \sim (a_1,\cdots,(a_i+1)\bmod k,\cdots,a_{d/2}),\qquad 1\leq i\leq d/2.$$

Fig. 3 illustrates $K_{4,9}$.

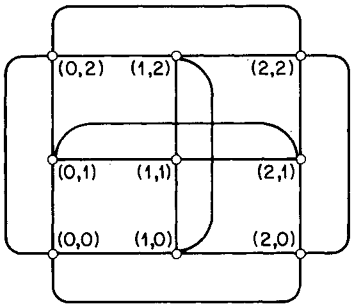


Fig. 3. Graph $K_{4,9}$ with $k = 3$.

Note that the vertices differing only in the ith coordinate comprise a ring of k vertices. We have the following result for the forwarding index.

Theorem 3: With d even and n restricted to the set $\{k^{d/2}\}$ for integers $k > 1$ we have

$$\xi(K_{d,n}) = \frac{d}{2}k^{d/2-1}\xi_{2,k+2} - (k^{d/2} - 1)$$

and hence

$$\xi(K_{d,n}) = \left(\frac{d}{8} + o(1)\right)n^{1+2/d}$$

where $o(1)$ refers to the limit $n \to \infty$ with d fixed.

Proof: It is easy to verify that a k-cube is symmetric, and therefore (3.1) holds. To compute this lower bound, we let $\delta_i(\boldsymbol{u}, \boldsymbol{v})$ be the distance between u_i and v_i in a k-element ring and write

$$\sum_{\boldsymbol{v}}\delta(\boldsymbol{u}, \boldsymbol{v}) = \sum_{\boldsymbol{v}}\sum_{i=1}^{d/2}\delta_i(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{d/2}\sum_{\boldsymbol{v}}\delta_i(\boldsymbol{u}, \boldsymbol{v}).$$

Now of the $k^{d/2}$ vertices of the second sum, $2/k$ of them are at distance j in the ith coordinate ring for each $j = 1,\cdots,\lfloor(k-1)/2\rfloor$, $1/k$ of them are at distance 0, and if k is even, $1/k$ of them are at distance $k/2$. Thus if k is odd,

$$\sum_{\boldsymbol{v}}\delta_i(\boldsymbol{u}, \boldsymbol{v}) = \frac{2}{k}k^{d/2}\sum_{j=1}^{(k-1)/2} j = k^{d/2-1}\frac{(k-1)(k+1)}{4},$$

and if k is even,

$$\sum_{\boldsymbol{v}}\delta_i(\boldsymbol{u}, \boldsymbol{v}) = \frac{2}{k}k^{d/2}\sum_{j=1}^{(k/2)-1} j + \frac{1}{k}k^{d/2}\frac{k}{2} = k^{d/2-1}\frac{k^2}{4}.$$

Thus from Theorem 2 we introduce $\xi_{2,k+2}$ and obtain

$$\sum_{\boldsymbol{v}}\delta_i(\boldsymbol{u}, \boldsymbol{v}) = k^{d/2-1}\xi_{2,k+2},$$

whereupon summing on i gives

$$\sum_{\boldsymbol{v}}\delta(\boldsymbol{u}, \boldsymbol{v}) = \frac{d}{2}k^{d/2-1}\xi_{2,k+2}.$$

Substituting into (3.1) we get

$$\xi(K_{d,n}) \geq \frac{d}{2}k^{d/2-1}\xi_{2,k+2} - (k^{d/2} - 1).$$

To verify that this bound is achieved, we note first that a shortest path between two vertices in a k-cube can be composed of the edges in shortest paths along coordinate rings. Thus we let the routing P for $K_{d,n}$ be a collection of shortest paths whose edges are ordered as follows. The path $p(\boldsymbol{u}, \boldsymbol{v})$ consists first of the edges, if any, in a shortest path from $\boldsymbol{u}$ to $(v_1, u_2, \cdots, u_{d/2})$ in the first coordinate ring, then the edges in a shortest path from $(v_1, u_2, \cdots, u_{d/2})$ to $(v_1, v_2, u_3, \cdots, u_{d/2})$ in the second coordinate ring, and so on with the final edges being those in a shortest path from $(v_1, \cdots, v_{d/2-1}, u_{d/2})$ to $\boldsymbol{v}$. With the coordinate-ring paths defined as in Theorem 2 the reader will find little difficulty in verifying that P creates the same forwarding index for each vertex. The lower bound is therefore achieved, as in Theorem 2.

Note that Theorem 3 with $d = 2$ is consistent with Theorem 2. Although simple rings for $d = 2$ are within a factor of four of being worst possible by Theorem 1, the forwarding index of k-cubes decreases substantially as d increases. In the next subsection we exhibit networks whose forwarding indexes grow even more slowly with n.

C. Generalized de Bruijn Graphs

To improve upon k-cubes, it is reasonable to try graphs with an average distance between vertices which is smaller than the $\Theta(dn^{2/d})$ value for k-cubes. To this end we consider de Bruijn graphs [4] which were analyzed originally in connection with the problem of finding minimum diameter graphs.

The de Bruijn graphs $B_{d,n}$ are introduced by considering first the simplest case $d = 3$ which differs somewhat from the general case $d \geq 4$. We assume that $n = 2^k$ for some $k > 1$, and represent the vertices by the k-tuples of integers

$$V(B_{3,n}) = \{(a_1, \cdots, a_k) : a_i = 1 \text{ or } 2, 1 \leq i \leq k = \log_2 n\}.$$

An edge exists between two vertices if the first $k - 1$ elements are equal or if the elements of one represent a cyclic shift of one position of the elements of the other, i.e., the vertex $(a_1, \cdots, a_k)$ has the three edges

$$\begin{aligned} &(a_1, \cdots, a_k) \sim (a_1, \cdots, a_{k-1}, a_k'), \qquad a_k' \neq a_k \\ &(a_1, \cdots, a_k) \sim (a_2, \cdots, a_k, a_1) \\ &(a_1, \cdots, a_k) \sim (a_k, a_1, \cdots, a_{k-1}). \end{aligned} \tag{3.3}$$

The graph $B_{3,8}$ is shown in Fig. 4.

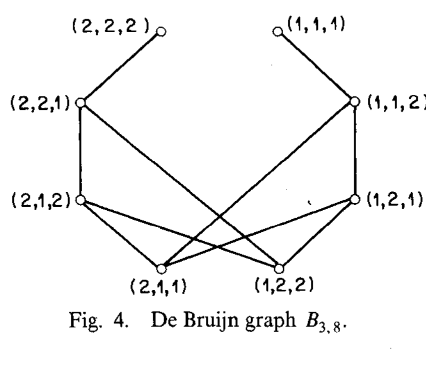


Fig. 4. De Bruijn graph $B_{3,8}$.

Although obvious symmetries exist in such graphs, they are not themselves symmetric. It is sufficient to observe that the vertices $(1,1,\cdots,1)$ and $(2,2,\cdots,2)$ have but one edge. For convenience the first of the edges in (3.3) is called a *change* edge, while the remaining two are called *shift* edges.

Theorem 4: For n restricted to the set $\{2^k: k>1\}$ we have

$$\xi(B_{3,n}) \le 2n\log_2(n/2).$$

Remark: We show in the next section that this upper bound is asymptotically at most a factor of two larger than $\xi(B_{3,n})$.

Proof: For the routing P we define the path from $(a_1,\cdots,a_k)$ to $(b_1,\cdots,b_k)$ as the sequence of at most $2k$ vertices obtained by successively changing the rightmost element, if necessary, and then shifting left one position, i.e.,

$$\begin{aligned}(a_1,\cdots,a_k) &\sim (a_1,\cdots,a_{k-1},b_1)\\ &\sim (a_2,\cdots,a_{k-1},b_1,a_1)\\ &\sim (a_2,\cdots,a_{k-1},b_1,b_2)\\ &\sim (a_3,\cdots,a_{k-1},b_1,b_2,a_2)\\ &\sim \cdots \sim (b_1,\cdots,b_k).\end{aligned}$$

The diameter of $B_{3,n}$ is readily seen to be $2k = 2\log_2 n$. Note that it is possible for the path to reach $(b_1,\cdots,b_k)$ with fewer than k shifts, e.g., if for some $1\le j\le k-1$,

$$(b_1,\cdots,b_k) = (a_j,\cdots,a_{k-1},b_1,\cdots,b_j).$$

Although in practice such a path should stop at this point, we assume that the path continues so that all paths have exactly k shift edges. This clearly gives an upper bound on the forwarding index that would be achieved if a "more careful" routing were considered. On the other hand, the multiplicative constant in our bound would not be affected by these latter routings.

To determine the forwarding index, consider an arbitrary vertex $\boldsymbol{v}$ and suppose the path from $\boldsymbol{u}=(u_1,\cdots,u_k)$ to $\boldsymbol{w}=(w_1,\cdots,w_k)$ passes through $\boldsymbol{v}$. Then by definition of P an i must exist, $1\le i\le k-1$, such that $\boldsymbol{v}=(u_i,\cdots,u_{k-1},w_1,\cdots,w_i)$ or $\boldsymbol{v}=(u_{i+1},\cdots,u_{k-1},w_1,\cdots,w_i,u_i)$. In both cases there are i elements of $\boldsymbol{u}$ and $k-i$ elements of $\boldsymbol{w}$ unspecified, so $2^i2^{k-i}=2^k=n$ ways exist to choose $\boldsymbol{u}$ and $\boldsymbol{w}$. Since $k-1=\log_2 n-1=\log_2(n/2)$ ways exist to choose i, we have for all $\boldsymbol{v}$, $\xi_{\boldsymbol{v}}\le 2n\log_2(n/2)$.

In the generalization to $d\ge 4$ we can improve our basic construction by combining the shift and change edges into a single edge. Specifically, assume $d\ge 4$ is even, and represent the vertices of $B_{d,n}$ by the k-tuples of integers in

$$V(B_{d,n}) = \{(a_1,\cdots,a_k): 1\le a_i\le d/2, 1\le i\le k=\log_{d/2} n\}.$$

An edge exists between two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ if the first $k-1$ elements of $\boldsymbol{u}$ (respectively $\boldsymbol{v}$) are the last $k-1$ elements of $\boldsymbol{v}$ (respectively $\boldsymbol{u}$), i.e., the edges incident to $\boldsymbol{u}=(u_1,\cdots,u_k)$ are given by

$$\begin{aligned}(u_1,\cdots,u_k) &\sim (v,u_1,\cdots,u_{k-1}) \quad \text{for all } v \text{ in } \{1,\cdots,d/2\}\\ (u_1,\cdots,u_k) &\sim (u_2,\cdots,u_k,v) \quad \text{for all } v \text{ in } \{1,\cdots,d/2\}.\end{aligned}$$

As can be seen, precisely $d/2+d/2=d$ edges are incident into each vertex, except those invariant under cyclic shifts. Specifically, the $d/2$ vertices of the form $(u,u,\cdots,u)$ each have only $2(d/2-1)=d-2$ edges incident to them. In practice the vertices $(u,u,\cdots,u)$ can be connected into a ring. For $d\ge 5$ odd we simply define $B_{d,n}=B_{d-1,n}$.

Theorem 5: For $d\ge 4$ and n restricted to the set $\{\lfloor d/2\rfloor^k: k>1\}$ we have

$$\xi(B_{d,n}) \le n\log_{\lfloor d/2\rfloor} n.$$

Proof: In P the path from $\boldsymbol{u}$ to $\boldsymbol{v}$ is defined by the sequence of vertices reached by the k shifts starting from $(u_1,\cdots,u_k)$ which successively generate $v_1,v_2,\cdots,v_k$ in the last position of the vertex names, i.e., the path is defined by the sequence

$$\begin{aligned}(u_1,\cdots,u_k) &\sim (u_2,\cdots,u_k,v_1) \sim (u_3,\cdots,u_k,v_1,v_2)\\ &\sim \cdots \sim (u_k,v_1,\cdots,v_{k-1}) \sim (v_1,\cdots,v_k).\end{aligned}$$

It is easy to verify that the diameter of $B_{d,n}$ is $\log_{\lfloor d/2\rfloor} n$, $d\ge 4$.

To determine the forwarding index, consider an arbitrary vertex $\boldsymbol{v}$ and suppose the path from $\boldsymbol{u}=(u_1,\cdots,u_k)$ to $\boldsymbol{w}=(w_1,\cdots,w_k)$ passes through $\boldsymbol{v}$. Then by definition of P an i, $1\le i\le k-1$, must exist such that $\boldsymbol{v}=(u_{i+1},\cdots,u_k,w_1,\cdots,w_i)$. This leaves $u_1,\cdots,u_i$ and $w_{i+1},\cdots,w_k$ unspecified. Since there are $\lfloor d/2\rfloor$ choices for each of these elements and $k-1$ choices for i, we have

$$\xi_{\boldsymbol{v}} = (k-1)\left[\lfloor d/2\rfloor^i \lfloor d/2\rfloor^{k-i}\right].$$

Using $n=\lfloor d/2\rfloor^k$, this leads to $\xi_{\boldsymbol{v}}\le n\log_{\lfloor d/2\rfloor} n$.

IV. Asymptotic Bounds on the Forwarding Index

We first prove a lower bound on forwarding indexes which is analogous to the Moore bound [2] for the diameter problem.

Theorem 6: For any given $d\ge 3$

$$\xi_{d,n} \ge [1+o(1)]\, n\log_{d-1} n.$$

Proof: Let G be an arbitrary graph with $\deg G=d$, and let $\boldsymbol{v}$ be any vertex in G. Adjacent to each vertex at distance $i\ge 1$ from $\boldsymbol{v}$ there are at most $d-1$ vertices at distance $i+1$ from $\boldsymbol{v}$. Thus since at most d vertices are at distance 1 from $\boldsymbol{v}$, the number n_i of vertices at a distance no greater than $i \geq 1$ from v must satisfy

$$n_i \leq d + d(d-1) + \cdots + d(d-1)^{i-1} = \frac{d}{d-2}\left[(d-1)^i - 1\right].$$

The number n_i' of vertices at a distance greater than i from v must therefore satisfy

$$n_i' \geq n - 1 - \frac{d}{d-2}\left[(d-1)^i - 1\right], \qquad i \geq 1.$$

In addition, $n_i' \geq 0$ if

$$n - 1 \geq \frac{d}{d-2}\left[(d-1)^i - 1\right],$$

and hence $1 \leq i \leq \log_{d-1} n$.

Now observe that n_i' paths exist from v to vertices at a distance greater than i which pass through vertices at distance i. Thus since $n_i' \geq 0$, $1 \leq i \leq \log_{d-1} n - 2$, the total number ζ_v of instances in which paths starting at v pass through other vertices must satisfy

$$\zeta_v \geq \sum_{i=1}^{\log_{d-1} n} \left(n - 1 - \frac{d}{d-2}\left[(d-1)^i - 1\right]\right) = (1 + o(1)) n \log_{d-1} n.$$

Thus summing over all vertices v yields

$$\zeta \geq [1 + o(1)] n^2 \log_{d-1} n$$

for the total number of instances of paths passing through vertices. It remains only to observe as before that the maximum number of paths passing through a vertex must satisfy $\max_v \xi_v \geq \zeta/n$.

Theorem 5 on de Bruijn graphs provides upper bounds to be compared with Theorem 6 for restricted sequences of n. Specifically, for n a power of $d/2$ the asymptotic ratio of the upper bound to the lower bound is

$$\log_{d/2}(d-1) \leq 1 + \frac{1}{\log_2 d/2}$$

and tends to one with increasing d.

Note also that the upper bound of Theorem 5 can be extended easily to many other sequences for n. For example, if $n = (r/2)^k$ for some $k > 1$ and even r, $4 \leq r \leq d$, then the asymptotic ratio is $\log_{r/2}(d-1)$. For example, with any $n = (r/2)^k$ where $r \geq 2\sqrt{d-1}$ the asymptotic ratio increases to at most two.

Our methods for obtaining graphs with simple structures and good forwarding indexes appear to require that n be restricted to integer powers. However, using a recursive construction based on de Bruijn graphs, it is easily seen from the following result that a constant asymptotic ratio exists that is valid for all $d \geq 6$ and n.

Theorem 7: For $d \geq 6$ we have[1]

$$\xi_{d,n} \leq \left[3 + O\left(\frac{1}{\log d}\right)\right] n \log_{d-1} n.$$

Proof: For arbitrary n and $d \geq 6$ we describe a graph satisfying the bound. Although a direct formal definition of such a graph can be given, a more transparent description is provided in the following recursive procedure for constructing the graph. For notational convenience we assume that $d \geq 6$ is a multiple of three. (By taking the largest multiple of three in an arbitrary d, it will be easy to see that the theorem holds for all $d \geq 6$.) We also assume that the recursive procedure to follow is invoked initially with G being a set of n vertices and no edges.

CONNECT (d, G)($d \geq 6$ is a multiple of 3 and $n = |V(G)| > 0$):

1) If $n \leq d + 1$, then add edges to vertices in G so as to make G a complete graph, and then stop.

2) Let

$$r(d/3)^k \leq n < (r+1)(d/3)^k \tag{4.1}$$

where $1 \leq r \leq d/3 - 1$. Add edges to any subset G' of $(d/3)^k$ vertices in G so that it has the edges of a de Bruijn graph with degree constraint $2d/3 \geq 4$ (since $d \geq 6$).

3) Let G'' be the vertices not in G'. If G'' is empty, then stop. Otherwise, to each vertex in G'' add an edge to a vertex in G' in such a way that at most $r \leq (d/3) - 1$ new edges are added to any vertex in G'.

4) If $d = 6$, then stop; otherwise, CONNECT $(d - 3, G'')$.

On the first recurrence of the procedure vertices in G' acquire $2d/3 + r \leq d$ edges, and no edges are added to these vertices subsequently. Since the new degree constraint on the second recurrence is $d - 3$, the vertices in G'' acquire at most $2d/3 - 2 + d/3 - 2 = d - 4$ edges in the second recurrence to go along with the one edge received in the first recurrence. Inductively, it is easy to see that no more than d edges are added to any vertex throughout the procedure.

Let $G_0 \equiv G, G_1, \cdots, G_j$, $j \geq 0$ be the sequence of graphs on which the procedure is invoked, and define the corresponding parameters n_i, r_i, and k_i, where $n_0 = n$. Let G_i', $0 \leq i \leq j$ denote the de Bruijn subgraph with degree constraint $2d_i/3$ and $(d_i/3)^{k_i}$ vertices. We now define and analyze the paths having at least one terminal vertex in G_i', $0 \leq i < j$, assuming $j > 0$. Since $V(G) = V(G_0') \cup \cdots \cup V(G_{j-1}') \cup V(G_j)$, only the vertices in G_j will remain to be considered.

Between vertices in G_i', $0 \leq i < j$, the paths are simply the de Bruijn paths defined in Theorem 5. From a vertex u in G_i' to a vertex w in G_{i+1} we have first the de Bruijn path from u to the vertex v in G_i' for which $v \sim w$. The path then concludes with the edge $v \sim w$. Similarly, the path from w to u begins with the edge $w \sim v$, which is then followed by the de Bruijn path from v to u. Note that no path having a terminal vertex in G_l', $l < i$, can pass through a vertex in G_i'.

Thus the only paths through vertices in G_i' are those connecting two vertices in G_i' and G_{i+1}, at least one of which is in G_i'. A path through v in G_i' must, therefore, consist of a de Bruijn path in G_i' plus at most one edge

[1] The base of a logarithm will be omitted when it is immaterial.

incident to a vertex in G_{i+1}. Using Theorem 5, there are at most $k_i(d_i/3)^{k_i}$ paths wholly within G_i' passing through v, at most $r_i k_i(d_i/3)^{k_i}$ paths starting in G_{i+1} and ending in G_i', and at most $r_i k_i(d_i/3)^{k_i}$ paths starting in G_i' and ending in G_{i+1}. Thus

$$\xi_v \le (2r_i + 1)k_i(d_i/3)^{k_i}.$$

Now $r_i(d_i/3)^{k_i} \le n_i$, so

$$\xi_v \le \left(2 + \frac{1}{r_i}\right) n_i \log_{d_i/3} \frac{n_i}{r_i}$$

whereupon $r_i \ge 1$ implies

$$\xi_v \le 3n_i \log_{d_i/3} n_i.$$

We now want to bound this function for all i. First from (4.1) and $r_i \le (d_i/3) - 1$ we have $n_i \le (d_i/3)^{k_i+1}$ or $(d_i/3)^{k_i} \ge n_i/(d_i/3)$. Thus since $d_i = d - 3i$,

$$n_{i+1} = n_i - (d_i/3)^{k_i} \le n_i - \frac{n_i}{d_i/3} = n_i\left[\frac{d/3 - i - 1}{d/3 - i}\right]$$

with $n_0 = n$. Routinely, we obtain

$$n_i \le \frac{d/3 - i}{d/3} n = (1 - 3i/d)n, \qquad 0 \le i < j.$$

Thus

$$3n_i \log_{d_i/3} n_i \le 3(1 - 3i/d)n \log_{(d/3)-i}(1 - 3i/d)n$$

and

$$3n_i \log_{d_i/3} n_i \le \ln(1 - 3i/d)n \frac{3n(1 - 3i/d)}{\ln \frac{d}{3}(1 - 3i/d)}, \qquad 0 \le i < j. \quad (4.2)$$

By inspection of its first derivative the function $(1 - x)/(\ln(d/3)(1 - x))$ is found to be positive and decreasing from $x = 0$ to the solution of $\ln(d/3)(1 - x) = 1$, viz. $x = 1 - 3e/d$. Now $0 \le i < j$ means that $d - 3i \ge 9$ from step 4 of the procedure. Since $1 - x = 1 - 3i/d \ge 9/d$ and $x \le 1 - 9/d \le 1 - 3e/d$, the right-hand side of (4.2) is maximum at $i = 0$ and

$$\xi_v \le 3n \log_{d/3} n \quad (4.3)$$

for any v in G_i', $0 \le i < j$.

It remains to consider G_j. Paths between vertices, at least one of which is in G_j', are defined as before. Paths between vertices in G_{j+1} consist simply of the appropriate de Bruijn paths in G_j' prefixed and suffixed by edges incident to vertices in G_{j+1}. Since $d_j = 6$, it is a simple matter to verify that $\xi_v = O((n/d)\log_2(n/d))$, and hence that the theorem is proved by (4.3) and $\log_{d/3} n = \log_{d/3}(d - 1)\log_{d-1} n = [1 + O(1/\log d)]\log_{d-1} n$.

V. Compounding Methods

In practice it is desirable to be able to compose larger networks using an existing smaller network as a building block. Of course, it is assumed in such circumstances that ports are available at the nodes to accommodate the new connections in the larger network. In this section we illustrate ways of doing this so that only two extra ports per node are needed and a $O(n \log n)$ growth in the forwarding index is assured. These techniques are known as compounding methods in the literature on the diameter problem [2].

Let (G, P) be a given network of n_0 vertices, each satisfying the degree constraint d. Suppose we want to construct a network (G_n, P_n), $\deg G_n = d + 2$, composed of $t > 1$ copies of (G, P), and let the $n = tn_0$ vertices of G_n be denoted by the vectors $(a, \boldsymbol{u})$, $0 \le a \le t - 1$, $\boldsymbol{u} \in V(G)$. A simple approach would be to connect the t vertices $(a, \boldsymbol{u})$, $0 \le a \le t - 1$ into a ring for each $\boldsymbol{u}$ in G. A path from $(a, \boldsymbol{u})$ to $(b, \boldsymbol{v})$ could then be defined as a path from $(a, \boldsymbol{u})$ to $(a, \boldsymbol{v})$ in copy a of G, which is followed by a shortest path to $(b, \boldsymbol{v})$ in the ring containing $(a, \boldsymbol{v})$ and $(b, \boldsymbol{v})$. It is not difficult to verify that the forwarding index of G would grow as $\xi_{2,d}\xi_0 \approx (t^2/4)\xi_0 = \Omega(tn \log n)$, and hence the forwarding index of (G, P) will be preserved in (G_n, P_n) for only small values of t.

The main problem with this approach is that the traversal of rings and vertices within copies is sequential. Depending on the structure of G, it may be possible to make these traversals in parallel and thus preserve the forwarding index for larger values of t. For example, suppose $G = B_{3,n_0}$ as defined in Section III, and suppose t is no larger than the diameter $2\log_2 n_0$. We define a ring of t de Bruijn graphs as follows.

Let $n_0 = 2^k$, and denote the $n = t2^k$ vertices as before by the pairs $(a, \boldsymbol{u})$, $0 \le a \le t - 1$, where $\boldsymbol{u} = (u_1, \cdots, u_k)$ is a vertex of B_{3,n_0}. The edges of G_n are defined by the following. For all a, $0 \le a \le t - 1$,

$$(a, \boldsymbol{u}) \sim (a, \boldsymbol{v}) \text{ if } \boldsymbol{u} \sim \boldsymbol{v} \text{ in } B_{3,n_0}$$

$$(a; u_1, \cdots, u_k) \sim ((a + 1) \bmod t;\ u_2, \cdots, u_k, u_1) \quad \text{for all } (u_1, \cdots, u_k) \text{ in } B_{3,n_0}.$$

Clearly, each vertex in G_n has a degree of at most 5. The two additional edges at each vertex $(a, \boldsymbol{u})$ link that vertex with those vertices in adjacent de Bruijn copies which are obtained by shifts of $\boldsymbol{u}$. These edges linking copies will be called global shift edges

Theorem 8: Let G_n be a ring of t de Bruijn graphs B_{3,n_0}. For any fixed $1 \le t \le 2\log_2 n_0$ we have

$$\xi(G_n) = 2n \log_2(n/t)$$

where $n = tn_0$ is the number of vertices in G_n.

Proof: We define the following routing P_n for G_n.

1) The path from $(a, \boldsymbol{u})$ to $(a, \boldsymbol{v})$ in G_n is simply the path from $\boldsymbol{u}$ to $\boldsymbol{v}$ in copy a, $0 \le a \le t - 1$ of B_{3,n_0}.

2) The path from $(a, \boldsymbol{u})$ to $(b, \boldsymbol{v})$, $a \ne b$ consists first of a shortest path (alternating change and global shift edges) along the ring from $(a, \boldsymbol{u})$ to a vertex, say $(b, \boldsymbol{u}')$, in copy b of B_{3,n_0} and then the path from $\boldsymbol{u}'$ to $\boldsymbol{v}$ in copy b of B_{3,n_0}.

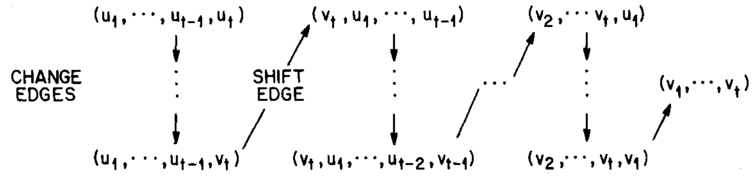


Fig. 5. Paths in P_n.

Observe that at most $\lfloor t/2 \rfloor$ global shift edges are taken along the ring in a path from $(a, \boldsymbol{u})$ to $(b, \boldsymbol{v})$. Since exactly $k = \log_2 n_0$ (local) shift edges are taken from $\boldsymbol{u}$ to $\boldsymbol{v} \neq \boldsymbol{u}$ in B_{3, n_0}, our assumption that $t \leq 2k$ suffices to ensure that the path from $(a, \boldsymbol{u})$ to $(b, \boldsymbol{v})$ reaches a vertex in copy b using at most the number of edges in the path from $\boldsymbol{u}$ to $\boldsymbol{v}$ in a copy of B_{3, n_0}.

To find $\xi(G_n, P_n)$, consider an arbitrary vertex $(b, \boldsymbol{v})$ in G_n. As before we want to find the largest number of vertex pairs $(a, \boldsymbol{u})$ and $(c, \boldsymbol{w})$ such that the path between these vertices passes through $(b, \boldsymbol{v})$. Assume that exactly $0 \leq j \leq k-1$ shift (local or global) edges have been taken in the path from $(a, \boldsymbol{u})$ to $(b, \boldsymbol{v})$. By an argument similar to that in Theorem 4 we must have at most 2^{k+1} possibilities for the pair $\boldsymbol{u}, \boldsymbol{w}$. (The extra factor of two is due to the ability to shift in either of two directions.)

Now if $b = c$, i.e., the path from $(a, \boldsymbol{u})$ to $(c, \boldsymbol{w})$ has already reached copy c, then at most $\lfloor t/2 \rfloor$ possible choices exist for a. However, if $b \neq c$, then only global shifts could have been taken from $(a, \boldsymbol{u})$ to $(b, \boldsymbol{v})$, and hence a is completely determined by j, b, and $\boldsymbol{u}$. Thus for a given j we have at most $((t/2) + (t/2))2^{k+1} = 2tn_0$ possible pairs $(a, \boldsymbol{u})$, $(c, \boldsymbol{w})$. Since j takes on at most k values, we have easily

$$\xi(G_n, P_n) = 2n \log_2 (n/t).$$

The approach of using shift edges for global connections can be generalized with an additional structure having the simplicity of de Bruijn graphs. The following method places no constraints on the basis graph G, and no upper bound exists on n. However, consistent with our remarks following Theorem 6, n is restricted to integer powers of n_0.

We define networks (G_n, P_n) composed of n_0^{t-1}, $t > 1$ copies of (G, P) as follows. The $n = n_0^t$ vertices in G_n are represented by t-tuples $(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_t)$ where each $\boldsymbol{v}_i$, $1 \leq i \leq t$ names one of the n_0 vertices in G. Vertices $(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_t)$ and $(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_t)$ are adjacent in G_n if and only if

1) $\boldsymbol{u}_i = \boldsymbol{v}_i, \quad 1 \leq i \leq t-1$ and $\boldsymbol{u}_t \sim \boldsymbol{v}_t$ in G or
2) $(\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_t) = (\boldsymbol{u}_t, \boldsymbol{u}_1, \cdots, \boldsymbol{u}_{t-1})$ or symmetrically $(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_t) = (\boldsymbol{v}_t, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_{t-1})$.

From a construction standpoint we associate a distinct $(t-1)$-tuple with each of the n_0^{t-1} copies of G. A vertex $\boldsymbol{v}$ in the copy identified by $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{t-1})$ is the vertex $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{t-1}, \boldsymbol{v})$ in G_n. By assumption we can add two edges to each vertex. Thus we establish a connection between $\boldsymbol{v}$ and vertex $\boldsymbol{x}_{t-1}$ in copy $(\boldsymbol{v}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{t-2})$ and one between $\boldsymbol{v}$ and vertex $\boldsymbol{x}_1$ in copy $(\boldsymbol{x}_2, \cdots, \boldsymbol{x}_{t-1}, \boldsymbol{v})$; i.e., we establish the two edges between $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{t-1}, \boldsymbol{v})$ and vertices $(\boldsymbol{v}, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{t-1})$ and $(\boldsymbol{x}_2, \cdots, \boldsymbol{x}_{t-1}, \boldsymbol{v}, \boldsymbol{x}_1)$ of G_n, respectively. The shift edges in 2) perform global connections between copies of G, and the change edges in 1) correspond to the existing connections local to a given copy of G.

We construct the routing P_t using the routing P of each copy of (G, P). The path $p(\boldsymbol{u}, \boldsymbol{v})$ from $\boldsymbol{u} = (\boldsymbol{u}_1, \cdots, \boldsymbol{u}_t)$ to $\boldsymbol{v} = (\boldsymbol{v}_1, \cdots, \boldsymbol{v}_t)$ in G_n begins with a path from $\boldsymbol{u}_t$ to $\boldsymbol{v}_t$ in the copy $(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_{t-1})$ of G. (The length of this path degenerates to zero if $\boldsymbol{u}_t = \boldsymbol{v}_t$.) Then the next edge of $p(\boldsymbol{u}, \boldsymbol{v})$ is the shift edge

$$(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_{t-1}, \boldsymbol{v}_t) \sim (\boldsymbol{v}_t, \boldsymbol{u}_1, \cdots, \boldsymbol{u}_{t-1})$$

to vertex $\boldsymbol{u}_{t-1}$ in the copy $(\boldsymbol{v}_t, \boldsymbol{u}_1, \cdots, \boldsymbol{u}_{t-2})$ of G. As already mentioned, the next segment of $p(\boldsymbol{u}, \boldsymbol{v})$ is a path in the copy $(\boldsymbol{v}_t, \boldsymbol{u}_1, \cdots, \boldsymbol{u}_{t-2})$ from vertex $\boldsymbol{u}_{t-1}$ to vertex $\boldsymbol{v}_{t-1}$ of G. Again as mentioned earlier, the next edge in $p(\boldsymbol{u}, \boldsymbol{v})$ is the shift edge

$$(\boldsymbol{v}_t, \boldsymbol{u}_1, \cdots, \boldsymbol{u}_{t-2}, \boldsymbol{v}_{t-1}) \sim (\boldsymbol{v}_{t-1}, \boldsymbol{v}_t, \boldsymbol{u}_1, \cdots, \boldsymbol{u}_{t-2})$$

to the copy $(\boldsymbol{v}_{t-1}, \boldsymbol{v}_t, \boldsymbol{u}_1, \cdots, \boldsymbol{u}_{t-3})$ of G. This process continues as shown in Fig. 5 until vertex $\boldsymbol{v}$ of G_n is reached after the tth shift edge is taken. The asymptotic forwarding index of (G_n, P_n) is determined by the following result.

Theorem 9: Let ξ_0 be the forwarding index of (G, P), let $\deg G = d$, and let $n = n_0^t$ be the number of vertices in G_n. Then

$$\xi(G_n, P_n) \leq n \log_{d-1} n \left\lceil \frac{\xi_0 + 2n_0 - 1}{n_0 \log_{d-1} n_0} \right\rceil.$$

If (G, P) is a de Bruijn network with $d \geq 4$, then for n restricted to the set $\{n_0^t, t \geq 1\}$

$$\xi(G_n, P_n) = \left[1 + O\left(\frac{1}{\log d}\right)\right] n \log_{\lfloor d/2 \rfloor} n.$$

Proof: Let $\boldsymbol{v} = (\boldsymbol{v}_1, \cdots, \boldsymbol{v}_t)$ be an arbitrary vertex in G_n, and consider how many paths in P_n can possibly pass through $\boldsymbol{v}$. Suppose that $\boldsymbol{v}$ is in the path from $\boldsymbol{a} = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_t)$ to $\boldsymbol{b} = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_t)$. In moving from $\boldsymbol{a}$ to $\boldsymbol{b}$ along $p(\boldsymbol{a}, \boldsymbol{b})$ suppose that when $\boldsymbol{v}$ is encountered, exactly i, $0 \leq i \leq t-1$ shift edges have already been taken. Then the last i elements of $\boldsymbol{b}$ have already been established in the first i elements of $\boldsymbol{v}$, so we must have $(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_i) = (\boldsymbol{b}_{t-i+1}, \cdots, \boldsymbol{b}_t)$. In addition, since $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{t-i-1}$ have been shifted right i positions but not changed, they must still be in the vertex $\boldsymbol{v}$ in positions $\boldsymbol{v}_{i+1}, \cdots, \boldsymbol{v}_{t-1}$; i.e., $(\boldsymbol{v}_{i+1}, \cdots, \boldsymbol{v}_{t-1}) = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{t-i-1})$. Thus, with i given, a count of the maximum number of vertex pairs $(\boldsymbol{a}, \boldsymbol{b})$ such that $p(\boldsymbol{a}, \boldsymbol{b})$ can pass through $\boldsymbol{v}$ must include n_0 independent choices for each of $\boldsymbol{a}_{t-i+1}, \cdots, \boldsymbol{a}_t$ and each of $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_{t-i-1}$ for a total of $n_0^i n_0^{t-i-1} = n_0^{t-1}$.

Now consider the choices for $\boldsymbol{a}_{t-i}$ and $\boldsymbol{b}_{t-i}$. We could have $\boldsymbol{a}_{t-i} = \boldsymbol{v}_t$ and n_0 possibilities for $\boldsymbol{b}_{t-i}$ or $\boldsymbol{b}_{t-i} = \boldsymbol{v}_t$ and n_0 possibilities for $\boldsymbol{a}_{t-i}$. This gives us a total of $2n_0 - 1$ distinct possibilities. The number of choices for $(\boldsymbol{a}_{t-i}, \boldsymbol{b}_{t-i})$ such that $\boldsymbol{a}_{t-i} \neq \boldsymbol{v}_t$ and $\boldsymbol{b}_{t-i} \neq \boldsymbol{v}_t$ can be at most the number of paths in P passing through $\boldsymbol{v}_t$. However, this is at most ξ_0 by definition. Therefore, for i given, at most $n_0^{t-1}(\xi_0 + 2n_0 - 1)$ choices exist for the pairs $(\boldsymbol{a}, \boldsymbol{b})$. Since t choices exist for i, we have

$$\xi_t \leq t n_0^{t-1}(\xi_0 + 2n_0 - 1).$$

Using $\log n = \log n_0^t = t \log n_0$, we obtain the theorem for arbitrary G. The result for de Bruijn graphs then follows from Theorem 5.

VI. Final Remarks

We conclude by mentioning a few of the many open problems related to the forwarding-index problem. Of course, there are the problems of sharpening our results and developing more effective networks for arbitrary values of n; the problems of designing and analyzing networks for small n or $d > 2$ which have minimum or near-minimum forwarding indexes have been left open. The extensive literature on the diameter problem (see [1]–[3] for surveys) may be useful here since small-diameter networks such as the de Bruijn networks appear to have small forwarding indexes.

We have also not dealt directly with complexity issues. For example, the question exists of whether the problem of finding a routing minimizing ξ for a given graph is NP-complete. Specialization of these questions to symmetric graphs would also be of interest as would their extension to a problem complementary to ours: given a positive integer k, what is the size of a smallest network with forwarding index k?

In the network capacity applications the assumption of asymmetric, i.e., arbitrary, transmission requirements is a generalization of practical interest. The problem of maximizing network capacity would appear to be substantially more difficult in this model since an appropriate measure of the transmission requirements at node i must now involve the sum of the rates on the paths passing through node i rather than simply the number of such paths. Switching networks are a particular example of this generalization. In such networks the nodes are switches defined as before by a capacity and port constraint. The network establishes communication among a set of devices that are connected, not necessarily symmetrically, to some subset of the switches. Our forwarding index results apply to such networks only under the assumptions of an equivalent set of devices connected to each switch and symmetric transmission requirements.

Another example of the more general model is afforded by arrays of microcomputers (see [5] for recent performance analyses and a survey of such networks). In these models spatial locality accompanies a locality in transmission requirements, i.e., the communication between neighboring devices is at a higher rate than that between remote devices.

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