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Cross-Monotone Subsequences

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Abstract. Two finite real sequences $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ are cross-monotone if each is nondecreasing and $a_{i+1} - a_i \geqslant b_{i+1} - b_i$ for all $i < k$. A sequence $(\alpha_1, \ldots, \alpha_n)$ of nondecreasing reals is in class $\mathrm{CM}(k)$ if it has disjoint k-term subsequences that are cross-monotone. The paper shows that $f(k)$, the smallest n such that every nondecreasing $(\alpha_1, \ldots, \alpha_n)$ is in $\mathrm{CM}(k)$, is bounded between about $k^2/4$ and $k^2/2$. It also shows that $g(k)$, the smallest n for which all $(\alpha_1, \ldots, \alpha_n)$ are in $\mathrm{CM}(k)$ *and* either $a_k \leqslant b_1$ or $b_k \leqslant a_1$, equals $k(k-1)+2$, and that $h(k)$, the smallest n for which all $(\alpha_1, \ldots, \alpha_n)$ are in $\mathrm{CM}(k)$ *and* either $a_1 \leqslant b_1 \leqslant \cdots \leqslant a_k \leqslant b_k$ or $b_1 \leqslant a_1 \leqslant \cdots \leqslant b_k \leqslant a_k$, equals $2(k-1)^2 + 2$.

The results for f and g rely on new theorems for regular patterns in (0, 1)-matrices that are of interest in their own right. An example is: Every upper-triangular $k^2 \times k^2$ (0, 1)-matrix has either k 1's in consecutive columns, each below its predecessor, or k 0's in consecutive rows, each to the right of its predecessor, and the same conclusion is false when k^2 is replaced by $k^2 - 1$.

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1. Introduction

A $2 \times k$ array of real numbers

$$a_1\, a_2 \ldots a_k, \qquad b_1\, b_2 \ldots b_k$$

is *cross-monotone* (CM) if $a_1 \leqslant \cdots \leqslant a_k$, $b_1 \leqslant \cdots \leqslant b_k$, and

$$a_{i+1} - a_i \geqslant b_{i+1} - b_i \quad \text{for } 1 \leqslant i \leqslant k-1.$$

If, in addition, either $a_k \leqslant b_1$ or $b_k \leqslant a_1$, then it is *separated cross-monotone* (SCM); and if either

$$a_1 \leqslant b_1 \leqslant \ldots \leqslant a_k \leqslant b_k \quad \text{or} \quad b_1 \leqslant a_1 \leqslant \cdots \leqslant b_k \leqslant a_k,$$

then it is *alternating cross-monotone* (ACM).

Our objective for each type of cross-monotonicity is to establish the smallest n such that every sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nondecreasing reals has index-disjoint subsequences $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ that form a CM $2 \times k$ array. To be precise, let $\mathscr{A}_n = \{(\alpha_1, \ldots, \alpha_n) : \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n\}$, and for each $k \geqslant 2$ let CM(k) be the family of α in $\cup_{n \geqslant 2k} \mathscr{A}_n$ for which there are disjoint index sets $\{i_1 < \cdots < i_k\}$ and $\{j_1 < \cdots < j_k\}$ such that

$$\alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_k}, \quad \alpha_{j_1} \alpha_{j_2} \ldots \alpha_{j_k}$$

is CM. Also let SCM(k) [ACM(k)] be the subfamily of CM(k) for which either $i_k < j_1$ or $j_k < i_1$ [either $i_1 < j_1 < i_2 < j_2 < \cdots$ or $j_1 < i_1 < j_2 < i_2 < \cdots$] can be satisfied. Define $f, g, h: \{2, 3, \ldots\} \to \{4, 5, \ldots\}$ by

$$f(k) = \min\{n : \mathscr{A}_n \subseteq \mathrm{CM}(k)\}$$
$$g(k) = \min\{n : \mathscr{A}_n \subseteq \mathrm{SCM}(k)\}$$
$$h(k) = \min\{n : \mathscr{A}_n \subseteq \mathrm{ACM}(k)\}.$$

Our main result is

THEOREM 1. *For all* $k \in \{2, 3, \ldots\}$,

$$\tfrac{1}{4}(k+1)^2 \leqslant f(k) \leqslant \tfrac{1}{2}[k^2 + 4k^{3/2} + 8k + 8k^{1/2}]$$
$$g(k) = k(k-1) + 2$$
$$h(k) = 2(k-1)^2 + 2.$$

The bounds on $f(k)$ are established in Section 7. Known exact values are $f(2) = 4$, $f(3) = 7$ and $f(4) = 11$. We remark on this further in the final section.

The proofs for Theorem 1 begin with the simplest case, h, then consider g and f in turn. The proof for each function is given in two sections, the first of which presents an auxiliary result that is used to bound the function from above. The upper bound itself is derived in the second section of the pair, followed by a construction for the lower bound.

The auxiliary results (Sections 2, 4 and 6) do not address cross-monotonicity directly and are of interest in themselves. The one in the next section is a familiar lemma (every arrangement of $n^2 + 1$ integers has a monotone subsequence of $n + 1$ integers) that is related to various results on regular patterns in sequences ([2, 3, 5, 6, 7, 9, 11]). Section 4 proves that every $k^2 \times k^2$ upper-triangular (0, 1)-matrix contains a 'monotone sequence' of k 0's or k 1's, and Section 6 does something similar for off-diagonal skew-symmetric square (0, 1)-matrices. We refer the reader to those sections for definitions and details.

The present study was motivated by the following open question in Jogdeo and Molenaar [8]: For every $(m, n) \geqslant (2, 2)$ and every real sequence $(c_1, \ldots, c_{mn})$, is there a bijection τ from $\{1, \ldots, m\} \times \{1, \ldots, n\}$ onto $\{1, \ldots, mn\}$ such that

$$c_{\tau(i+1, j+1)} - c_{\tau(i+1, j)} \geqslant c_{\tau(i, j+1)} - c_{\tau(i, j)}$$

for all $(i, j) \leqslant (m - 1, n - 1)$? We do not resolve this question.

2. Basic Lemma

LEMMA 0. *Every linear arrangement of* $\{1, 2, \ldots, n^2 + 1\}$ *has either an increasing subsequence or a decreasing subsequence of at least* $n + 1$ *integers.*

An early published proof of this is in Erdös and Szekeres [4]. Later proofs are in Kruskal [9] and Seidenberg [10]. It was recently generalized by Chung [1] for unimodal (up, then down; *or* down, then up) subsequences.

3. Alternating Cross-Monotonicity

Here, and later,

$$\Delta_i = \alpha_{i+1} - \alpha_i \quad \text{for} \quad \alpha = (\alpha_1, \ldots, \alpha_n) \text{ in } \mathscr{A}_n.$$

We shall often write a nondecreasing sequence of Δ_i or α_i using only their subscripts. Thus, for the Δ_i, 371... could denote $\Delta_3 \leqslant \Delta_7 \leqslant \Delta_1 \ldots$ A sequence $c_1, c_2, c_3, \ldots$ of positive reals is *superincreasing* if $\Sigma_{i<j} c_i < c_j$ for each $j \geqslant 2$. Superincreasing Δ_i sequences will be used in our lower-bound constructions.

We prove $h(k) = 2(k-1)^2 + 2$ by two lemmas, the first of which uses Lemma 0 for the upper bound.

LEMMA 1. $h(k) \leqslant 2(k-1)^2 + 2$.

Proof. Given $n = 2(k-1)^2 + 2$, let α be any sequence in $\mathscr{A}_n$. There are $(k-1)^2 + 1$ of the Δ_i with odd i. Hence, by Lemma 0, a natural ($\leqslant$) ordering of these Δ_i has a k-term subsequence whose *indices* are monotone increasing or decreasing. Let $i_1, i_2, \ldots, i_k$ be such a subsequence of indices. If $i_1 < \cdots < i_k$, then $(\alpha_{i_1+1}, \alpha_{i_2+1}, \ldots, \alpha_{i_k+1})$ above $(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k})$ shows that $\alpha \in \mathrm{ACM}(k)$; if $i_1 > \cdots > i_k$, then $(\alpha_{i_1+1}, \ldots, \alpha_{i_k+1})$ beneath $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ verifies $\alpha \in \mathrm{ACM}(k)$. □

In the lower-bound lemma we shift the index by 1 for notational convenience.

LEMMA 2. $h(k+1) > 2k^2 + 1$.

Proof. Given $n = 2k^2 + 1$, let $\alpha \in \mathscr{A}_n$ have superincreasing Δ_i according to the following subscript order:

$$2k\ 2k-1 \ldots 1 \mid 4k\ 4k-1 \ldots 2k+1 \mid \ldots \mid 2k^2\ 2k^2-1 \ldots (2k-2)k+1.$$

Denote this order by $<_o$ so that $i <_o j \Rightarrow \Delta_i < \Delta_j$. Finally, let $(a_1, a_2, \ldots, a_{k+1})$ above $(b_1, b_2, \ldots, b_{k+1})$ be a $2 \times (k+1)$ array from α that allegedly verifies $\alpha \in \mathrm{ACM}(k+1)$. Two cases show that this allegation is false. We speak of each left-to-right subsequence of $2k$ integers in $<_o$ as a block. Vertical bars separate blocks.

Case 1. Suppose $b_1 < a_1 < \cdots < b_{k+1} < a_{k+1}$. Then $\mathrm{CM} \Rightarrow a_1 - b_1 \leqslant \cdots \leqslant a_{k+1} - b_{k+1}$. Suppose $a_i - b_i$ *spans* one or more Δ_j in the pth block of $<_o$, i.e., if $\Delta_j = \alpha_{j+1} - \alpha_j$, then $a_i \leqslant \alpha_j$ and $\alpha_{j+1} \leqslant b_k$. Then, by construction, including superincreasingness, $a_{i+1} - b_{i+1}$ must span a Δ_j in a later block. But this is impossible since there are $k + 1$ differences $a_i - b_i$ and only k blocks.

Case 2. Suppose $a_1 < b_1 < \cdots < a_{k+1} < b_{k+1}$. Then $\mathrm{CM} \Rightarrow b_1 - a_1 \geqslant \cdots \geqslant b_{k+1} - a_{k+1}$. Observe that $b_1 - a_1$ is the sum of one or more contiguous Δ_i; $b_2 - a_2$ is the sum of one or more contiguous Δ_i for i that exceed those for $b_1 - a_1$ by 2 or more; and so forth. Therefore, every Δ subscript of $b_2 - a_2$ is $<_o$ every Δ subscript for $b_1 - a_1$, and similarly for each $b_{i+1} - a_{i+1}$ versus $b_i - a_i$. Moreover, all Δ subscripts used must be in the same block. But this too is impossible since at most k of the i's in any block are separated by 2 or more from each other. □

4. Monotone Sequences in (0, 1)-Matrices

Let $\mathscr{C}_n$ denote the class of all $n \times n$ (0, 1)-matrices $C_n = [c(i, j)]$. We shall say that C_n has a *monotone sequence* of length k if either

(0) $\exists\ 1 \leqslant i \leqslant n-k+1$ and $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n$ such that $c(i, j_1) = c(i+1, j_2) = \cdots = c(i+k-1, j_k) = 0$

or

(1) $\exists\ 1 \leqslant j \leqslant n-k+1$ and $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ such that $c(i_1, j) = c(i_2, j+1) = \cdots = c(i_k, j+k-1) = 1$.

The (i, j) cell identifications are understood to be part of monotone sequences (and paths, defined below) along with the c values, but will often be implicit.

A type (0) monotone sequence [0's in successive rows, each to the right of its predecessor] of length k is abbreviated $\mathrm{MS}_0(k)$, and a type (1) monotone sequences [1's in successive columns, each below its predecessor] of length k is abbreviated $\mathrm{MS}_1(k)$. An $\mathrm{MS}(k)$ is either an $\mathrm{MS}_0(k)$ or an $\mathrm{MS}_1(k)$. Let

$$f_1(k) = \min\{n : \text{every } C_n \in \mathscr{C}_n \text{ has an MS}(k)\}$$

$$f_2(k) = \min\{n : \text{every } C_n \in \mathscr{C}_n \text{ has an MS}(k) \text{ lying on or below the main diagonal}\}.$$

The rest of this section proves

THEOREM 2. *$f_1(k) = f_2(k) = k^2$ for every $k \geqslant 1$.*

Since $f_2(k) \geqslant f_1(k)$, we need only show that $f_1(k) \geqslant k^2$ and $k^2 \geqslant f_2(k)$, but we will treat both fully for expository purposes.

Given $n = k^2 - 1$, the matrix with all 0's in columns tk for $t = 1, 2, \ldots, k-1$ and 1's elsewhere has $\mathrm{MS}_0(k-1)$'s and $\mathrm{MS}_1(k-1)$'s, but no $\mathrm{MS}(k)$. Hence, $f_1(k) > k^2 - 1$ and $f_2(k) > k^2 - 1$.

We next prove $f_1(k) \leqslant k^2$ and then refine the proof to show that $f_2(k) \leqslant k^2$. Some definitions are needed.

DEFINITIONS 1. A *vertical path* through C_n is a sequence $c(1, j_1), c(2, j_2), \ldots, c(n, j_n)$ with

$$j_i \leqslant j_{i+1} \leqslant j_i + 1 \quad \text{for } i = 1, \ldots, n-1.$$

Two such paths, one with $j_1, \ldots, j_n$, the other with $j'_1, \ldots, j'_n$, are *column disjoint* if $\{j_1, \ldots, j_n\} \cap \{j'_1, \ldots, j'_n\} = \emptyset$. A vertical path $c(1, j_1), \ldots, c(n, j_n)$ is a *leading monotone sequence* of 1's of length l, abbreviated $\text{LMS}_1(l)$, if $l = j_n - j_1 + 1$ and if, for all $i < n$,

$$j_i = j_n \Rightarrow c(i, j_i) = 0,$$
$$j_i < j_n \Rightarrow [c(i, j_i) = 1 \Leftrightarrow j_i < j_{i+1}].$$

Figure 1 illustrates some $\text{LMS}_1(l)$'s on the upper left. Each has embedded within it an $\text{MS}_1(l)$ if $c(n, j_n) = 1$, and an $\text{MS}_1(l-1)$ if $c(n, j_n) = 0$. Note that a column of 0's is an $\text{LMS}_1(1)$ and contains an $\text{MS}_1(0)$.

Given $n = k^2$, assume an arbitrary $C_n \in \mathscr{C}_n$ has no $\text{MS}_1(k)$. We show that it must have an $\text{MS}_0(k)$. Let $\mathscr{L}$ be a family of k *column disjoint* LMS_1's through C_n. Since

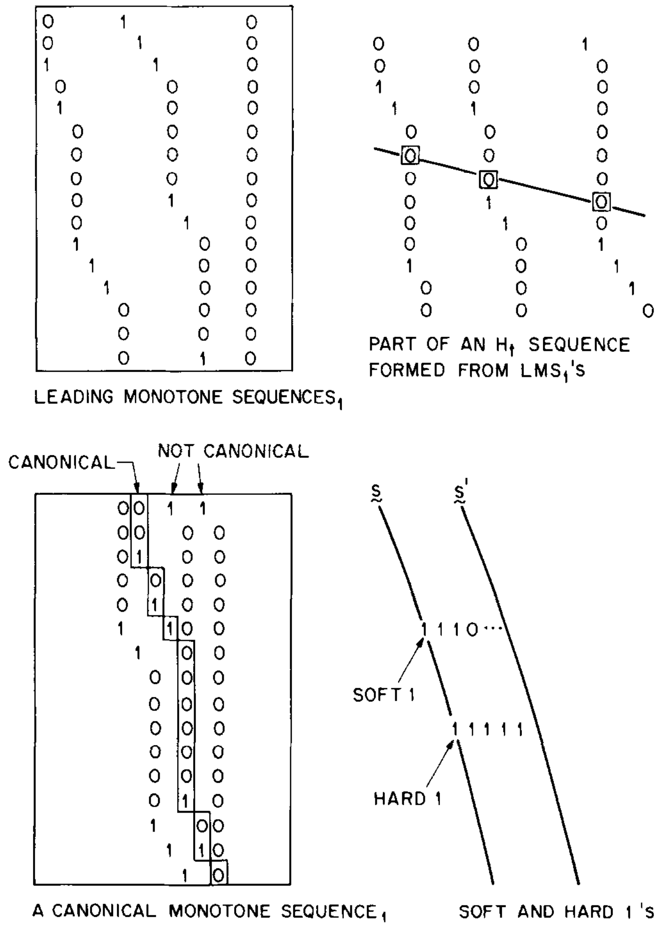

Fig. 1.

each of these has $l \leqslant k$, such a family exists. For each t with $0 \leqslant t \leqslant k^2 - k$, let H_t be the k-term sequence

$$c(t+1, x(t, 1)), c(t+2, x(t, 2)), \ldots, c(t+k, x(t, k)),$$

where $x(t, i)$ is the column corresponding to row $t+i$ in the ith LMS_1 (reading left to right) in $\mathscr{L}$. If H_t is an all-0 sequence, as suggested on the upper right of Figure 1, then it is an $\mathrm{MS}_0(k)$. Suppose no H_t is an $\mathrm{MS}_0(k)$. Then every H_t has a 1, for a total of at least $k^2 - k + 1$ different 1's from the k LMS_1's in $\mathscr{L}$. However, since C_n has no $\mathrm{MS}_1(k)$, the number of 1's in $\mathscr{L}$ can be at most $k(k-1) = k^2 - k$. Consequently, some H_t must be an $\mathrm{MS}_0(k)$, so $f_1(k) \leqslant k^2$.

DEFINITIONS 2. Let $\mathscr{L}_1$ be the family of all LMS_1's in C_n. Define $\mathscr{J} \in \mathscr{L}_1$ as *canonical* if it has the smallest initial column of all $\mathscr{J}' \in \mathscr{L}_1$ that end in the same column as $\mathscr{J}$. (See lower left in Figure 1.) Let $\mathscr{C}\mathscr{L}_1$ be the family of canonical sequences in $\mathscr{L}_1$. (The members of $\mathscr{C}\mathscr{L}_1$ do not touch or cross.) The *distance* between $\mathscr{J} = [c(1, j_1), \ldots, c(n, j_n)]$ and $\mathscr{J}' = [c(1, j_1'), \ldots, c(n, j_n')]$ in $\mathscr{C}\mathscr{L}_1$ with $\mathscr{J}'$ adjacent to $\mathscr{J}$ and to the right of $\mathscr{J}$ ($j_i < j_i'$ for all i) is

$$d(\mathscr{J}, \mathscr{J}') = j_1' - j_1 - 1.$$

A 1 in $\mathscr{J} \in \mathscr{C}\mathscr{L}_1$ is *soft* if $\mathscr{J}$ is not the rightmost member of $\mathscr{C}\mathscr{L}_1$ and the row cells strictly between this 1 and the next $\mathscr{J}' \in \mathscr{C}\mathscr{L}_1$ to the right of $\mathscr{J}$ contain at least one 0; otherwise the 1 is *hard*. (See lower right in Figure 1.)

LEMMA 3. *If $\mathscr{J}$ and $\mathscr{J}'$ are adjacent members of $\mathscr{C}\mathscr{L}_1$ with $j_1 < j_1'$, then $\mathscr{J}$ has at least $d(\mathscr{J}, \mathscr{J}')$ soft* 1's.

Lemma 3 is proved in the Appendix.

We now prove that $f_2(k) \leqslant k^2$. Given $n = k^2$, let C_n be an arbitrary matrix in $\mathscr{C}_n$. Modify C_n to C_n' by replacing every 1 strictly above the diagonal by 0. Assume that C_n' does not contain an $\mathrm{MS}_1(k)$. We shall show that C_n' contains an $\mathrm{MS}_0(k)$ on or below the diagonal. The applications of Definitions 1 and 2 are made to C_n', not to C_n, in what follows.

Let $\mathscr{J}_1, \mathscr{J}_2, \ldots, \mathscr{J}_k$ be the leftmost k members in $\mathscr{C}\mathscr{L}_1$, let w_i be the number of 1's in $\mathscr{J}_i$, and let $d_i = d(\mathscr{J}_i, \mathscr{J}_{i+1})$ for $i < k$. By hypothesis, $w_i \leqslant k - 1$ for each i. Moreover, it is easily seen that $\mathscr{J}_1$ begins in cell $(1, 1)$, so in fact there are at least k members of $\mathscr{C}\mathscr{L}_1$.

Each $\mathscr{J}_i$ proceeds vertically from its initial cell in row 1 until it hits the diagonal. Therefore the initial column for $\mathscr{J}_k$ is $k + \Sigma_1^{k-1} d_i$, and $(k + \Sigma d_i,\ k + \Sigma d_i)$ is the first cell for $\mathscr{J}_k$ on (or below) the diagonal. The number of rows from this point to the lower boundary of the matrix, and including row $k + \Sigma d_i$, is $k^2 + 1 - k - \Sigma d_i = R$.

For each $0 \leqslant t < R$ let H_t be the k-term sequence

$$c(k^2 - t - k + 1, x(t, 1)), c(k^2 - t - k + 2, x(t, 2)), \ldots, c(k^2 - t, x(t, k)),$$

where $x(t, i)$ is the column corresponding to row $k^2 - t - k + i$ in $\mathscr{J}_i$. These R sequences are disjoint (no cells in common) and lie on or below the diagonal. If any one of them consists entirely of 0's and soft 1's then it yields an $MS_0(k)$ on or below the diagonal. (Soft 1's on $\mathscr{I}_k$ are not used here. In the next paragraph, all 1's on $\mathscr{I}_k$ are treated as hard. Assuming that $\mathscr{I}_k$ has only hard 1's, the diagonal entry for $(k+\Sigma_1^{k-1}d_i - 1, k+\Sigma_1^{k-1}d_i - 1)$ must be 0. Hence, if $c(k^2-R, x(R-1, k-1))$ is 1, it is soft and can be used to obtain an $MS_0(k)$ on or below the diagonal.)

It follows that we get an $MS_0(k)$ on or below the diagonal unless every H_t has at least one hard 1, so that at least R hard 1's are needed in $\mathscr{I}_1$ through $\mathscr{I}_k$ to prevent such an $MS_0(k)$. Let R' be the total number of hard 1's in $\mathscr{I}_1$ through $\mathscr{I}_k$, including all 1's in $\mathscr{I}_k$. Then, by Lemma 3,

$$R' \leqslant \sum_{i=1}^{k-1} (w_i - d_i) + w_k \leqslant k(k-1) - \Sigma d_i < R.$$

Since $R' < R$, some H_t must yield an $MS_0(k)$ on or below the diagonal, and the proof of Theorem 2 is complete. □

5. Separated Cross-Monotonicity

We now use Theorem 2 to establish

LEMMA 4. $g(k+1) \leqslant (k+1)k + 2$.

Proof. Let α be any sequence in $\mathscr{A}_{(k+1)k+2}$, and let $<_o$ be a total order of the (k^2+k+1) $\Delta_i = \alpha_{i+1} - \alpha_i$ that has $\Delta_i <_o \Delta_j$ whenever $\Delta_i < \Delta_j$. For each $1 \leqslant i \leqslant j \leqslant k^2$ let

$$c(i,j) = \begin{cases} 0 & \text{if } \Delta_i <_o \Delta_{j+k+1} \\ 1 & \text{if } \Delta_{j+k+1} <_o \Delta_i. \end{cases}$$

These $c(i, j)$ give an upper-triangular $k^2 \times k^2$ (0, 1)-matrix. By $f_2(k) = k^2$ in Theorem 2, and symmetry (rotate by 180°), this matrix either has an $MS_0(k)$ of type (0) with $i \leqslant j_1$ or an $MS_1(k)$ of type (1) with $i_k \leqslant j+k-1$. (See first paragraph of Section 4.) In either case, we get a $2 \times (k+1)$ array of α_t that satisfies SCM. By subscripts of the α_t, the array is

$$\begin{array}{lllll} j_1+k+1 & j_2+k+1 & \ldots & j_k+k+1 & j_k+k+2 \\ i & i+1 & \ldots & i+k-1 & i+k \end{array}$$

for type (0), where $i \leqslant j_1$ in the matrix implies $i+k < j_1+k+1$, and

$$\begin{array}{lllll} i_1 & i_2 & \ldots & i_k & i_k+1 \\ j+k+1 & j+k+2 & \ldots & j+2k & j+2k+1 \end{array}$$

for type (1), with $i_k + 1 < j+k+1$. □

The proof for g in Theorem 1 is completed by

LEMMA 5. $g(k+1) > (k+1)k + 1$.

Proof. Let $n = (k+1)k + 1$, and consider $\alpha \in \mathscr{A}_n$ whose Δ_i are superincreasing according to the subscript order

$$<_0 = 1 \ldots k-1 \mid k+1 \ldots 2k-1 \mid 2k+1 \ldots 3k-1 \mid \ldots$$
$$\mid (k-1)k+1 \ldots k^2-1 \mid^* k^2+1 \ldots k^2+k \mid k^2\ (k-1)k \ldots 2k\ k.$$

This has k blocks of $k-1$ contiguous i's up to *, then a block of k contiguous i's, and finally a block of k decreasing indices that are multiples of k. Suppose $\alpha \in \mathrm{SCM}(k+1)$ as shown by $(a_1, \ldots, a_{k+1})$ above $(b_1, \ldots, b_{k+1})$, with either $a_{k+1} < b_1$ or $b_{k+1} < a_1$. The b_i span a subsequence of Δ_i with contiguous i's in their natural order (similarly for the a_i) and $\min\{|i-j| : \Delta_i$ is spanned by the b_t, Δ_j is spanned by the $a_t\} \geqslant 2$. We show that this supposition is false.

We note first that none of the final k Δ_i in $<_0$ can be involved with $(b_1, \ldots, b_{k+1})$. This will be shown for $i = k^2$; proofs for other multiples of k in the last block of $<_0$ are similar. If Δ_{k^2} is spanned by the b_i, then superincreasingness forces the a_i to span some Δ_j for $k^2 <_0 j$. Since all such j precede k^2 in the natural order $<$, we have $a_1 < \cdots < a_{k+1} < b_1 < \cdots < b_{k+1}$. Since all $i <_0 k^2$ are in the natural order, it follows that the only Δ_j that can be used for the a_i are those with $k^2 <_0 j$. But there are only $k-1$ such j, and since a_1 through a_{k+1} must span at least k of the Δ_i, we get a contradiction.

Hence, the b_i span no Δ_i for the last k elements in $<_0$. Because at least k contiguous Δ_i must be spanned by the b_i, the only other possibility is to use the Δ_i for $k^2 + 1 \leqslant i \leqslant k^2 + k$, so that

$$(b_1, \ldots, b_{k+1}) = (\alpha_{k^2+1}, \ldots, \alpha_{k^2+k+1}).$$

By CM and superincreasingness, this would force the a_i to span the k Δ_i for the i in the final block of $<_0$. But this is impossible since one of these is

$$\Delta_{k^2} = \alpha_{k^2+1} - \alpha_{k^2},$$

and the first of these α_i's has already been used for b_1. □

6. Monotone Sequences in Skew-Symmetric Matrices

We now return to square (0, 1)-matrices to prepare for the upper-bound proof for f. Let $\mathscr{S}_n$ denote the set of all matrices in $\mathscr{C}_n$ that are *off-diagonal skew-symmetric* in the sense that, for all $i, j \in \{1, \ldots, n\}$,

$$i \neq j \Rightarrow c(i, j) + c(j, i) = 1.$$

Also let

$$f_3(k) = \min\{n : \text{every } C_n \in \mathscr{S}_n \text{ has an } \mathrm{MS}(k)\}.$$

The rest of this section proves

THEOREM 3. $\frac{1}{2}(k^2+k+2) \leqslant f_3(k) \leqslant \frac{1}{2}(k^2+3k)$ *for all* $k \geqslant 2$.

For the lower-bound proof ($k \geqslant 2$), take $n = \frac{1}{2}(k^2+k)$ and let the diagonal of C_n begin with $k-1$ 0's, then one 1, then $k-2$ 0's, then one 1, ..., then two 0's, then one 1, and finally 010. Every row to the right of the diagonal has the diagonal element throughout; every column below the diagonal has the nondiagonal element throughout. Hence $C_n \in \mathscr{S}_n$, and it is easily seen that C_n has no MS(k). Therefore $f_3(k) > \frac{1}{2}(k^2+k)$ for $k \geqslant 2$.

Our upper-bound proof for Theorem 3 uses the notion of canonical leading monotone sequences of 1's and auxiliary ideas from Definitions 2. We shall also need the following.

DEFINITIONS 3. Given $C_n \in \mathscr{S}_n$, let $p = |\mathscr{CL}_1|$, and order the members of $\mathscr{CL}_1$ left to right as $\mathfrak{s}_1$ [begins in cell (1, 1)], $\mathfrak{s}_2, \ldots, \mathfrak{s}_p$ [ends in bottom row near lower right corner]. Let w_i be the number of 1's in $\mathfrak{s}_i$, let r_i be the number of soft 1's in $\mathfrak{s}_i$ ($i < p$), and let $d_i = d(\mathfrak{s}_i, \mathfrak{s}_{i+1})$. Also let $\mathbf{t}_i$ for $i = 1, \ldots, p$ be the *diagonal-symmetric image* of $\mathfrak{s}_i$, so that

$$\mathfrak{s}_i = [c(1, j_1), \ldots, c(n, j_n)] \Rightarrow \mathbf{t}_i = [c(j_1, 1), \ldots, c(j_n, n)].$$

(Each $\mathbf{t}_i$ is a horizontal path through C_n. The pair $(\mathfrak{s}_i, \mathbf{t}_i)$ may share more than one cell along the diagonal; when $i \neq j$, $\mathfrak{s}_i$ and $\mathbf{t}_j$ have exactly one cell in common – proof left to the reader – and this intersection cell is not on the diagonal.) For $i \neq j$, the cell that $\mathfrak{s}_i$ and $\mathbf{t}_j$ have in common is called an *$\cap$-cell.* (By skew-symmetry, if the $\cap$-cell for $\mathfrak{s}_i$ and $\mathbf{t}_j$ has a 0, then the $\cap$-cell for $\mathfrak{s}_j$ and $\mathbf{t}_i$ has a 1, and if the former has a 1 then the latter has a 0.) Soft 1's in $\cap$-cells along $\mathfrak{s}_i$ for $i < p$ are partitioned into three types:

Type A: there is a hard 1 in $\mathfrak{s}_i$ below the $\cap$-cell soft 1, and this hard 1 occurs before the next $\cap$-cell or the diagonal (whichever comes first);

Type B: given that the soft 1 is in the $\cap$-cell for $\mathfrak{s}_i$ and $\mathbf{t}_j$, there is a hard 1 in $\mathfrak{s}_j$ below the $\cap$-cell for $\mathfrak{s}_j$ and $\mathbf{t}_i$ that occurs before the next $\cap$-cell for $\mathfrak{s}_j$ or the diagonal (whichever comes first);

Type C: all others – neither type A nor type B.

In this section only, let a_i, b_i, and c_i be respectively the number of soft 1's in $\cap$-cells along $\mathfrak{s}_i$ of type A, type B, and type C for $i < p$.

The new ideas introduced in Definitions 3 are illustrated in Figure 2. The top diagram portrays a matrix in $\mathscr{S}_n$ that has $p = 4$. The middle diagrams picture type A and type B soft 1's in $\cap$-cells. Although it is not quite obvious, a soft 1 in an $\cap$-cell cannot be both of type A and type B. Otherwise, the row of 1's to the right of the hard 1 on $\mathfrak{s}_i$ below the $\cap$-cell would have to cut across $\mathbf{t}_j$ above the initial 0 on $\mathbf{t}_j$ in the vertical column of 0's that ends just above $\mathbf{t}_{j+1}$ (see middle right diagram); but this is impossible since as soon as $\mathbf{t}_j$ hits a 1 in the hard-1 row, it continues horizontally eastward along this row until it runs into $\mathfrak{s}_{j+1}$. Consequently, with y_i the total number of soft 1's in $\cap$-cells along $\mathfrak{s}_i$, we have $y_i = a_i + b_i + c_i$.

The following lemma, which is similar to Lemma 3 and is also proved in the Appendix, adheres to the notations of Definitions 3.

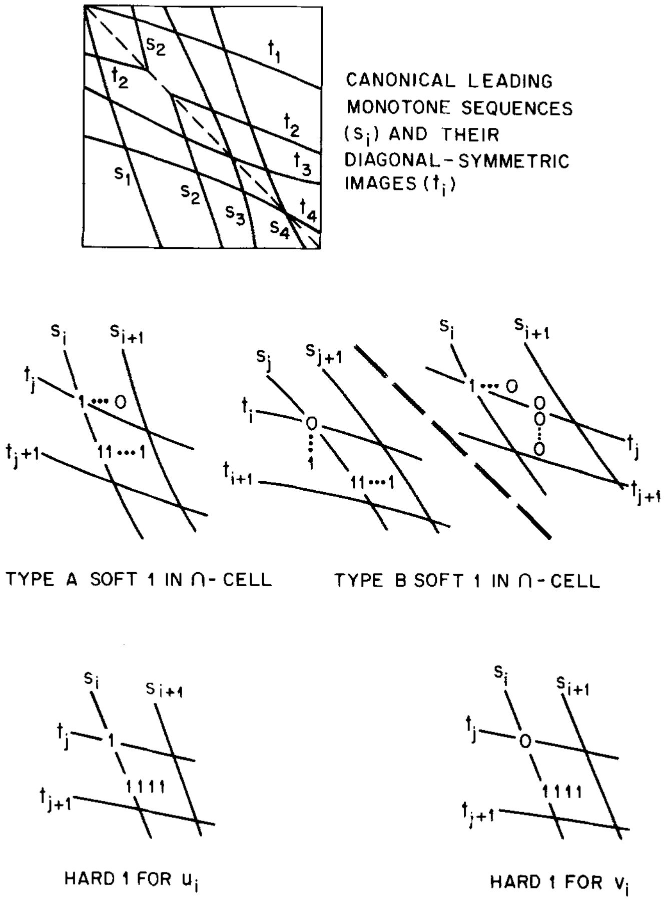

Fig. 2.

LEMMA 6. *For each* $i<p$, $r_i - c_i \geqslant d_i$.
We need one final construction and a few more definitions before completing the proof that $f_3(k) \leqslant \frac{1}{2}(k^2+3k)$. The construction step is to add a column of n 0's at the right edge of C_n, thus giving an enhanced $n \times (n+1)$ matrix. We refer to the new column as the *dummy*. Using the dummy, we obtain a $(p+1)$st canonical leading monotone sequence – call it $\mathcal{S}_{p+1}$ – that is completely to the right of $\mathcal{S}_p$ and ends in the dummy. It may begin in a column of C_n or [if $\mathcal{S}_p$ ends in cell $(n, n), \ldots$] consist entirely of the dummy.

Using $\mathcal{S}_{p+1}$, the definitions of r_p, d_p, a_p, b_p, c_p and y_p are exactly the same as the definitions of these terms for $i<p$. In particular, note that hard and soft 1's in $\mathcal{S}_p$ only involve cells in C_n since these terms are defined with respect to the row cells up to, but not including, the cells in $\mathcal{S}_{p+1}$. (We do not define $\mathbf{t}_{p+1}$.)

Moreover, Lemma 6 holds for $i=p$ since its proof is oblivious to whether or not a dummy is present.

We thus have w_i, d_i, r_i, a_i, b_i, c_i and $y_i = a_i + b_i + c_i$ defined for all $i \leqslant p$, with $r_i - c_i \geqslant d_i$ for each $i \leqslant p$. In addition, let x_i be the number of 1's in $\cap$-cells of $\mathscr{I}_i$, let z_i be the number of hard 1's in $\mathscr{I}_i$ that are *not* in $\cap$-cells, and partition z_i into two parts as follows:

u_i: the number of hard 1's in $\mathscr{I}_i$ not in $\cap$-cells whose nearest northwest $\cap$-cells (or diagonal intersections, if immediately above the hard 1) contain 1's (see lower left diagram in Figure 2);

v_i: the number of hard 1's in $\mathscr{I}_i$ not in $\cap$-cells whose nearest northwest $\cap$-cells (or diagonal intersections...) contain 0's, plus the number of non-$\cap$-cell hard 1's in $\mathscr{I}_i$ that lie above all $\cap$-cells of $\mathscr{I}_i$ (lower right in Figure 2).

It is easily seen that $u_i \geqslant a_i$ for all $i \leqslant p$, and, by reflections about the diagonal, that

$$\sum_{i=1}^{p} v_i \geqslant \sum_{i=1}^{p} b_i .$$

Moreover, for each $i \leqslant p$,

$$\begin{aligned} w_i = {} & \#(\text{1's in } \cap\text{-cells of } \mathscr{I}_i) \\ & + \#(\text{soft 1's of } \mathscr{I}_i \text{ not in } \cap\text{-cells}) \\ & + \#(\text{hard 1's of } \mathscr{I}_i \text{ not in } \cap\text{-cells}) \\ = {} & x_i + (r_i - y_i) + z_i \\ = {} & x_i + (r_i - c_i) + (u_i - a_i) + (v_i - b_i). \end{aligned}$$

Now, with $p = |\mathscr{CL}_1|$ as in Definitions 3, assume that $n = \frac{1}{2}(k^2 + 3k)$. We shall prove that C_n contains an $\mathrm{MS}_1(k)$. Suppose to the contrary that C_n has no $\mathrm{MS}_1(k)$, so that $w_i \leqslant k - 1$ for all $i \leqslant p$. Then the preceding equation summed over i gives

$$\begin{aligned} 0 & = \sum_{i=1}^{p} [w_i - x_i - (r_i - c_i) - (u_i - a_i) - (v_i - b_i)] \\ & \leqslant \sum_{i=1}^{p} [w_i - x_i - (r_i - c_i)] \\ & \leqslant p(k-1) - \frac{p(p-1)}{2} - \sum_{i=1}^{p} d_i \end{aligned}$$

by skew-symmetry for Σx_i, and Lemma 6 extended for $\Sigma(r_i - c_i)$. Moreover, $p = |\mathscr{CL}_1|$ and $w_i \leqslant k - 1$ imply that

$$\sum_{i=1}^{p} (d_i + 1) + k - 1 \geqslant n,$$

so that

$$-\Sigma d_i \leqslant -(n + 1 - k - p).$$

Substitution in the preceding inequality then gives

$$0 \leqslant p(k-1) - \frac{p(p-1)}{2} - (n+1-k-p),$$

which we rewrite as

$$n \leqslant -\frac{p^2}{2} + p(k+\tfrac{1}{2}) + k - 1.$$

We claim that this inequality is false. Its left-hand side is $n = \frac{1}{2}(k^2+3k)$. Its right-hand side is maximized at either $p = k$ or $p = k+1$, where its value is $\frac{1}{2}(k^2+3k-2)$.

Thus, with $n = \frac{1}{2}(k^2+3k)$, C_n must contain an $\mathrm{MS}_1(k)$. □

We suspect that $f_3(k)$ equals the lower-bound value in Theorem 3 for $k \geqslant 2$.

7. Regular Cross-Monotonicity

This section uses the upper bound in Theorem 3 to obtain an upper bound on $f(k) = \min\{n : \text{every } \alpha \in \mathscr{A}_n \text{ is in } \mathrm{CM}(k)\}$, then verifies the lower bound on f in Theorem 1.

Our upper bound on f in Theorem 1 will be obtained from an upper bound for a type of cross-monotonicity that is intermediate between SCM and regular CM and that is more directly related to Theorem 3 than is CM. To define the intermediate type, let $\alpha \in \mathscr{A}_n$ be in SCM*(k) if it admits a $2 \times k$ CM array

$$a_1\, a_2 \ldots a_k, \quad b_1\, b_2 \ldots b_k$$

that satisfies either SCM ($a_k \leqslant b_1$ or $b_k \leqslant a_1$) or has

$$a_i \leqslant b_1 \leqslant \cdots \leqslant b_k \leqslant a_{i+1} \quad \text{for some} \quad 1 \leqslant i \leqslant k-1.$$

Also let

$$g^*(k) = \min\{n : \mathscr{A}_n \subseteq \mathrm{SCM}^*(k)\}.$$

LEMMA 7. *Let*

$$\begin{aligned} \sigma(k) &= 2\lfloor k^{1/2} \rfloor + 1 \quad \text{if} \quad q^2 \leqslant k < q^2 + q, \\ &= 2\lfloor k^{1/2} \rfloor + 2 \quad \text{if} \quad q^2 + q \leqslant k < (q+1)^2 \end{aligned}$$

for an integer q. Then, for all $k \geqslant 1$,

$$g^*(k+1) \leqslant \tfrac{1}{2}[k + \sigma(k) + 2]\,[k + \sigma(k) + 1].$$

COROLLARY 1. $f(k+1) \leqslant \frac{1}{2}[k^2 + 4k^{3/2} + 10k + 12k^{1/2} + 9]$ *for* $k \geqslant 1$.

Proof. The Corollary follows directly from Lemma 7, $f(k+1) \leqslant g^*(k+1)$, and $\sigma(k) \leqslant 2k^{1/2} + \frac{3}{2}$. □

When $k+1$ in Corollary 1 is replaced by k, and we use the facts that $(k-1)^{3/2} \leqslant k^{3/2} - k^{1/2}$ and $(k-1)^{1/2} < k^{1/2}$, it follows that

$$f(k) \leqslant \tfrac{1}{2}[k^2 + 4k^{3/2} + 8k + 8k^{1/2}],$$

which is the upper bound on f used in Theorem 1.

Proof of Lemma 7. Given $\sigma(k)$ as defined in the lemma, let

$$n = \tfrac{1}{2}[k + \sigma(k) + 2][k + \sigma(k) + 1] - 1$$
$$= \tfrac{1}{2}([k + \sigma(k)]^2 + 3[k + \sigma(k)]).$$

By Theorem 3, every skew-symmetric (0, 1)-matrix $C_n \in \mathscr{S}_n$ has a monotone sequence of 0's, or of 1's, of at least $k + \sigma(k)$ terms.

Let α be any sequence in $\mathscr{A}_{n+1}$, and let $<_o$ be a total order of the n $\Delta_i = \alpha_{i+1} - \alpha_i$ that has $\Delta_i <_o \Delta_j$ whenever $\Delta_i < \Delta_j$. Construct $C_n \in \mathscr{S}_n$ as follows:

(i) for all $1 \leqslant i \leqslant j - (k+1) \leqslant n - (k+1)$,

$$c(i, j) = 0 \quad \text{if} \quad \Delta_i <_o \Delta_j$$
$$= 1 \quad \text{if} \quad \Delta_j <_o \Delta_i;$$

(ii) for all $1 \leqslant j \leqslant i - (k+1) \leqslant n - (k+1)$,

$$c(i, j) = 1 - c(j, i);$$

(iii) with $m = \lceil k^{1/2} \rceil$, let diagonal cell (i, i) have a 1 if i is a multiple of $(m + 1)$, and have 0 otherwise. Let the k cells immediately east of the diagonal (or $< k$ cells if the

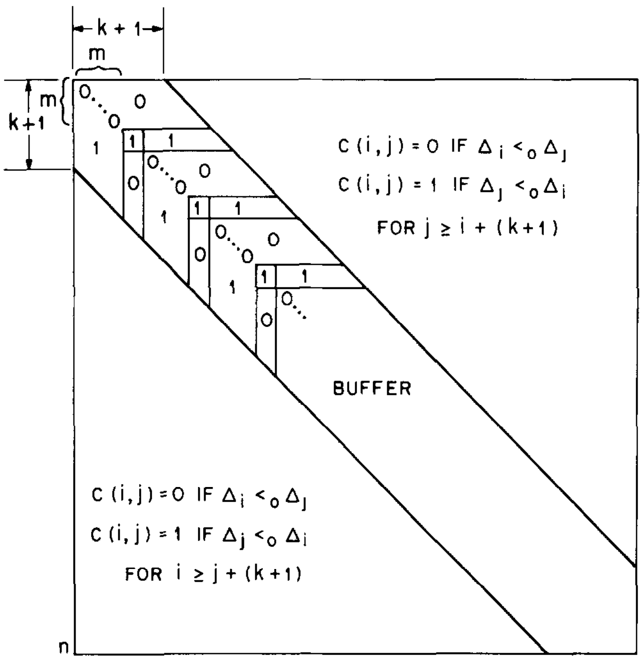

Fig. 3.

right boundary is hit first) contain the diagonal element, and let the k cells immediately south of the diagonal contain the non-diagonal element.

Figure 3 illustrates C_n. Part (iii) bears no relationship to $<_0$. It is a buffer between the upper right triangle (i) and the lower left triangle (ii) that is designed to minimize the number of 1's, or 0's, that can lie within the buffer in a monotone sequence in C_n.

It is not hard to verify that, when the diagonal through the buffer has m 0's, then one 1, then m 0's, then one 1, ..., the longest MS_1 completely *within* the buffer has $m + 1 + \lfloor k/m \rfloor$ terms, and the longest MS_0 completely *within* the buffer has a $m + \lfloor (k-1)/m \rfloor$ terms. For example, a longest MS_1 within the buffer uses m 1's for a below-diagonal block of 1's, one 1 on the diagonal, and then as many 1's as possible from the above-diagonal horizontal strips of 1's.

In particular, when $m = \lceil k^{1/2} \rceil$, straightforward calculations show that $\sigma(k) = m + 1 + \lfloor k/m \rfloor$. Consequently, it follows from skew-symmetry and the first paragraph of this proof that C_n contains an $\mathrm{MS}_1(k + \sigma(k))$, at least k terms of which are *not* in the buffer.

Consider the case in which $1 \leq x \leq k-1$ terms in an MS_1 lie above the buffer in consecutive columns, and $k - x$ lie below the buffer in consecutive columns. If there are γ columns between these two segments, then we have, say,

$$c(i_1, j) = c(i_2, j+1) = \cdots = c(i_x, j+x-1) = 1,$$
$$c(i_{x+1}, j+x+\gamma) = c(i_{x+2}, j+x+\gamma+1) = \cdots = c(i_k, j+k+\gamma-1) = 1,$$

with $i_x \leq j+x-1-(k+1)$ for the first group, and $i_{x+1} \geq j+x+\gamma+(k+1)$ for the second group. The first group (using subscripts for the α_i) gives the SCM

$$\begin{matrix} i_1 & i_2 & \ldots & i_x & i_x+1 & [i_x+1<j] \\ j & j+1 & \ldots & j+x-1 & j+x & \end{matrix}$$

and the second group gives the SCM

$$\begin{matrix} i_{x+1} & i_{x+2} & \ldots & i_k & i_k+1 & [j+k+\gamma<i_{x+1}] \\ j+x+\gamma & j+x+\gamma+1 & \ldots & j+k+\gamma-1 & j+k+\gamma. & \end{matrix}$$

Since $\gamma \geq 0$, these two fit together to yield a $2 \times (k+1)$ [or $2 \times (k+2)$] array from α which proves that $\alpha \in \mathrm{SCM}^*(k+1)$.

If all the 1's from an $\mathrm{MS}_1(k + \sigma(k))$ that are not in the buffer lie above the buffer, or if they all lie below the buffer, then an $\mathrm{SCM}(k+1)$ pattern obtains.

Hence $\alpha \in \mathrm{SCM}^*(k+1)$, and it follows that $g^*(k+1) \leq n+1$. □

Our lower-bound construction for f proceeds as follows. First, divide the α_i into left-to-right blocks of lengths $1, 2, 3, \ldots, x-1, x, \ldots, x$ with y blocks of length x so that

$$\text{number of blocks} = x - 1 + y,$$
$$n = x(x-1)/2 + xy.$$

Given $\alpha \in \mathscr{A}_n$, let d_j denote the Δ_i between the last member of block j and the first member of block $j+1$, so $d_1 = \Delta_1$, $d_2 = \Delta_3$, $d_3 = \Delta_6$, and so forth. Arrange the Δ_i in a superincreasing sequence as follows:

the Δ_i in the final block, left to right;
the Δ_i in the penultimate block, left to right;
.
.
.
Δ_2 (from the second block);
$d_1, d_2, \ldots, d_{x+y-2}$.

This is illustrated in Figure 4, where each heavy dot is an α_i and solid arrows show the direction of increase in the Δ_i.

Consider a $2 \times k$ CM array from α:

$$a_1\ a_2\ \ldots a_k, \qquad b_1\ b_2\ \ldots b_k.$$

This has $a_1 - b_1 \leqslant a_2 - b_2 \leqslant \ldots \leqslant a_k - b_k$. The $i \in \{1, \ldots, k\}$ for this array can be partitioned into four classes:

$C1 = \{i : a_i < b_i, a_i \text{ and } b_i \text{ in different blocks}\}$
$C2 = \{i : a_i < b_i, a_i \text{ and } b_i \text{ in same block}\}$
$C3 = \{i : b_i < a_i, a_i \text{ and } b_i \text{ in same block}\}$
$C4 = \{i : b_i < a_i, a_i \text{ and } b_i \text{ in different blocks}\}$.

By $a_1 - b_1 \leqslant \ldots \leqslant a_k - b_k$, $\max(C1 \cup C2) < \min(C3 \cup C4)$. Moreover, the superincreasing order of the Δ_i implies that, when the sets involved are not empty, max $C1 <$ min $C2$ and max $C3 <$ min $C4$. Therefore, the partition defining $C1$ through $C4$ is a left-to-right partition of $1, 2, \ldots, k$.

The definition of the Δ_i sequence reveals (as can be seen using Figure 4) that

$S1$. All b_i for $i \in C1$ are in the same block;
$S2$. Different b_i for $i \in C2$ are in different blocks;
$S3$. All a_i and b_i for $i \in C3$ are in the same one block;
$S4$. Different a_i for $i \in C4$ are in different blocks.

Given x and y, let $m(x, y)$ denote the largest k such that $\alpha \in \mathrm{CM}(k)$. It is easily seen that we can do no better than to use an x block for $C3$ (see $S3$), with $C1 \cup C2$ coming up to this block and $C4$ beginning immediately thereafter. There is a minor difference

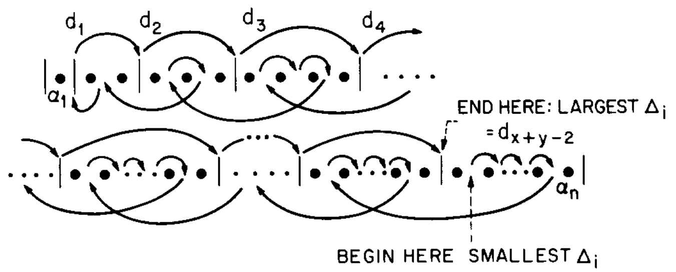

Fig. 4.

between x odd and x even, but since x odd generally gives better results we shall presume that case. Given x odd, we achieve $m(x, y)$ by using the penultimate block for $C3$ (one α_i left over from this block is used for $C4$) to get

$$m(x, y) = [x + y - 3] \text{ (by } C1 \cup C2) + \frac{x-1}{2} \text{ (by } C3) + 1 \text{ (by } C4)$$
$$= \tfrac{3}{2}x + y - \tfrac{5}{2}.$$

We use this to establish

LEMMA 8. $f(k) \geqslant (k+1)^2/4$ *for all* $k \geqslant 1$.

Proof. Let x_k be the largest odd integer strictly less than $(k+4)/2$. It is easily verified that x_k maximizes

$$n(x, y) = x(x-1)/2 + xy$$

over all odd x for which (x, y) satisfies

$$k = m(x, y) = \tfrac{3}{2}x + y - \tfrac{5}{2}.$$

Thus, subject to $m(x, y) = k$, and x odd, the largest n, say n_k, that is given by our construction is

$$n_k = x_k(x_k - 1)/2 + x_k(k + \tfrac{5}{2} - \tfrac{3}{2}x_k)$$
$$= x_k(k+2) - x_k^2.$$

Since the indicated values of x_k and $y = k + \frac{5}{2} - \frac{3}{2}x_k$ do not admit a $2 \times (k+1)$ CM array, it follows that $f(k+1) > n_k$. Moreover, simple analysis of cases ($k = 4w + j$, $j = 0, 1, 2, 3$) shows that $n_k \geqslant k^2/4 + k$ in all cases. Therefore

$$f(k+1) \geqslant n_k + 1 \geqslant k^2/4 + k + 1 = \tfrac{1}{4}(k+2)^2,$$

which proves the lemma. □

8. Discussion

As shown by Theorem 1, we know g and h precisely, but have only been able to bound $f(k)$ between about $k^2/4$ and $k^2/2$. The actual bounds on $f(4)$ given by the theorem are $7 \leqslant f(4) \leqslant 48$. The exact value is $f(4) = 11$. It is not hard to show that $n = 11$ always yields a 2×4 CM array, and the $\alpha \in \mathscr{A}_{10}$ given by

$$\alpha = (0, 0, 2, 4, 7, 14, 15, 21, 21, 21)$$

allows the conclusion that $f(4) > 10$ since it admits no 2×4 CM array.

In sum, the general case of cross-monotonicity seems exceedingly complex, and new methods appear to be needed to sharpen the present bounds on f.

Appendix

LEMMA 3. *If* $\mathscr{S}$ *immediately precedes* $\mathscr{S}'$ *in* $\mathscr{C}\mathscr{L}_1$, *then* $\mathscr{S}$ *has at least* $d(\mathscr{S},\mathscr{S}')$ *soft* 1's.

Proof. To the contrary, suppose $\mathscr{S}$ has $r < d = d(\mathscr{S},\mathscr{S}')$ soft 1's. Consider

$$\mathrm{x} = \{c(x_t, y_t) : 1 \leqslant t \leqslant w - r\},$$

where w is the number of 1's in $\mathscr{S}$,

$$\begin{aligned} x_t &= \text{row index for } t\text{th hard 1 in } \mathscr{S}, \\ y_1 &= j_1 + d = j_1' - 1 \text{ when } \mathscr{S} \text{ begins at } c(1, j_1) \text{ and} \\ &\qquad\qquad \mathscr{S}' \text{ begins at } c(1, j_1'); \\ y_t &= y_1 + t - 1 \text{ for } t = 2, \ldots, w - r. \end{aligned}$$

We claim that the cells of x are disjoint from those of $\mathscr{S}'$. If not, let $u + 1$ be the smallest t for which (x_t, y_t) is used for $\mathscr{S}'$. Then $y_{u+1} = j_v'$ for some v so that the first u terms of x and the last $w' - u + 1$ terms of $\mathscr{S}'$ that contain 1's form an $\mathrm{MS}_1(w' + 1)$ – where w' is the number of 1's in $\mathscr{S}'$. But this contradicts the construction of canonical sequences in $\mathscr{C}\mathscr{L}_1'$.

It follows that $c(x_t, y_t) = 1$ for all t. Therefore either $\mathscr{S}$ ends with $c(n, j_n) = 1$, in which case

$$y_{w-r} = y_1 + w - r - 1 = j_1 + d + w - r - 1 = j_n + d - r > j_n,$$

or $\mathscr{S}$ ends with $c(n, j_n) = 0$, whence

$$y_{w-r} = j_1 + w + d - r - 1 = j_n + d - r - 1 \geqslant j_n.$$

If equality holds in the latter case then, since $c(x_{w-r}, y_{w-r}) = 1$ and $c(n, j_n) = 0$, it must be true that $x_{w-r} < n$, and the cell immediately southeast of (x_{w-r}, y_{w-r}) is in the next column after column j_n.

Let x* be the LMS_1 that begins in column y_1. Then, as just indicated (since x* can never be to the left of x′), the last part of x* is to the right of the last part of $\mathscr{S}$. Moreover, the definition of canonical sequence and the fact that x* begins before $\mathscr{S}'$ imply that the lower part of x* is to the left of the lower part of $\mathscr{S}'$. But then x* is strictly between $\mathscr{S}$ and $\mathscr{S}'$, which contradicts the lemma's hypothesis that $\mathscr{S}$ and $\mathscr{S}'$ are adjacent members of $\mathscr{C}\mathscr{L}_1$. □

LEMMA 6. *For each* $i < p$, $r_i - c_i \geqslant d_i$.

Proof. To the contrary, suppose $\mathscr{S}_i$ has $r_i - c_i < d_i$. Consider the hard 1's in $\mathscr{S}_i$ (there are $w_i - r_i$ of these) along with the c_i soft 1's in ∩-cells of type C. Call this set of 1's (and their cell positions) S, ordered from top to bottom: $|S| = w_i - r_i + c_i$. Consider

$$\mathrm{x} = \{c(x_t, y_t) : 1 \leqslant t \leqslant w_i - r_i + c_i\},$$

where x_t is the row containing the tth member of S, $y_1 = j_1 + d_i = j_1' - 1$ [$\mathscr{S}_i$ begins at

$c(1, j_1)$, $\mathscr{I}_{i+1}$ at $c(1, j_1')$], and $y_t = y_1 + t - 1$ for $t = 2, \ldots, w_i - r_i + c_i$.

Next, let

$$\mathrm{x}' = \{c(x_t', y_t) : 1 \leqslant t \leqslant w_i - r_i + c_i\},$$

where $x_t' = x_t$ if the tth member of S is a hard 1, and, otherwise – for type C,

$$x_t' = \min\{q : q \geqslant x_t \text{ and } c(q, y_t) = 1\}.$$

Let τ be the smallest t for which there is no q as just defined for x_t' for type C, if such exists, i.e., $c(q, y_t) = 0$ for all $q \geqslant x_t$. Then, because the soft 1 in the $\cap$-cell is type C and not type B, the lower 0's in column y_t must be part of $\mathscr{I}_{i+1}$. But this contradicts the canonical nature of $\mathscr{I}_{i+1}$ since x' begins before $\mathscr{I}_{i+1}$ begins. Therefore there is no such τ and every x_t' is well defined.

Moreover, when x_t' is defined for type C by the min expression, the fact that the soft 1 in the $\cap$-cell is not type A ensures that $x_t' < x_{t+1}'$ when $t < w_i - r_i + c_i$. Consequently, $x_1' < x_2' < \cdots < x_z'$ $(z = w_i - r_i + c_i)$, so that x' is an MS_1. Since $y_1 < j_1'$ (for $\mathscr{I}_{i+1}$), x' starts to the left of $\mathscr{I}_{i+1}$ and is therefore disjoint from $\mathscr{I}_{i+1}$. In addition,

$$\begin{aligned} y_{w_i - r_i + c_i} &= y_1 + w_i - r_i + c_i - 1 = j_1 + d_i + w_i - r_i + c_i - 1 \\ &= (j_1 + w_i - 1) + d_i - r_i + c_i = j_n + d_i - (r_i - c_i) > j_n \end{aligned}$$

if $\mathscr{I}_i$ ends with $c(n, j_n) = 1$, and

$$y_{w_i - r_i + c_i} = j_n - 1 + d_i - (r_i - c_i) \geqslant j_n$$

if $\mathscr{I}_i$ ends with $c(n, j_n) = 0$. In the latter case, if equality holds then, since $c(x_z', y_z) = 1$, x_z' must be less than n, and the cell immediately southeast of (x_z', y_z) must be in a column to the right of column j_n.

It follows that if x^* is the LMS_1 that begins in column y_1, then x^* is strictly between $\mathscr{I}_i$ and $\mathscr{I}_{i+1}$. However, this would imply that there is a canonical LMS_1 between $\mathscr{I}_i$ and $\mathscr{I}_{i+1}$, a contradiction. Therefore $r_i - c_i \geqslant d_i$. □

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