Triangle-Free Graphs with Restricted Bandwidth

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ABSTRACT

The bandwidth of a graph G is the smallest integer m for which the vertices in G can be labeled $v_1, v_2, \ldots, v_n$ so that $|i-j| \le m$ whenever $\{v_i, v_j\}$ is an edge of G. Essentially, it gives the smallest edge length achievable when the vertices of G are arranged at distinct integer-valued points on the line. In this paper we study a Turán-type extremal graph problem: What is the maximum number $t(n,m)$ of edges in a triangle-free graph having n vertices and bandwidth at most m? We show that

$$.586\, nm = (2-\sqrt{2})\, nm \le t(n,m) \le \frac{5+\sqrt{3}}{11}\, nm = .612\, nm.$$

I. Introduction

Let $G = (V,E)$ be a graph having n vertices. The *bandwidth* of G is the smallest integer m for which the vertices of G can be labelled $v_1, v_2, \ldots, v_n$ so that $|i-j| \le m$ whenever $\{v_i, v_j\}$ is an edge of G. In this paper, we investigate the problem of determining the maximum number $t(n,m)$ of edges in a triangle-free graph having n vertices and bandwidth at most m. When m is small, it may be possible to determine $t(n,m)$ precisely. For example, it is easy to see that $t(n,1) = n-1$ and $t(n,2) = \lfloor 3n/2 - 5/2 \rfloor$. On the other hand, the well-known theorem of Turán gives $t(n,m) = \lfloor n^2/4 \rfloor$ when $m \ge n-1$. However, it appears to be a difficult problem to determine $t(n,m)$ for all values of the parameters so we will be concerned primarily with providing inequalities for $t(n,m)$. We will prove

* Research supported in part by NSF grants ISP-8011451 and MCS-8202172.

ISBN 0-12-114320-1

$$.586\ nm = (2-\sqrt{2})nm \le t(n,m) \le \frac{5+\sqrt{3}}{11}\ nm = .612\ nm.$$

II. Inequalities for $t(n,m)$

We begin with the following elementary result.

THEOREM 1: $\frac{1}{2}\ nm \le t(n,m) \le (\frac{3}{4} + o(1))\ nm$

PROOF: Consider the graph H containing all edges of the form $\{i,j\}$ where $|i-j| \le m$ and i and j have opposite parity. H is triangle-free and has $\frac{1}{2}\ nm$ edges, so $\frac{1}{2}\ nm \le t(n,m)$.

On the other hand, let G be any triangle-free graph having $t(n,m)$ edges. Then let d_i denote the number of edges $\{i,j\}$ where $i < j \le i + m$. Then $t(n,m) = \sum_{i=1}^{n} d_i$. For each $i=1,2,\ldots,n-m$, let G_i denote the restriction of G to the vertices $\{i,i+1,\ldots,i+m\}$. Since G_i is triangle-free, it has at most $m^2/4$ edges. However, G_i has at least $\sum_{k=0}^{m-1} d_{i+k}-k = (\sum_{k=0}^{m-1} d_{i+k})-m^2/2$ edges. Therefore,

$$m\ t(n,m) \le \sum_{i=1}^{n-m+1} \sum_{k=0}^{m-1} d_{i+k}+m^3 \le \frac{3}{4}nm^2 + \frac{1}{4}m^3$$

so $t(n,m) \le (\frac{3}{4} + o(1))nm$. □

The remainder of the paper is devoted to improving these elementary inequalities. We begin with a construction for improving the lower bound.

THEOREM 2: $(2-\sqrt{2})nm = .586nm \le t(n,m)$.

PROOF: Let c be a real number with $\frac{1}{2}<c<1$. Partition the set $\{1,2,\ldots,n\}$ into $s = n/mc$ blocks $B_1,B_2,\ldots,B_s$ each consisting of mc consecutive integers.[1] Then let G_c be the graph containing all edges of the form $\{i,j\}$ where $|i-j| \le m$ and i and j come from distinct consecutive blocks. Since G_c is bipartite, it is triangle-free. A simple computation shows that G_c has $(2 - c - \frac{1}{2c})\ nm$ edges, and that this function is maximized when $c = \sqrt{2}/2$. For this value, G_c has

1. We remark that although n/mc or mc may not be integers, such statements are always made with the implicit understanding that the graphs (and quantities) involved may have to be adjusted slightly by adding or deleting (asymptotically) trivial subgraphs (and amounts) so as to make the stated inequalities true.

$(2-\sqrt{2})nm$ edges.□

It will take a bit more work to improve the elementary upper bound $t(n,m) \le (\frac{3}{4} + O(1))nm$.

THEOREM 3: $t(n,m) \le \frac{5+\sqrt{3}}{11} nm = .612nm$.

PROOF: Let G be a triangle-free graph with $t(n,m)$ edges. Let A be the adjacency matrix of G, i.e., A is the $n \times m$ matrix where $a_{ij} = 1$ when $\{i,j\}$ is an edge, and $a_{ij} = 0$ otherwise. For convenience, we let $\bar{A}$ denote the adjacency matrix of the complement of G, i.e., $\bar{a}_{ij} = 1 - a_{ij}$. For each $i = 1,2,\ldots,n$, we let R_i denote the vector of length n formed by the entries in the i^{th} row of A. We then let $R_i \circ R_j$ denote the inner product of the vectors. Note that $R_i \circ R_j$ counts the number of paths of the following form:

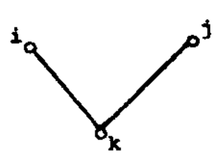

Figure 1

Next, let $S = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \ne j}}^{n} R_i \circ R_j$. In view of our previous comment on counting paths, it follows that $S = \sum_{k=1}^{n} r_k^2 - r_k$ where r_k is the degree of vertex k. Note that r_k is also the sum of the entries in R_k so that $t(n,m) = \frac{1}{2} \sum_{k=1}^{n} r_k$. We now proceed to obtain an upper bound on S.

First we note that $R_i \circ R_j = 0$ unless $|i-j| \le 2m$ and $a_{ij}=0$. We write $S = S_1 + S_2$ where

$$S_1 = \sum_{i=1}^{n} \left(\sum_{j=i-2m}^{i-m-1} R_i \circ R_j + \sum_{j=i+m+1}^{i+2m} R_i \circ R_j \right) \text{ and } S_2 = \sum_{i=1}^{n} \sum_{\substack{j=i-m \\ j \ne i}}^{i+m} R_i \circ R_j.$$

We observe that S_1 denotes the number of paths of the type shown in Figure 1 where $m < |i-j| \le 2m$. Therefore, we can provide an alternate expression for S_1.

$S_1 = 2\sum_{k=1}^{n} w_k$ where w_k denotes the number of pairs (i,j) with $k - m \le i < k < j \le k + m$, $i + m < j$, and $a_{ki} = a_{kj} = 1$. Hereafter, we refer to the positions a_{ki} and a_{kj} as being *separated* in R_k when $k - m \le i < k$, $k < j \le k + m$ and $i + m < j$. We find it helpful to view w_k as counting the number of "separated ones" in the vector R_k. We then let w_k' count the number of separated zeroes in R_k.

Now let i be fixed. If $i - m \le j < i$, then

$$\begin{aligned}
R_i \circ R_j &= \sum_{k=1}^{n} a_{ik} a_{jk} \\
&= \bar{a}_{ij} \sum_{k=i-m}^{j+m} a_{ik} a_{jk} \\
&\le \bar{a}_{ij} \sum_{k=i-m}^{j+m} a_{ik} \\
&= \bar{a}_{ij} \sum_{k=i-m}^{i+m} a_{ik} - \bar{a}_{ij} \sum_{k=j+m+1}^{i+m} a_{ik} \\
&= \bar{a}_{ij} r_i - \bar{a}_{ij} \sum_{k=j+m+1}^{i+m} (1 - \bar{a}_{ik}) \\
&= \bar{a}_{ij} r_i - \bar{a}_{ij} |i-j| + \sum_{k=j+m+1}^{i+m} \bar{a}_{ij} \bar{a}_{ik}
\end{aligned}$$

Similarly, if $i < j \le i + m$, then

$$R_i \circ R_j \le \bar{a}_{ij} r_i - \bar{a}_{ij} |i-j| + \sum_{k=i-m}^{j-m-1} \bar{a}_{ij} \bar{a}_{ik}.$$

Therefore,

$$\begin{aligned}
\sum_{\substack{j=i-m \\ j \ne i}}^{i+m} R_i \circ R_j \le (2m - r_i) r_i - \sum_{j=i-m}^{i+m} \bar{a}_{ij} |i-j| \\
+ \sum_{j=i-m}^{i-1} \sum_{k=j+m+1}^{i+m} \bar{a}_{ij} \bar{a}_{ik} \\
+ \sum_{j=i+1}^{i+m} \sum_{k=i-m}^{j-m-1} \bar{a}_{ij} \bar{a}_{ik}.
\end{aligned}$$

However, $\sum_{j=i-m}^{i-1} \sum_{k=j+m+1}^{i+m} \bar{a}_{ij} \bar{a}_{ik} = \sum_{j=i+1}^{i+m} \sum_{k=i-m}^{j-m-1} \bar{a}_{ij} \bar{a}_{ik} = w_i'$.

Thus we have

$$S_2 = \sum_{i=1}^{n}(2m-r_i)r_i - \sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\bar{a}_{ij}|i-j| + 2\sum_{i=1}^{n}w_i'.$$

It follows that

$$\begin{aligned} S &= S_1 + S_2 \\ &\le \sum_{i=1}^{n}(2m-r_i)r_i - \sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\bar{a}_{ij}|i-j| + 2\sum_{i=1}^{n}(w_i+w_i') \\ &= \sum_{i=1}^{n}\left(2mr_i - r_i^2 + E_i\right) \text{ where } E_i = 2(w_i+w_i') - \sum_{j=i-m}^{i+m}\bar{a}_{ij}|i-j| \end{aligned}$$

In order to complete our argument we need to establish an upper bound on the expression $E_i = 2(w_i+w_i') - \sum_{j=i-m}^{i+m}\bar{a}_{ij}|i-j|$ for all vectors R_i with fixed row sum r_i. The proof of the following Lemma will be given later.

LEMMA

$$E_i \le \begin{cases} -\dfrac{r_i^2}{4} & \text{if } r_i \le \left(\dfrac{2\sqrt{23}+30}{101}\right)m \\ \dfrac{39r_i^2 - 30mr_i + 4m^2}{46} & \text{if } \left(\dfrac{2\sqrt{23}+30}{101}\right)m \le r_i \le 8m/7 \\ 3mr_i - 2m^2 - 3r_i^2/4 & \text{if } r_i \ge 8m/7 \end{cases}$$

From the previous theorem, we know that $t = t(n,m) = \frac{1}{2}\sum_{i=1}^{n} r_i \ge .58nm$ so that the average value of r_i is at least $1.16m$. It follows that we should expect to use the upper bound $E_i \le 3mr_i - 2m^2 - 3r_i^2/4$ for most values of i. However, this inequality does not hold when r_i is small so we will have to correct for these terms. To accomplish this, we define $A = \{i: r_i > 8m/7\}$, $B = \{j: (2\sqrt{23}+30)m/101 < r_j \le 8m/7\}$ and $C = \{k: r_k \le (2\sqrt{23}+30)m/101\}$. We then let $a = |A|$, $b = |B|$, and $c = |C|$. Note that $a + b + c = n$.

Now we return to the inequality $S \le \sum_{i=1}^{n}\left(2mr_i - r_i^2 + E_i.\right)$ We substitute $S = \sum_{i=1}^{n}\left(r_i^2 - r_i\right)$ replace the expression E_i by the upper bounds. The resulting inequality is

$$\sum_{i=1}^{n} r_i^2 - r_i \le \sum_{i=1}^{n}(2mr_i - r_i^2 + 3mr_i - 2m^2 - 3r_i^2/4)$$
$$+ \sum_{j\in B}((39r_j^2 - 30mr_j + 4m^2)/46 - (3mr_j - 2m^2 - 3r_j^2/4))$$
$$+ \sum_{k\in C}((-r_k^2/4) - (3mr_k - 2m^2 - 3r_k^2/4)).$$

Simplifying, we obtain

$$\frac{11}{4}\sum_{i=1}^{n} r_i^2 - \sum_{j\in B}(147r_j^2 - 336mr_j + 192m^2)/92$$
$$- \sum_{k\in C}\left[\frac{1}{2}r_k^2 - 3mr_k + 2m^2\right] \le \sum_{i=1}^{n}((5m+1)r_i - 2m^2).$$

At this point, we need to establish the following inequality for real nonnegative values r_i satisfying $\sum_{i=1}^{n} r_i$ = constant.

$$\frac{11}{4n}\left[\sum_{i=1}^{n} r_i\right]^2 \le \frac{11}{4}\sum_{i=1}^{n} r_i^2 - \sum_{j\in B}(147r_j^2 - 336mr_j + 192m^2)/92$$
$$- \sum_{k\in C}\left[\frac{1}{2}r_k^2 - 3mr_k + 2m^2\right] \qquad (*)$$

The standard method for proving inequalities of this type is to use a "local exchange" to uniformize the r_i's. In order to establish (*), we first show that it suffices to establish the inequality when $r_j = 8m/7$ for all $j \in B$ and $r_k = (2\sqrt{23}+30)m/101$ for all $k \in C$. To see that this statement is valid, let F denote the expression on the right hand side of (*). Then suppose that $r_k < (2\sqrt{23}+30)m/101$ for some $k \in C$. Next choose $i \in A$ with r_i as large as possible. We know that $r_i \ge 1.16m$. Then let F' denote the value of the expression on the right hand side of (*) when we decrease r_j by ϵ and increase r_k by ϵ where ϵ is a positive value. A simple calculation shows

$$F - F' > \epsilon\left[\frac{11}{2}r_i - \frac{9}{2}r_k - 3m\right] > 0$$

Thus if (*) holds for F', it also holds for F. This exchange can be applied as many times as required to increase each r_k to $(2\sqrt{23}+30)m/101$ where $k \in C$.

Next, we show that it suffices to prove that (*) is valid when C is empty. For suppose C is nonempty but that $r_k = (2\sqrt{23}+30)m/101$ for every $k \in C$. We then let F denote the value of the expression on the right hand side of (*). We then choose $i \in A$ with r_i as large as possible and let F' be the value of the right hand side of (*) when we decrease r_i by ϵ and increase r_k by ϵ. Note that this moves k from C to B. It can be shown that $F - F' = \left[\frac{11}{2}r_i - \frac{53}{23}r_k - \frac{84}{23}m\right]\epsilon$ which is positive. We may then apply this exchange as many times as is required to remove all elements of C.

Under the assumption that $C = \phi$, we may then show that it suffices to show that (*) is valid when $r_j = 8m/7$ for every $j \in B$. To see that this is true we observe that decreasing r_i by ϵ when $i \in A$ and increasing r_j by ϵ when $j \in B$ and $r_j < 8m/7$ produces a net change of $\left[\frac{11}{2}r_i - \frac{53}{23}r_k - \frac{84}{23}m\right]\epsilon$ which is positive.

So we have reduced the problem of establishing (*) in the general case to the problem of proving (*) is valid when $C = \phi$ and $r_j = 8m/7$ for every $j \in B$. But when $r_j = 8m/7$, the term $2r_j^2 - 4mr_j + 2m^2$ is zero, so in this case, the inequality reduces to

$$\frac{11}{4n}\left[\sum_{i=1}^{n} r_i\right]^2 \leq \frac{11}{4}\sum_{i=1}^{n} r_i^2.$$

This is now the well-known Cauchy-Schwarz inequality. With this observation, our proof that the inequality (*) is valid is complete. We may conclude that

$$\frac{11}{4n}\left[\sum_{i=1}^{n} r_i\right]^2 \leq \sum_{i=1}^{n} (5m+1)r_i - 2m^2$$

We can then solve this quadratic inequality to obtain $\sum_{i=1}^{n} r_i \leq \frac{20 + 4\sqrt{3}}{22}nm$ and thus

$$t(n,m) = \frac{1}{2}\sum_{i=1}^{n} r_i \leq \frac{5+\sqrt{3}}{22}nm = .612nm.$$

It remains to prove the Lemma.

LEMMA

$$E(R_i) = E_i \le \begin{cases} -\dfrac{r_i^2}{4} & \text{if } r_i \le \left(\dfrac{2\sqrt{23}+30}{101}\right) m \\ \dfrac{39r_i^2 - 30mr_i + 4m^2}{46} & \text{if } \left(\dfrac{2\sqrt{23}+30}{101}\right) m \le r_i \le 8m/7 \\ 3mr_i - 2m^2 - 3r_i^2/4 & \text{if } r_i \ge 8m/7 \end{cases}$$

for a row vector R_i with row sum r_i.

PROOF: The proof here involves some calculation which is done by the symbolic computation system VAXIMA. The details of manipulations are sometimes omitted.

Let R_i denote a vector with r_i 1's such that (a) E_i is maximized; (b) $\sum_{j=i-m}^{i+m} \bar{a}_{ij}|i-j|$ is as small as possible among all R_i satisfying (a).

We will prove a sequence of facts on R_i from which the Lemma will then follow.

CLAIM 1: If $a_{ij} = 1$ and $a_{ik} = 0$ for $i < j < k \le i+m$, then we have $3(k-j) < 4 \sum_{l=j-m}^{k-m-1} \bar{a}_{il}$

PROOF: Let R_i' be the vector obtained from R_i by exchanging the values of the entries a_{ij} and a_{ik} in R_i. Since $\sum_{j=i-m}^{i+m} \bar{a}_{ij}|i-j| = (k-j) + \sum_{j=i-m}^{i+m} \bar{a}'_{ij}|i-j| > \sum_{j=i-m}^{i+m} \bar{a}'_{ij}|i-j|$, we have $E(R_i) > E(R_i')$, where a_{ij}' denotes the jth entry of R_i'. This implies

$$\begin{aligned} E(R_i) > E(R_i') &= E(R_i) + (k-j) + 2\sum_{l=j-m}^{k-m-1} (a_{il} - \bar{a}_{il}) \\ &= E(R_i) + 3(k-j) - 4\sum_{l=j-m}^{k-m-1} \bar{a}_{il} \end{aligned}$$

Therefore

$$3(k-j) < 4\sum_{l=j-m}^{k-m-1} \bar{a}_{il}$$

CLAIM 2: If $a_{ij} = 0$ and $a_{ik} = 1$ for $i < j < k \le i+m$, then we have $3(k-j) \ge 4 \sum_{l=j-m}^{k-m-1} \bar{a}_{il}$.

PROOF: Let R_i' be the vector obtained from R_i by exchanging the values of the entries a_{ij} and a_{ik} in R_i.

Since $\sum_{j=i-m}^{i+m} \bar{a}_{ij}|i-j| < \sum_{j=i-m}^{i+m} \bar{a}_{ij}'|i-j|$, we have $E(R_i) \geq E(R_i')$, which implies then

$$3(k-j) \geq 4\sum_{l=j-m}^{k-m-1} \bar{a}_{lk}.$$

Claims 3 and 4 can be proved in a similar way.

CLAIM 3: If $a_{ij} = 0$ and $a_{ik} = 1$ for $i - m \leq j < k < i$, then we have $3(k-j) < 4 \sum_{l=j+m+1}^{k+m} \bar{a}_{il}$.

CLAIM 4: If $a_{ij} = 1$ and $a_{ik} = 0$ for $i - m \leq j < k < i$, then we have $3(k-j) \geq 4 \sum_{l=j+m+1}^{k+m} \bar{a}_{il}$.

Now we partition R_i into "blocks" $B_1, B_2, \ldots, B_s$ where a block is a set of consecutive entries a_{il}, $j \leq l < k$, all of value 0 or all of value 1. R_i is chosen so that the number of (maximal) blocks in $\{i-m,..,i+m\} - \{i\}$ is minimized among all R_i satisfying (a) and (b) (see p. 182).

CLAIM 5: Suppose B is a block consisting of entries a_{il}, $i \leq l < k$ for fixed $i < j < k \leq i+m+l$ and all a_{il} are 0's. Then we have

$$0 = a_{i,j-m-1} = a_{i,j-m} = \cdots = a_{i,j-m+s-1} \text{ for some } s, \qquad \text{(i)}$$

$$\text{and } 1 = a_{i,j-m+s} = \cdots = a_{i,k-m-1};$$

$$\text{if } i+1 < j < k \leq i+m \text{ then } s = \frac{3(k-j)}{4.} \qquad \text{(ii)}$$

PROOF: (i) is an immediate consequence of Claim 1. We only have to prove (ii).

Since $a_{i,j-1} = 1$ and $a_{i,k} - 1 = 0$ by Claim 1 we have $3(k-j) < 4 \sum_{l=j-m-1}^{k-m-2} \bar{a}_{il}$. Since $a_{i,j} = 0$ and $a_{i,k} = 1$, by Claim 2 we have

$$3(k-j) \geq 4\sum_{l=j-m}^{k-m-1} \bar{a}_{il}.$$

This implies $\bar{a}_{i,j-m-1} = 0$ and $\bar{a}_{i,j-m-1} = 1$ and

$$3(k-j) = 4\sum_{l=j-m}^{k-m-2} \bar{a}_{il}.$$

Suppose $a_{i,j'} = 1$ and $a_{i,k'} = 0$ for $k-m-1 \leq j' < k' < j-m-1$. Then by Claim 3 we have

$$3(k'-j') \le 4 \sum_{l=j'+m+1}^{k'+m} \bar{a}_{il} = 0$$

which is impossible. Thus we have

$$0 = a_{i,j-m-1} = a_{i,j-m} = \dots a_{i,j+m+s-1} \text{ where } s = \frac{3}{4}(k-j)$$

$$1 = a_{i,j+m+s} = a_{i,j+m+s+1} = \cdots a_{i,j-m-1}$$

Similarly we can prove

CLAIM 6: Suppose B is a block consisting of entries a_{il}, $i < j \le l < k$, for fixed $j < k$. If all a_{il} are $1's$, then for some s we have

$$1 = a_{i,j-m-1} = a_{i,j-m} = \dots = a_{i,j-m+s-1},$$

$$0 = a_{i,j-m+s-1} = a_{i,k-m-1}.$$

If $i+1 < j < k \le i+m$, then $s = \dfrac{k-j}{4}$.

CLAIM 7: Suppose B is a block consisting of entries a_{il}, $j < l \le k$, for fixed $j < k \le i$. If all a_{il} are 0's, then for some k we have

$$1 = a_{i,j+m+1} = \dots = a_{i,j+m+s}$$

$$0 = a_{i,j+m+s+1} = \dots = a_{i,k+m+1}$$

If $i-m \le j < k < i$, then we have $s = \dfrac{(k-j)}{4}$.

CLAIM 8: Suppose B is a block consisting of entries a_{il}, $j < l \le k$, for fixed $j < k \le i$. If all a_{il} are 1's then for some s we have

$$0 = a_{i,j+m+1} = \dots = a_{i,j+m+s+1},$$

$$1 = a_{i,j+m+s+1+} = \cdots = a_{i,k+m+1}.$$

If $i-m \le j < k < i$, then $s = \dfrac{3(k-j)}{4}$.

Now suppose the entries $\{a_{il}: i<l\le i+m\}$ are partitioned into blocks $B_1,\dots,B_r$ and entries $\{a_{il}: i-m\le l<i\}$ are partitioned into blocks $B_1', \dots, B_{r'}'$. Let ϵ_l be 0 if B_l consists only of 0's and be 1 otherwise. ϵ_l' is the corresponding value for B_l'. Then the sequence $(\epsilon_1', \epsilon_2', \dots, \epsilon_{r'}'; \epsilon_1, \epsilon_2, \dots, \epsilon_r)$ is called the block pattern of R_i. $B_1, B_r, B_1', B_{r'}'$ are called boundary blocks and the rest are interior blocks.

CLAIM 9: All interior 0-blocks are of the same size t. All interior blocks are of the size $3t$.

PROOF: Suppose B_u is an interior 1-block consisting of entries $\{a_{il}: j \le l < k\}$. The entries $\bar{B}_u = \{a_{i,l-m}: a_{il} \in B_u\}$ are in a 1-block B_w' and a 0-block B'_{w+1}. Then $3|B_w' \cap \bar{B}_u| = |B'_{w+1} \cap \bar{B}_u|$. If B_u is an interior 0-block and the entries in $\bar{B}_u$ can be partitioned into a 0-block B_w' and a 1-block B'_{w+1}, then

$$|B_w' \cap \bar{B}_u| = 3|B'_{w+1} \cap \bar{B}_u|.$$

Now suppose B_u is an interior 1-block with $\bar{B}_u \subseteq B_w' \cup B'_{w+1}$. If B'_{w+1} is an interior block, then we have

$$|B'_{w+1} \cap \bar{B}_u| = \frac{3}{4}|B_u|$$

$$|B'_{w+1} \cap \bar{B}_{u+1}| = \frac{3}{4}|B_{w+1}|$$

Since B'_{w+1} is also an interior block, we have

$$|B'_{w+1} \cap \bar{B}_u| = \frac{1}{4}|B'_{w+1}|$$

$$|B'_{w+1} \cap \bar{B}_{u+1}| = \frac{3}{4}|B'_{w+1}|$$

Therefore $3|B_u| = |B_{u+1}| = |B'_{w+1}|$ and Claim 9 will follow.

CLAIM 10: Suppose B_u is interior in $\{i,\ldots,i+m\}$. If $\bar{B}_u \subseteq B_w' \cup B'_{w+1}$, then B_w' is not interior.

PROOF: Suppose $B_u = \{a_{il}: j \le l < k\}$ and $B_w' = \{a_{i,l}: j' \le l < k'\}$ are interior. We now consider R_i' obtained from R_i by exchanging the entries $a_{i,j-t}$ with $a_{i,k-t}$ and the entries $a_{i,j'-t}$ with $a_{i,t}$ for $1 \le t \le p$ where B'_{w-1} consists of $\{a_{i,l}: j'' - p \le l < j''\}$. It is easy to verify that $E(R_i') \ge E(R_i)$ and the number of blocks in R_i' is fewer than that of R_i, which is a contradiction.

It follows from Claim 10 and the symmetric version of Claim 10 that

CLAIM 11: $r \le 3$ and $r' \le 3$. If $r=3$ and $r'=3$, then there is one interior 0-block and one interior 1-block.

CLAIM 12: One of r and r' is less than 3.

PROOF: If $r = r' = 3$, we may assume without loss of generality that the block pattern is $(0,1,0{:}1,0,1)$ and $\bar{B}_2 \subset B_1' \cup B_2'$. Now we consider R_i' with entries a'_{il} such that

$$a'_{i,l} = a_{i,l-1} \text{ if } l \neq i+1 \text{ and } l \neq i-m;\ a'_{i,i+1} = a_{i,i+m};$$

and $a'_{i,i-m} = a_{i,i-1}$. It can be easily checked that $E(R_i') > E(R_i)$, which is impossible.

From now on, we may assume $r \le 2$. From Claims 5, 6 and 11, we conclude that the block pattern for R_i is one of the following

(i) (0,1:1,0)

(ii) (1,0,1:1,0)

(iii) (0,1,0:0,1)

(iv) (1,0:0,1)

CLAIM 13: If the block pattern in (0,1:1,0), then $E(R_i) \le -\frac{r_i^2}{4}$.

PROOF: If $||B_1| - |B_2'|| \ge 2$, we can adjust the size of B_1, B_2' and obtain an R_i' with larger $E(R_i')$ value, which is impossible. We may then assume $||B_1| - |B_2'|| \le 1$. By straightforward calculation we have $E(R_i) \le -\frac{r_i^2}{4}$.

CLAIM 14: If the block pattern is (1,0:0,1), then $E(R_i) \le 3mr_i - 2m^2 - 3r_i^2/4$.

PROOF: It can be similarly shown that $||B_i'| - |B_2|| \le 1$. Thus we can calculate $E(R_i)$ and obtain

$$E(R_i) \le 3mr_i - 2m^2 - 3r_i^2/4.$$

The remaining two cases are more complicated.

CLAIM 15: If the block pattern is (1,0,1:1,0), then $r_i \le m/2$ and $E(R_i) \le \frac{39r_i^2 - 30mr_i + 7m^2}{98}$ for $r_i \ge \frac{31}{79}m$ and $E(R_i) \le \frac{17r_i^2 - 10mr + 2m^2}{18}$ otherwise.

PROOF: In this case the pattern is (1,0,1:1,0). Let b_i denote $|B_i'|$. Then we have,

$$2b_1 + b_3 + \frac{b_2}{4} = r_i \text{ (by Claim 7)}$$

$$b_1 + b_2 + b_3 = m$$

Therefore $b_1 = \frac{3}{4}b_2 + r_i - m \ge 0$ and $b_3 = 2m - r_i - \frac{7}{4}b_2 \ge 0$. This implies $\frac{4(m-r_i)}{3} \le b_2 \le \frac{4}{7}(2m-r_i)$. Now by straightforward calculation $E(R_i) = -\frac{16m^2 - 32b_2b_3 - 48b_1^2 - 24b_1b_2 - 15b_2^2}{32}$. Substituting for b_1 and b_3 and setting $b = b_2$, we obtain

$$E(R_i) = f(b) = \frac{12r_i^2 + (16b - 24m)r_i + 8m^2 - 8bm + b^2}{8}$$

Since $\frac{d^2f}{db^2} = \frac{1}{4} > 0$, there is no interior maximum. Using Claim 1, we have $b_1 \le \frac{1}{4}|B_1| = \frac{1}{4}(b_1 + \frac{b}{4})$ and $b_3 \le \frac{1}{4}|B_2| = \frac{1}{4}(\frac{3}{4}b+b_3)$. Thus $b_1 \le \frac{1}{12}b$ and $b_3 \le \frac{b}{4}$. This implies $\frac{2m-r_i}{2} \le b \le \frac{(3m-r_i)}{2}$ (It can be easily shown that in this case $r_i \le \frac{m}{2}$).

We then have

$$E(R_i) \le \begin{cases} f\left[\frac{(3m-r_i)}{2}\right] = \frac{39r_i^2 - 30mr_i + 7m^2}{48} & \text{if } r_i \ge \frac{31}{79}m \\ f\left[\frac{4}{3}(m-r_i)\right] = \frac{-17r_i^2 + 10mr_2 - 2m^2}{18} & \text{otherwise} \end{cases}$$

CLAIM 16: If the block pattern is (0,1,0:0,1), then $r_i \le \frac{5m}{4}$,

$$E(R_i) \le \frac{39r_i^2 - 30mr_i + 4m^2}{46} \text{ if } \frac{8}{7}m \ge r_i \text{ and}$$

$$E(R_i) \le \frac{75r_i^2 + 174mr_i - 100m^2}{2} \text{ if } r_i < \frac{8}{7}m.$$

PROOF: Let b_i denote $|B_i'|$ for $i=1,2,3$. Then we have

$$b_2 + \frac{b_2}{4} + b_3 = r_i$$

$$b_1 + b_2 + b_3 = m$$

Therefore

$$b_3 = r_i - \frac{5}{4}b_2 \ge 0$$

$$b_1 = m - r_i + \frac{b_2}{4} \ge 0$$

This implies $4(r_i - m) \le b = b_2 \le \frac{4}{5}r_i$ and $r_i \le \frac{5}{4}m$. Also

$$b_3 \le \frac{3}{4}|B_2| = \frac{3}{4}\left[\frac{1}{4}b+b_3\right]$$

and

$$b_1 \le \frac{3}{4}\left[b_1+\frac{3}{4}b\right]$$

i.e.

$$b_3 \le \frac{3}{4}b\ ,\ b_1 \le \frac{9}{4}b$$

$$r_i - \frac{5}{4}b \le \frac{3}{4}b\ ,\ m - r_i + \frac{b}{4} \le \frac{9}{4}b$$

Therefore $b \ge \frac{r_i}{2}$, $b \ge \frac{m-r_i}{2}$.

By straightforward computation we have

$$E(R_i) \le \frac{4r_i^2 + 16br_i - 8mr_i + 8bm - 23b^2}{8} = f(b)$$

Since $\frac{d^2f}{db^2} < 0$, the maximum value of $f(b)$ is achieved at $\frac{df}{db}(b) = 0$, i.e., $b = \frac{8r + 4m}{23}$. We have

$$E(R_i) \le f\left[\frac{8r+4m}{23}\right] = \frac{39r_i^2 - 30mr_i + 4m^2}{46} \text{ if } \frac{8}{7}m \ge r_i \text{ and}$$

$$E(R_i) \le f(4(r_i-m)) = \frac{75r_i^2 + 174mr_i - 100m^2}{2} \text{ if } \frac{5}{4}m \ge r_i > \frac{8}{7}m.$$

Now we can summarize the following

$$E(R_i) \le \begin{cases} -\frac{r_i^2}{4} & \text{if } r_i \le \left(\frac{2\sqrt{23}+30}{101}\right)m \\ \frac{39r_i^2 - 30mr_i + 4m^2}{46} & \text{if } \left(\frac{2\sqrt{23}+30}{101}\right)m \le r_i \le 8m/7 \\ 3mr_i - 2m^2 - 3r_i^2/4 & \text{if } r_i \ge 8m/7 \end{cases}$$

This completes the proof of our theorem.

III. Related problems

The function $t(n,m)$ we studied in this paper came up in connection with the following problem. For an integer $n \geq 2$, let $T(n)$ denote the directed graph whose vertex set is $\{1,2,\ldots,n\}$ and whose edge set is $\{(i,j): 1 \leq i < j \leq n\}$. Recall that a directed graph is said to be *unipathic* when it contains at most one directed path between any two vertices. We then let $u(n,m)$ denote the maximum number of edges in a subgraph of $T(n)$ so that the restriction to any m consecutive vertices is unipathic. Since a unipathic directed graph is triangle-free, it follows that $u(n,m) \leq t(n,m-1)+\binom{n-m+1}{2}$. Unlike the situation with $t(n,m)$, it is possible to provide an explicit formula for $u(n,m)$.

This formula was provided by Maurer, Rabinovitch, and Trotter [2] who showed that the problem of determining $u(n,m)$ was equivalent to determining the rank of a certain class of partially ordered sets.

THEOREM 4: Let n and m be integers with $n \geq m \geq 2$. Then $u(n,m) = \binom{q}{2}(m-1)^2+q(m-1)r+\lfloor\frac{r^2}{4}\rfloor$ where $n = q(m-1)+r$ and $\lceil\frac{1}{2}(m-1)\rceil \leq r < \lceil\frac{3}{2}(m-1)\rceil$. □

We refer the reader to [2] for the argument for this theorem and to [1] for a discussion of the combinatorial theory of rank for partially ordered sets.

Here we studied a Turán type problem with restrictions imposed on the bandwidth. There are numerous variations that can be formulated in the following framework.

Let **H** denote a family of graphs and **F** denote another family of graphs. Define $t(n,\mathbf{H},\mathbf{F})$ to be the maximum number of edges in a graph G in **F** on n vertices which does not contain any graph in **H**. If we take **H** to contain only one graph K_s and **F** to be the family of all graphs, then $t(n,\mathbf{H},\mathbf{F})$ is just the Turán number. The problem of determining $t(n,\mathbf{H},\mathbf{F})$ is of interest for various families **H** and **F**. Here we mention a few:

(i) $\mathbf{H} = \{K_s\}$ and $\mathbf{F} = \{$all graphs with bandwidth $\leq m\}$;

(ii) $\mathbf{H} = \{$all graphs with chromatic number $k\}$ and $\mathbf{F} = \{$all graphs with bandwidth $m\}$;

(iii) $\mathbf{H} = \{$all graphs with chromatic number $k\}$ and $\mathbf{F} = \{$all graphs with chromatic number $l\}$,

(iv) $\mathbf{H} = \{$cycles of length $\leq k\}$ and $\mathbf{F} = \{$all graphs with chromatic number $k\}$.

In this paper we studied the case that **H** consists of one graph K_3

and $\mathbf{F} = B_m$ is the set of all graphs with bandwidth $\leq m$. We proved that

$$(2-\sqrt{2})nm \leq t(n,\{K_3\},B_m) \leq \frac{5+\sqrt{3}}{11}nm$$

Probably the lower bound is closer to the "truth" here. However, the likely candidate for an extremal graph (see Theorem 2) does not have constant degree and cannot achieve the upper bound obtained here by using averaging arguments. Some new idea is need to close up the gap.

References

1. S. B. Maurer, I. Rabinovitch, and W. T. Trotter, Jr., "Large minimal realizers of partial order, II", Discrete Math. 31 (1980) 297-313.

2. S. B. Maurer, I. Rabinovitch, and W. T. Trotter, Jr., "A Generalization of Turán's Theorem to Directed Graphs", Discrete Math. 32 (1980) 167-189.