70

A Survey of Bounds for Classical Ramsey Numbers

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ABSTRACT

This paper is a survey of the methods used for determining exact values and bounds for the classical Ramsey numbers in the case that the sets being colored are two-element sets. Results concerning the asymptotic behavior of the Ramsey functions $R(k,l)$ and $R_m(k)$ are also given.

1. INTRODUCTION AND NOTATION

In this paper, we give the main methods which have been used to calculate bounds and exact values for classical Ramsey numbers. We shall concentrate our attention on the case where the sets being colored are the edges of K_n, the complete graph on n vertices, since very little is known about the other cases.

We define $R(k_1, k_2, \ldots, k_m)$ to be the smallest integer n such that no matter how the edges of K_n are colored with m colors, there exists some i such that there is a complete subgraph of size k_i, all of whose edges are of color i. In the case $k_1 = k_2 = \cdots = k_m = k$, we will write $R_m(k)$ instead of $R(k_1, k_2, \ldots, k_m)$. Ramsey's theorem [39] states that all of these numbers exist.

2. TWO COLORS

In most cases, the problem of determining a Ramsey number splits into two problems, these being the problems of finding upper and lower bounds for the Ramsey number. A new Ramsey number is found in the unlikely event that the upper and lower bounds can be shown to be equal. Table I gives all

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TABLE 1. Known Exact Values and Bounds for $R(k,l)$ with References Given as Subscripts.

k \ l	3	4	5	6	7	8	9	10
3	6_{23}	9_{23}	14_{23}	18_{31}	23_{22}	28_{25}–29_{25}	36_{25}	39_{31}–44_{41}
4		18_{23}	25_{32}–28_{42}	34_{31}–44_{42}				
5			42_{27}–55_{42}	57_{28}–94_{42}				
6				102_{32}–169_{19}				
7					126_{29}–586_{19}			

known values for $R(k,l)$, where $3 \leq k \leq l$, together with the best known upper and lower bounds for a few of the other Ramsey numbers. Obviously, $R(k,l) = R(l,k)$, $R(1,l) = 1$, and $R(2,l) = l$.

In the case of two colors, a coloring of the edges of K_n is usually thought of as a subgraph of K_n, where the edges of the first color represent the edges of the subgraph, and those of the second color represent the nonedges. We define a (k,l,n,e)-graph to be a graph on n vertices which has e edges and which has no complete subgraphs on k vertices and no independent subgraphs of l vertices. If e is unspecified, the graph will be called a (k,l,n)-graph, and if n is also unspecified, then the graph will be called a (k,l)-graph. It is easy to see that $R(k,l) = n$ if and only if the largest (k,l)-graph has $n-1$ vertices.

Let G be a (k,l)-graph, and let v be any vertex in G. It is easy to check that the subgraphs of G generated by the neighbors and non-neighbors of v are $(k-1,l)$- and $(k,l-1)$-graphs, respectively. We denote these two subgraphs by $H_1(v)$ and $H_2(v)$, respectively. In such a partition of G, the vertex v is said to be *preferred*. This idea immediately gives the following recursive upper bound for $R(k,l)$:

$$R(k,l) \leq R(k-1,l) + R(k,l-1) \tag{1.1}$$

A close examination reveals a slight improvement. Suppose there is a (k,l)-graph G with $R(k-1,l) + R(k,l-1) - 1$ vertices. Then each vertex v of G must have exactly $R(k-1,l) - 1$ neighbors. Hence the number of edges of G is $(R(k-1,l) + R(k,l-1) - 1)(R(k-1,l) - 1)/2$, which must be an integer. This is impossible if both $R(k-1,l)$ and $R(k,l-1)$ are even. Thus, in this case, we have strict inequality in (1.1).

A method which generally gives much better upper bounds when $k = 3$ was developed by Graver and Yackel [22]. In a $(3,l)$- graph G, if a vertex v is preferred, then $H_1(v)$ is an independent set; hence every vertex has degree at most $(l-1)$. Furthermore, each edge of G is either in $H_2(v)$ or else is adjacent to exactly one neighbor of v. These ideas prompt the following definitions.

Definition 1. In a graph G, let $\|G\|$ be the number of edges of G. Also if v is any vertex of G, let $d(v)$ denote the degree of v, and let $Z(v)$ denote the sum of the degrees of the neighbors of v.

Thus, in any $(3,l)$-graph G with preferred vertex v, we have $\|G\| = Z(v) + \|H_2(v)\|$. Since $H_2(v)$ is a $(3,l-1)$-graph, if we knew a lower bound on the number of edges in such graphs, then we would have a lower bound on the difference $\|G\| - Z(v)$.

Definition 2. Let $e(k,l,n)$ be the least number of edges in any (k,l,n)-graph.

Definition 3. Let v be a vertex of degree d in a (k,l,n)-graph G. If $\|H_2(v)\| = e(k,l-1,n-d-1)$, then v is said to be a *full* vertex.

Definition 4. In a $(3,l)$-graph, let $d_i = l-1-i$, and let m_i be the number of vertices of degree d_i.

Since in a $(3,l)$-graph, no vertex can have degree greater than $l-1$, the subscript on the term d_i denotes "how far" a vertex of degree d_i is from maximum degree.

Theorem 2.1 [22]. Let G be a $(3,l,n,e)$-graph. Let

$$\Delta = ne - \sum_{i\geq 0} \{e(3,l-1,n-d_i-1) + d_i^2\}m_i.$$

Then $\Delta \geq 0$, and there are at least $n - \Delta$ full vertices in G.

We now give an example to show how this theorem can be used.

Theorem 2.2. If G is a (3,6,16,32)-graph, then either G has a full vertex or G is regular of degree 4.

Proof. It can easily be shown that G has no 2-vertices (vertices of degree 2), and so in order to apply the Theorem 2.1, we need to know the values of $e(3,5,n)$, where $n = 10,11,12$. These have been calculated (see [22]), and are 10,15, and 20, respectively. Applying the theorem we obtain:

$$\Delta = 512 - 35n_0 - 31n_1 - 29n_2 \geq 0.$$

We also know that

$$64 = 5m_0 + 4m_1 + 3m_2, \quad \text{and}$$
$$16 = m_0 + m_1 + m_2.$$

These equations are obtained by counting edges and vertices. If we solve this system, we obtain $m_0 = m_2$, and $\Delta = 16 - 2m_2$. So either $m_0 = m_2 = 0$, in which case G is regular of degree 4, or $\Delta < 16$, in which case there is a full vertex.

If G is a (3,6,16,34)-graph with a full 5-vertex d_1, for example, then $H_2(v)$ is a (3,5,10)-graph with $e(3,5,10)$ edges. There is only one such graph, namely the disjoint union of two pentagons. (In fact, it is generally the case that there are very few $(3,l,n)$-graphs with $e(3,l,n)$ edges.) This knowledge of the structure of G makes a computer search feasible. For a description of the algorithms used in such a search, the reader should consult [23]. We note that if G is regular of degree 4, then $H_2(v)$ is a (3,5,11,16)-graph for any vertex v. These graphs are not hard to find (there are six of them). However, as a general rule, if $H_2(v)$ is a (k,l,n)-graph, then as the difference $\|H_2(v)\| - e(k,l,n)$ increases, the number of such graphs increases very rapidly.

In 1968, Walker [41] produced a method which allowed him to improve the upper bound for $R(4,5)$, and to prove a general upper bound for $R(k,k)$ in terms of $R(k-2,k)$. The method uses a theorem first proved by Goodman [20].

Theorem 2.3. Let G be a graph with n vertices. Let n_j be the number of vertices of degree j, $0 \le j \le n-1$. Let $t(G)$ be the number of triangles in G. Then

$$t(G) + t(\bar{G}) = \binom{n}{3} - \frac{1}{2} \sum_{j=0}^{n-1} (n_j)(j)(n-1-j). \tag{2.1}$$

If G is a (k,l,n)-graph, and if v is preferred, with v of degree j, then $H_1(v)$ is a $(k-1,l,j)$-graph and $H_2(v)$ is a $(k,l-1,n-1-j)$-graph. If we let $E(k,l,n)$ be the maximum number of edges in a (k,l,n)-graph (and define it to be 0 if no such graph exists), then v can be in no more than $E(k-1,l,j)$ triangles. The triangles in $\bar{G}$ which contain v are the independent 3-sets of G which contain v, and these come from independent 2-sets in $H_2(v)$. Since $H_2(v)$ must contain at least $e(k,l-1,n-1-j)$ edges, there are at most $\binom{n-1-j}{2} - e(k,l-1,n-1-j)$ independent 3-sets in G containing v. Furthermore, we have $n - R(k,l-1) \le j \le R(k-1,l) - 1$ by considering the number of vertices in $H_1(v)$ and $H_2(v)$. Hence, we obtain the following upper bound for $t(G) + t(\bar{G})$.

$$t(G) + t(\bar{G}) \le \frac{1}{3} \sum_{j=n-R(k,l-1)}^{R(k-1,l)-1} (n_j)\Big[E(k-1,l,j) + \binom{n-1-j}{2} - e(k,l-1,n-1-j)\Big]. \tag{2.2}$$

This inequality, combined with the equality (2.1), gives a method for eliminating some integers as candidates for $R(k,l)$. We illustrate by showing (as Walker did) that $R(4,5) \le 28$.

Assume that G is a (4,5,28)-graph. Then the relevent E- and e-values are: $E(3,5,10) = 20$, $E(3,5,11) = 22$, $E(3,5,12) = 24$, $E(3,5,13) = 26$, $e(4,4,14) = 50$, $e(4,4,15) = 55$, $e(4,4,16) = 60$, and $e(4,4,17) = 68$. As the reader can check, the statements (2.1) and (2.2), together with the obvious fact that $\Sigma_j n_j = 28$, are not simultaneously satisfiable. (In 1968, Walter [41] showed that $R(4,5) \leq 29$. In 1972, he showed that $R(4,5) \leq 28$ by improving his estimates of $e(4,4,n)$ for $n = 14,15$, and 16 (see [42]). We use the exact values of these numbers, which were calculated in [13].)

The above method can also be used to prove the following theorem (see [41]).

Theorem 2.4. $R(k,k) \leq 4R(k-2,k) + 2$.

Very few constructive methods are known which yield good lower bounds for $R(k,l)$. Most methods rely on some assumption of symmetry, and the methods are useful only in specific cases. We will now give some of these methods.

The first constructive method is due to Greenwood and Gleason [23]. Their method allowed them to evaluate $R(3,5)$, $R(4,4)$, and $R(3,3,3)$. Since $R(3,4) = 9$, we know that $R(4,4) \leq R(3,4) + R(4,3) = 18$. To show that $R(4,4) = 18$, we must exhibit a (4,4,17)-graph G. This is done by labeling the vertices of G with the integers 1 through 17, and then joining two vertices with an edge if the absolute value of their difference is in the set $D = \{1,2,4,8.9,13,15,16\}$. We shall call such a graph a cyclic graph, and the set D will be called the determining set of the graph. Cyclic graphs may be used to obtain sharp lower bounds for some of the small Ramsey numbers (see Table II).

We note that the (3,3,5)-, (3,5,13)-, and (4,4,17)-graphs are unique. There are two other (3,4,8)-graphs (both are subgraphs of the cyclic one), and it is not known whether the (3,9,35)-graph is unique.

For certain other values of the parameters k and l, the largest cyclic (k,l,n)-graph is known. These values were calculated by Kalbfleisch [32], Graver and Yackel [22], Hanson [27], and others. They are given in Table III. We denote the largest n such that a cyclic (k,l,n)-graph exists by $C(k,l)$. It is easily seen that $R(k,l) \geq C(k,l) + 1$. In most cases, the number $C(k,l)$ is so far below the best known upper bound that it is unlikely that the cyclic (k,l)-graph on $C(k,l)$ vertices is the largest (k,l)-graph. Cyclic graphs and their variations can be used to generate lower bounds for Ramsey numbers of more than two colors. We will discuss this in Section 4.

TABLE II. Determining sets of the cyclic Ramsey graphs.

k	l	n	$R(k,l)$	D
3	3	5	6	{1,4}
3	4	8	9	{1,4,7}
3	5	13	14	{1,5,8,12}
3	9	35	36	{1,7,11,16,19,24,28,34}
4	4	17	18	{1,2,4,8,9,13,15,16}

TABLE III

			Best Known Bounds for $R(k,l)$	
k	l	$C(k,l)$	Lower	Upper
3	6	16	18	18
3	7	21	23	23
3	8	26	28	29
3	10	38	39	44
3	11	45	46	54
3	12	48	49	63
4	5	24	25	28
4	6	33	34	44
5	5	41	42	55
5	6	56	57	105
6	6	101	102	178

3. ASYMPTOTICS FOR TWO COLORS

In Section 2, we showed that

$$R(k,l) \leq R(k-1,l) + R(k,l-1),$$

which immediately gives

$$R(k,l) \leq \binom{k+l-2}{k-1}.$$

This implies that

$$R(k,k) \leq c4^k\, k^{-1/2}$$

for some constant c. From now on, c denotes some appropriate constant. Graver and Yackel [22] proved that

$$R(3,k) \leq c\,\frac{k^2 \log\log k}{\log k}.$$

This result can be extended to the Ramsey number $R(l,k)$ for fixed l and large k such that

$$R(l,k) \leq c\,\frac{k^{l-1} \log\log k}{\log k} \tag{3.1}$$

where the constant c depends on l.

We remark that for the case that l and k both are large, in particular $l = k$, it is not known that (3.1) holds. (The widely quoted result of Yackel [44] seems now in question [36].)

Recently, Ajtai, Komlós, and Szemerédi [3] proved that any triangle-free graph on n vertices with degree t contains an independent set of size $c(n \log t/t)$. Now suppose G is a graph on n vertices having no triangle and no independent set of size k. Every vertex is of degree at most k since the neighbors of a vertex form an independent set. Thus we have $n \leq c\, k^2/\log k$, which gives

$$R(3,k) \leq c\, k^2/\log k. \tag{3.2}$$

Griggs [24] shows that 5/12 is an acceptable value for c in (3.2).

There are two ways to obtain asymptotic lower bounds—the constructive method and the probabilistic method. The latter method, which proves the existence of finite structures having certain properties without actually constructing them, often yields better lower bounds. This leads to an intriguing situation. For n sufficiently large, a "good configuration" is assured by probabilistic arguments (in fact almost all the configurations are "good"), but no one can find a procedure which can produce one "good configuration."

P. Erdös, who played a predominant role in the development of the probabilistic method, first proved the following (see [9]):

Theorem 3.3.

$$R(k,k) > \frac{k\, 2^{k/2}}{e\sqrt{2}}. \tag{3.3}$$

Spencer [40] improved the lower bound by a factor of 2 using the Lovasz Local Theorem [11], which is a powerful tool in probabilistic methods.

It is not known whether $\lim_{k\to\infty} R(k,k)^{1/k}$ exists. The results stated above imply that

$$\sqrt{2} \leq \lim\ \inf\ R(k,k)^{1/k} \leq \lim\ \sup\ R(k,k)^{1/k} \leq 4.$$

The determination of the value of $\lim R(k,k)^{1/k}$, if it exists, is one of the major open problems in this area of Ramsey theory. For fixed $l > 3$, it is conjectured [12,40] that $R(k,l) = k^{l-1+o(1)}$, asympotically in k. Note that this is true for $l = 3$.

As was stated earlier, constructive methods give weaker results than the probabilistic method. Abbott [1] has given a recursive construction which shows that $R(k,k) \geq ck^{C}$, where $C' = \log 41/\log 4 = 2.679\ldots$. Nagy [38] has given a construction which shows that $R(k,k) \geq ck^3$. Frankl [14] has shown by construction that $R(k,k) \geq ck^m$ for any m and some constant c depending on m. Chung [7] improved Frankl's construction by showing that $R(k,k) \geq \exp(c(\log k)^{4/3}/(\log\log k)^{1/3})$. Recently, Frankl and Wilson [16] proved that $R(k,k) \geq \exp((1+o(1))(\log k)^2/(4\log\log k))$ by construction using set intersection theorems. Namely, they consider a graph with vertices v_i represented by subsets S_i of a set, so that the color of an edge $\{v_i, v_j\}$ depends solely upon the cardinality of $S_i \cap S_j$. The problem of finding a 2-coloring of K_n containing no monochromatic K_k, where $n = (1+\varepsilon)^k$ for some positive ε, remains open.

When k and l are distinct, the following result of Spencer [21,40] generalized Erdös' argument of (3.3):

Theorem 3.4. If for some p, $0 < p < 1$,

$$\binom{n}{k} p^{\binom{l}{2}} + \binom{n}{l} (1-p)^{\binom{l}{2}} < 1,$$

then

$$R(k,l) > n.$$

In the case that $l = 3$, Erdös [10] proved by the probabilistic method that

$$R(3,k) \geq c\frac{k^2}{(\log k)^2}.$$

This can also be obtained immediately by an application of the Lovász Local Theorem [40]. We note that this bound is very close to the best upper bound (see (3.2)).

4. MORE THAN TWO COLORS

The only known exact value for m-colored Ramsey numbers, $m \geq 3$, is $R_3(3) = 17$, determined by Greenwood and Gleason [23]. Known nontrivial bounds are also scarce. In fact, the only ones are:

$$51 \leq R_4(3) \leq 65 \ [6,14],$$
$$159 \leq R_5(3) \leq 322 \ [17,43],$$
$$128 \leq R_3(4) \leq 254 \ [29,19].$$

A general upper bound was obtained in [23].

$$R(k_1,k_2,\ldots,k_m) \le 2 + \sum_{i=1}^{m} (R(k_1,\ldots,k_{i-1},k_i-1,k_{i+1},\ldots,k_m)-1). \tag{4.1}$$

Although it is suspected that this upper bound is never tight for $m \ge 4$ and $k_i \ge 3$, this was shown only in the case $m = 4$, $k_1 = k_2 = k_3 = k_4 = 3$, by Folkman [14]. Folkman's result can be used to get better upper bounds. For example, it can be shown that $R_k(3) \le k!(e - \frac{1}{24})$, where $e = 2.7\ldots.$

The lower bound $R_3(3) > 16$ was obtained in [23] by considering the Galois field of 2^4 elements. The triangle-free coloring is determined by cosets of cubic residues in the multiplicative group of the non-xero field elements (i.e., an edge $\{u,v\}$ is in color i if $u - v$ is in the ith coset).

For general k, a triangle-tree k-coloring can be constructed recursively from a triangle-free $(k-1)$-coloring and a triangle-free $(k-3)$-coloring [6]. It can be shown that

$$R_k(3) \ge 3R_{k-1}(3) + R_{k-3}(3) - 3.$$

Another technique used in obtaining lower bounds is to consider sum-free partitions of integers. A set S of positive integers is said to be sum-free if whenever i and j are elements of S the $i + j$ is not an element of S. Suppose s_k is the largest integer such that the integers from 1 to s_k can be partitioned into k sum-free sets. We can then color an edge $\{u,v\}$ of $K_{s_k} + 1$ in the ith color if $|u - v|$ is in the ith sum-free set. This k-coloring does not contain any monochromatic triangle since, for three integers $a < b < c$, the values $c - b$, $b - a$, $c - a$ cannot all be in a sum-free set. Thus we have

$$R_k(3) \ge s_k + 2.$$

This method does not give very tight bounds for the Ramsey numbers since it only considers cyclic colorings. It is known [4] that $s_1 = 1$, $s_2 = 4$, $s_3 = 13$, and $s_4 = 44$. Both the constructions for $R_3(3) > 16$ and $R_4(3) > 50$ are not cyclic colorings. However, in order to make computer search for triangle-free colorings computationally feasible, additional conditions are required together with clever back-track methods [4]. Fredrickson [17] found by computer that $s_5 \ge 157$. Large sum-free sets can be constructed recursively from small ones. In particular, (see [2])

$$s_{k+l} \ge 2s_k s_l + s_k + s_l.$$

Thus we have, for a fixed l,

$$R_k(3) \ge s_k \ge C(2s_l + 1)^{k/l}$$

for an appropriate constant C and $k \geq l$. Using $s_5 \geq 157$, we have

$$R_k(3) \geq C(315)^{k/5} = C(3.16\ldots)^k. \tag{4.2}$$

Unlike $(R(k,k))^{1/k}$, the limit of $(R_k(3))^{1/k}$ is guaranteed to exist by

$$R_{k_1+k_2}(3) \geq R_{k_1}(3)R_{k_2}(3),$$

which can be obtained by constructing a triangle-free $(k_1 + k_2)$-coloring of K_{mn} using a triangle-free k_1-coloring of K_m and a triangle-free k_2-coloring of K_m. From (4.1) and (4.2) the value of $\lim_{k\to\infty} (R_3(3))^{1/k}$ is between 3.16 and ∞. Note that any new lower bounds for s_k might improve the lower bound of $\lim_{k\to\infty} (R_k(3))^{1/k}$. This could be an interesting task for those who have access to large amounts of free computer time.

We now give three generalizations of the sum-free partitions and the residue partitions which could be used to improve lower bounds for Ramsey numbers.

(G1) Whitehead [43] successfully partitioned $Z_4 \times Z_4$ into three sum-free sets and $Z_7 \times Z_7$ into four sum-free sets. Using these sum-free sets, we can, for example, construct a triangle-free 4-coloring of K_{49} by labeling the vertices by elements of $Z_7 \times Z_7$. Note that $49 > s_5 = 44$, and we are now considering a larger class of triangle-free colorings. Still, the Ramsey bound obtained here is $R_4(3) \geq 50$, one less than the best known lower bound.

(G2) A set A of positive integers is said to be i-difference-free, if, for any subset of i elements in A, there is some pair of these elements whose absolute difference is not contained in A. Suppose the integers from 1 to n can be partitioned into k i-difference-free subsets. We can then obtain a K_{i+1}-free k-coloring of K_{n+1} since, for any $i+1$ numbers $x_1, \ldots, x_{i+1}$, the set of pairwise differences for the i numbers $x_2 - x_1 \ldots, x_{i+1} - x_1$ cannot lie entirely in an i-difference-free set S if S contains $x_j - x_1$, $2 \leq j \leq i+1$. Partitions of the integers from 1 to 40 into 4-difference-free sets were found independently by Lin [37], Burling [5], Irving [30], Garcia [18], and Hanson [27]. Thus $R_2(5) \geq 42$. Hanson and Hanson [28] partitioned 1 to 55 into one 4-difference-free set and one 5-difference-free set to obtain $R(5,6) \geq 57$.

(G3) Instead of partitioning integers $1, \ldots, q$, we can consider partitions of residue classes into i-difference-free sets where q is a prime power and $x, -x$ are in the same class. In this way substantial computer search time can be saved at the expense of imposing further restrictions on the coloring scheme. Hill and Irving [29] used this method to obtain the lower bounds $R_2(7) > 125$ and $R_3(4) > 127$. Also, Guldan and Tomasta [26] obtained the bounds $R_2(10) > 457$ and $R_2(11) > 521$.

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