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Diameter Bounds for Altered Graphs

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ABSTRACT

The main question addressed in this article is the following: If t edges are removed from a $(t + 1)$ edge-connected graph G having diameter D, how large can the diameter of the resulting graph be? (The diameter of a graph is the maximum, over all pairs of vertices, of the length of the shortest path joining those vertices.) We provide bounds on this value that imply that the maximum possible diameter of the resulting graph, for large D and fixed t, is essentially $(t + 1) \cdot D$. The bulk of the proof consists of showing that, if t edges are *added* to an n-vertex path P_n, then the diameter of the resulting graph is at least $(n/(t + 1)) - 1$. Using a similar proof, we also show that if t edges are added to an n-vertex cycle C_n, then the least possible diameter of the resulting graph is (for large n) essentially $n/(t + 2)$ when t is even and $n/(t + 1)$ when t is odd. Examples are given in all these cases to show that there exist graphs for which the bounds are achieved. We also give results for the corresponding *vertex* deletion problem for general graphs. Such results are of interest, for example, when studying the potential effects of node or link failures on the performance of a communication network, especially for networks in which the maximum time-delay or signal degradation is directly related to the diameter of the network.

1. INTRODUCTION

For an undirected graph (in general, we follow the graph-theoretic terminology of [1]) G, let $V(G)$ denote the vertex set of G and let $E(G)$ denote the edge set of G. The *distance* $d_G(u,v)$ between two vertices u, $v \in V(G)$ is the length (in number of edges) of a shortest path in G joining u and v; if no such path exists, we set $d_G(u,v) = \infty$. The *diameter* $D(G)$ of G is the maximum value of $d_G(u,v)$ taken over all pairs of vertices u, $v \in V(G)$. Thus a graph has finite diameter if and only if it is connected.

Journal of Graph Theory, Vol. 8 (1984) 511–534
 CCC 0364-9024/84/040511-24$04.00

The main question studied in this paper is the following: Suppose that G is a graph with diameter D and that G' is obtained from G by removing t of its edges. How large can the diameter of G' be? To avoid the relatively uninteresting case in which G' can be disconnected, we shall require that G be $t + 1$ edge-connected, i.e., that every pair of vertices be joined by at least $t + 1$ edge-disjoint paths. Our results, however, do not use this requirement in any essential way, and they all continue to hold under the weaker requirement that G' simply be connected.

In addition to considering this edge-deletion question, we also address the analogous vertex-deletion question, which turns out to be substantially easier to resolve.

These questions are of interest, for example, when graphs are used to model the linkage structure of communication networks. Here the vertices and edges of the graph represent the nodes and links of the network, and the diameter of the graph corresponds to the maximum number of links over which a message between two nodes must travel. In cases where the number of links in a path is roughly proportional to the time delay or signal degradation encountered by messages sent along the path, one would therefore like to have the diameter of the network be relatively small. The particular questions discussed in this paper thus pertain to the performance of communication networks under the occurrence of link or node failures.

Notice that these questions are also related to the corresponding "augmentation" questions: Given a connected graph G', if we add t edges to G' to form a graph G, how small can we make the diameter of G? This version is of interest when considering adding facilities to a network to upgrade its performance.

Both the edge-deletion and vertex-deletion questions are closely related to an extremal graph problem, originally posed by Vijayan and Murty [17], of minimizing the number $f(n,D,D',t)$ of edges in a graph with n vertices and given diameter D having the property that, if any t edges (vertices) are deleted, the remaining graph has diameter no more than a given value D'. Although this problem has received quite a bit of attention in the literature [2–18], it seems to be quite difficult in general, and relatively little is known beyond cases with small values of t and D (primarily $t = 1$ and $D \leq 5$). A recent survey of work in this area appears in [6,9]. The questions we are concerned with can be viewed as asking about the extreme values of the parameter D', in terms of t and D, beyond which the optimal graphs do not change.

In Section 2 we deal with the edge-augmentation problem for two fundamental classes of graphs, the n-vertex paths P_n and the n-vertex cycles C_n. For the path case, we show that the least possible diameter of a graph obtained by adding t edges to the path P_n is $(n/(t + 1)) + O(1)$. This is the main result needed for proving our edge-deletion bound for general graphs, and its proof is fairly difficult. We include the cycle case, which is interesting in its own right, primarily because its proof is similar to that for the path case. The result itself is perhaps surprising, since it says that the least possible diameter of a graph obtained by adding t edges to the cycle C_n is $(n/(t + 2)) + O(1)$ if t is even and $(n/(t + 1)) + O(1)$ if t is odd.

In Section 3 we consider the edge-deletion problem for general graphs. Here we use the edge-augmentation result for paths from Section 2 to show that the maximum possible diameter for a graph obtained by deleting t edges from a $t + 1$ edge-connected graph G is $(t + 1) \cdot D(G) + O(t)$. For $t = 1$, we then strengthen this result to exactly $2 \cdot D(G)$.

In Section 4 we deal with the analogous vertex-deletion problem for general graphs, which is much simpler than the edge-deletion problem. In the vertex-deletion case, however, the corresponding maximum diameter is *unbounded* in terms of t and $D(G)$, but it *can* be bounded in terms of t, $D(G)$, $|V(G)|$, and the vertex-connectivity of G.

Finally, Section 5 concludes the paper with several open problems.

2. ADDING EDGES TO A PATH OR A CYCLE

In this section we address the following question: If t edges are added to an n-vertex path P_n or an n-vertex cycle C_n, how small can the diameter of the resulting graph be? Let $P(n,t)$ denote this least possible diameter for the path case, and let $C(n,t)$ denote the corresponding value for the cycle case.

Theorem 1. For all $t \geqslant 1$,

$$\frac{n}{t+1} - 1 \leqslant P(n,t) < \frac{n}{t+1} + 3.$$

Proof. We prove the lower bound first. Fix n and t, and let G be a graph formed by adding t edges to P_n satisfying

$$P(n,t) = D(G) = D.$$

Without loss of generality, we may assume that none of the t additional edges forms a loop, i.e., joins two identical vertices.

Let X denote the subgraph of G formed by the t extra edges, and let $c_1, c_2, \ldots, c_r$, with $r \leqslant t$, denote the connected components of X. The vertices of X determine a partition of the path P_n into subpaths, called *segments,* as shown in Figure 1. Each segment is a maximal subpath of P_n containing no endpoint of an edge in X in its interior. Thus every segment includes only two vertices from X, as the end vertices of the segment, and there are two special *tail segments* t_1 and t_2 that include

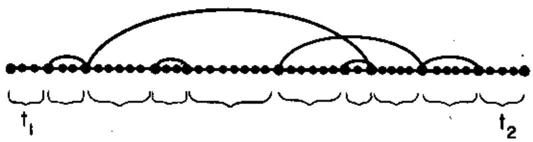

FIGURE 1. Partition of P_n into segments determined by the endpoints of edges in X, with tail segments t_1 and t_2.

only one vertex each from X (either or both of these may be empty, i.e., may be a one-vertex path P_1). We shall refer to the non-tail segments as *regular segments.*

Lemma 1. There are at most $t + r - 1$ regular segments.

Proof. The number of regular segments is one less than the number of vertices in the vertex set $V(X)$ for X. We can bound the size of $V(X)$ by first noting that, for each component c_i, the fact that c_i is connected implies that $|E(c_i)| \geqslant |V(c_i)| - 1$. Therefore,

$$\begin{aligned} |V(X)| &= \sum_{i=1}^{r} |V(c_i)| \\ &\leqslant \sum_{i=1}^{r} (|E(c_i)| + 1) \\ &= |E(x)| + r = t + r. \quad \blacksquare \end{aligned}$$

Our proof will ignore the interior structure of the components c_i, concentrating instead on the segments and how they are interconnected through common components. Using bounds on the lengths of certain collections of segments and the fact that the sum of the lengths of all the segments is exactly $n - 1$ (the length of P_n), we will obtain our desired bound on D.

To this end, we define G^* to be the graph obtained from G by merging the vertices of each component c_i into a single *component vertex* labeled by c_i and removing the t edges in X. See Figure 2. The edges from G that remain in G^* are exactly those edges in the original path P_n, so the segments all retain their identities in G^*, and they act much like edges joining the condensed vertices. Notice that in general G^* may contain loops and multiple edges, so, strictly speaking, it is actually a multigraph. More importantly, observe that any two vertices of G^* must still be connected by a path of length D or less.

By the *degree* of a component vertex in G^*, we will mean the number of time it occurs as an end vertex of a segment. This differs from the usual notion of degree only in that any *empty* tail segment contributes

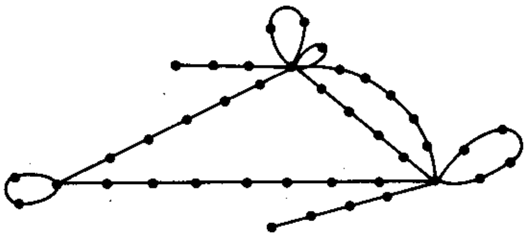

FIGURE 2. The condensed graph G^* obtained from the graph G of Figure 1 by shrinking each connected component of X to a single vertex.

1 to the degree of its single component vertex, a convention we use merely to simplify the presentation.

Lemma 2. The degree of each component vertex in G^* is even and at least four.

Proof. In G, each component c_i of X includes at least two vertices of the path P_n. Each such vertex is an end vertex of exactly two segments and hence contributes exactly two to the degree of c_i in G^*. ∎

Let s_0 denote a regular segment in G^* of maximum length, and let v_0 be a vertex on s_0 with distance at least $z = \lfloor |E(s_0)|/2 \rfloor$ to both end vertices of s_0, i.e., v_0 is in the center of s_0. We will use s_0 and v_0 as focal points for computing bounds on the lengths of the segments in G^*. In order to do this, we now define a tree T that contains, for each component vertex c_i, a shortest path in G^* from c_i to the segment s_0. We also will require that this "shortest path tree" T include the segment s_0, *unless* both end vertices of s_0 are the same in G^*, in which case we exclude s_0 from T. (These two possibilities will in fact be treated separately at a later point in the proof.) Specifically, we let T be any subgraph of G^* such that (1) T is a tree, (2) for each component vertex c_i, T contains a shortest path in G^* from c_i to s_0, (3) all leaves (vertices of degree 1) in T are component vertices, and (4) s_0 is contained in T if and only if the two end vertices of s_0 in G^* are distinct. It is straightforward to show by standard techniques that such a T exists. Figure 3 illustrates the two possible forms for T.

For each component vertex c_i, let p_i denote the path in T from c_i to the segment s_0. The key bound on segment lengths that we will be using is the following:

Lemma 3. For any regular segment $s \neq s_0$ joining component vertices c_i and c_j,

$$|E(s)| + |E(p_i)| + |E(p_j)| + |E(s_0)| \leq 2D + 2.$$

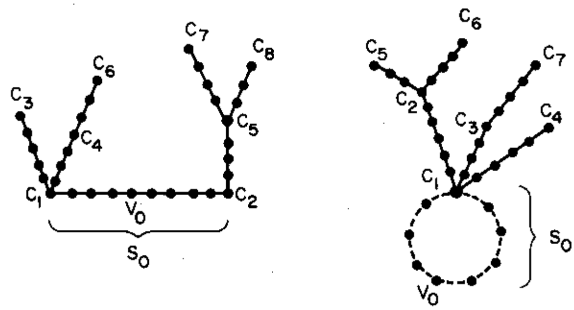

FIGURE 3. Two cases for the "shortest path tree" T. If the two end vertices of s_0 in G^* are distinct, then s_0 is included in T. If they are the same, then s_0 is excluded from T.

Proof. If s is contained in T, let c_j be the further of c_i and c_j from s_0. Then $z + |E(p_j)| \leq D$, so we have

$$|E(s)| + |E(p_i)| + |E(p_j)| + |E(s_0)| \leq 2 \cdot |E(p_j)| + 2z + 1$$
$$\leq 2D + 1.$$

If s is not contained in T, let $\{u,v\}$ be an edge in s such that there is a shortest path in G^* from v_0 to u that includes c_i and there is a shortest path in G^* from v_0 to v that includes c_j. If $c_i = c_j$, we require in addition that the distance in s from u to c_i be $\lfloor |E(s)|/2 \rfloor$ and the distance in s from v to $c_j = c_i$ be $\lfloor (|E(s)| - 1)/2 \rfloor$. The existence of such an edge $\{u,v\}$ follows from the fact that s is not contained in T, since then each of the vertices c_i and c_j has a shortest path to s_0 that does not contain any edge of s. The length of the shortest path from v_0 to u satisfies

$$d_s(u,c_i) + |E(p_i)| + z \leq D.$$

The length of the shortest path from v_0 to v satisfies

$$d_s(v,c_j) + |E(p_j)| + z \leq D.$$

Hence

$$2D + 2 \geq (1 + d_s(u,c_i) + d_s(v,c_j)) + |E(p_i)| + |E(p_j)| + (2z + 1)$$
$$\geq |E(s)| + |E(p_i)| + |E(p_j)| + |E(s_0). \quad ∎$$

Lemma 4. For $i \in \{1,2\}$, let c_i denote the single component vertex in the tail segment t_i. Then

$$2 \cdot |E(t_i)| + 2 \cdot |E(p_i)| + |E(s_0)| \leq 2D + 1.$$

Proof. Since the shortest path from v_0 to the far end of t_i must be no more than D, we have

$$|E(t_i)| + |E(p_i)| + z \leq D.$$

Hence

$$\begin{aligned} 2D + 1 &\geq 2(|E(t_i)| + |E(p_i)| + z) + 1 \\ &\geq 2 \cdot |E(t_i)| + 2 \cdot |E(p_i)| + |E(s_0)|. \quad \blacksquare \end{aligned}$$

Lemma 5. Suppose that s_0 is contained in T, and let T_1 and T_2 be the two subtrees obtained when s_0 is deleted from T. Let s_1 denote a shortest segment joining a vertex in T_1 to a vertex in T_2, with $s_1 = s_0$ if and only if it is the only such segment. Then for $i \in \{1,2\}$, the tail segment t_i satisfies

$$2 \cdot |E(t_i)| + |E(p_i)| + |E(s_1)| + |E(s_0)| \leq 2D + 1.$$

Proof. Let c and c' denote the two end vertices of s_0. Without loss of generality, assume that c and c_i belong to T_1. Then p_i is a shortest path joining c and the segment t_i. Let p_i' denote a shortest path in G^* from c' to the segment t_i. Then, by the same argument used for proving Lemma 3, with the far end of t_i playing the role of v_0 (note that the proof of Lemma 3 did not depend on s_0 being a longest segment), we have

$$2 \cdot |E(t_i)| + |E(p_i)| + |E(p_i')| + |E(s_0)| \leq 2D + 1.$$

Moreover, p_i' must contain some regular segment that crosses from T_1 to T_2, and hence $|E(p_i')| \geq |E(s_1)|$, from which the result follows. ∎

We now use the bounds in Lemmas 3, 4, and 5 to bound the total length of all the segments in G^*. Let S denote the set of all segments (including the tail segments), let L denote the set of leaves of T other than the endpoints of s_0, and let R denote the set of all regular segments not included in T. Note that, since T includes exactly $r - 1$ segments, it follows from Lemma 1 that $|R| \leq t$. We now divide the proof into cases according to whether or not s_0 is contained in T.

Case 1. s_0 is not contained in T.

$$\begin{aligned} 2 \cdot \sum_{s\in S}|E(s)| &= 2\sum_{s\subseteq T}|E(s)| + 2\sum_{s\in R}|E(s)| + 2(|E(t_1)| + |E(t_2)|) \\ &\leq 2\sum_{c_i\in L}|E(p_i)| + 2\sum_{s\in R}|E(s)| + 2(|E(t_1)| + |E(t_2)|) \end{aligned}$$

$$\leq \sum_{s \in R - \{s_0\}} (|E(s)| + |E(p_i)| + |E(p_j)|) + |E(p_1)| + |E(p_2)|$$
$$+ \sum_{s \in R - \{s_0\}} |E(s)| + 2(|E(s_0)| + |E(t_1)| + |E(t_2)|)$$

since every leaf in L has degree at least 2 (in fact, at least 3) in $G^* - T$; the paths p_i and p_j in the summation over $s \in R - \{s_0\}$ are those to the end vertices of s. By Lemmas 3 and 4, we then have

$$2 \cdot \sum_{s \in S} |E(s)| \leq (|R| - 1)(2D + 2 - |E(s_0)|) + (|R| - 1) \cdot |E(s_0)|$$
$$+ 2 \cdot (2D + 1)$$
$$= (|R| + 1)(2D + 2) - 2.$$

Thus

$$n - 1 = \sum_{s \in S} |E(s)| \leq (|R| + 1)(D + 1) - 1$$
$$\leq (t + 1)D + t$$

and

$$D \geq \frac{n - 1 - t}{t + 1}.$$

Case 2. s_0 is contained in T. Let T_1 and T_2 denote the two subtrees of T obtained when s_0 is removed, and let s_1 be a shortest segment joining a vertex in T_1 to a vertex in T_2 as in Lemma 5. We have two subcases depending on whether or not $s_1 = s_0$:

Subcase 2.1. $s_1 \neq s_0$.

$$2 \cdot \sum_{s \in S} |E(s) = 2 \sum_{s \subseteq T} |E(s)| + 2 \sum_{s \in R} |E(s)| + 2(|E(t_1)| + |E(t_2)|)$$
$$\leq 2 \sum_{c_i \in L} |E(p_i)| + 2|E(s_0)| + 2 \sum_{s \in R} |E(s)|$$
$$+ 2(|E(t_1)| + |E(t_2)|)$$
$$\leq \sum_{s \in R - \{s_1\}} (|E(s)| + |E(p_i)| + |E(p_j)|) + |E(p_1)| + |E(p_2)|$$
$$+ 2|E(s_0)| + 2|E(s_1)| + \sum_{s \in R - \{s_1\}} |E(s)|$$
$$+ 2(|E(t_1)| + |E(t_2)|),$$

since every leaf in L has degree at least 2 in $(G^* - T) - s_1$. By Lemmas 3 and 5, we then have

$$2 \cdot \sum_{s \in S} |E(s)| \leq (|R| - 1)(2D + 2 - |E(s_0)|) + (|R| - 1) \cdot |E(s_0)|$$
$$+ 2 \cdot (2D + 1)$$
$$= (|R| + 1)(2D + 2) - 1,$$

so again we have

$$n - 1 = \sum_{s \in S} |E(s)| \leq (t + 1)D + t$$

and

$$D \geq (n - 1 - t)/(t + 1).$$

Subcase 2.2. $s_1 = s_0$. We first claim that there exists a segment $s_2 \in R$ such that every leaf in L has degree at least two in $(G^* - T) - s_2$. Since every leaf in L has degree at least three in $G^* - T$, we are done if any segment in R has two distinct end vertices or has an end vertex not in L. So suppose that each segment in R is a "loop" ending at a vertex in L. Because both end vertices of s_0 have degree four in G^* and neither belongs to L, each of those end vertices must have degree at least four in $T \cup t_1 \cup t_2$. Thus there are at least four leaves in L, and at least one of them is not an end vertex for any tail segment t_i. Since that leaf has degree at least three in $G^* - T$ and has only "loops" from R incident on it in $G^* - T$, it must in fact have at least two such "loops" incident on it. Either of them suffices as s_2. The claim follows.

Let s_2 be such a segment. As in the previous subcase, we then have

$$2 \cdot \sum_{s \in S} |E(s)| \leq \sum_{s \in R - \{s_2\}} (|E(s)| + |E(p_i)| + |E(p_j)|)$$
$$+ |E(p_1)| + |E(p_2)| + 2|E(s_0)|$$
$$+ 2|E(s_2)| + \sum_{s \in R - \{s_2\}} |E(s)| + 2(|E(t_1)| + |E(t_2)|),$$

since every leaf in L has degree at least 2 in $(G^* - T) - s_2$. By Lemmas 3 and 5, with $s_1 = s_0$ in the latter, we obtain

$$2 \cdot \sum_{s \in S} |E(s)| \leq (|R| - 1)(2D + 2 - |E(s_0)|) + (|R| - 1) \cdot |E(s_0)|$$
$$+ 2|E(s_2)| + 2(2D + 1 - |E(s_0)|)$$
$$= (|R| + 1)(2D + 2) - 2 + 2(|E(s_2)| - |E(s_0)|)$$
$$\leq (|R| + 1)(2D + 2) - 2,$$

since s_0 is the longest regular segment. Therefore, in this final case we also have

$$n - 1 = \sum_{s \in S} |E(s)| \leqslant (t + 1)D + t$$

and

$$D \geqslant \frac{n - 1 - t}{t + 1},$$

so the proof for the lower bound is complete.

The following construction achieves the upper bound: Given n and t, choose

$$x = \lceil (n - t - 1)/2(t + 1) \rceil.$$

Let the vertices $u_0, u_1, \ldots, u_t$ in P_n be such that the distance from u_0 to one end vertex of P_n is no greater than x, the distance from u_t to the other end vertex of P_n is no greater than x, and, for $1 \leqslant i \leqslant t$, the distance from u_{i-1} to u_i is at most $2x + 1$. The choice of x ensures that this can be done. The t additional edges we add to form G are simply the edges $\{u_0, u_i\}$ for $1 \leqslant i \leqslant t$, i.e., they make up a star with center u_0. See Figure 4 for an example. To obtain an upper bound on the diameter D of G, observe that every vertex is within distance x of some vertex u_i and every pair of vertices u_i and u_j are within distance two of one another. Hence,

$$\begin{aligned} P(n,t) \leqslant D &\leqslant 2x + 2 \\ &< ((n - t - 1)/(t + 1)) + 4 \\ &= (n/(t + 1)) + 3 \end{aligned}$$

as required. The proof of Theorem 1 is complete. ∎

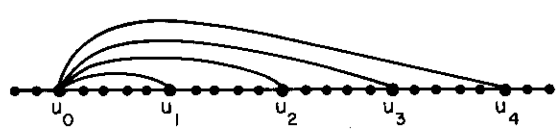

FIGURE 4. An example of the upper bound construction for $P(n,t)$, with $t = 4$ and $n = 25$. This graph has diameter 6.

Corollary 1. For all $t \geq 1$, $P(n,t) = n/(t + 1) + O(1)$.

We use similar arguments for bounding $C(n,t)$. Perhaps surprisingly, the result depends in a significant way on whether t is even or odd.

Theorem 2. If t is even,

$$\frac{n}{t+2} - 1 \leq C(n,t) < \frac{n}{t+2} + 3.$$

Proof. Since one way of adding $t + 1$ edges to a path P_n is to first add one edge joining the two end vertices of P_n and then to add t additional edges in the optimal way for the resulting cycle C_n, we immediately have the lower bound

$$C(n,t) \geq P(n,t + 1) \geq \frac{n}{t+2} - 1.$$

The following construction achieves the upper bound: Given n and t, choose

$$x = \lceil n/(t + 2) \rceil.$$

Choose vertices $u_0,u_1,u_2, \ldots, u_{t+1}$ on the cycle C_n such that the distance from u_i to u_{i+1}, $0 \leq i \leq t$, and from u_{t+1} to u_0 is at most x. The choice of x ensures that this can be done. The t additional edges added to form G are simply the edges joining u_0 to each of $u_2,u_4, \ldots, u_t$ and joining u_1 to each of $u_3,u_5, \ldots, u_{t+1}$, i.e., they make up two disjoint stars whose vertices are interlaced on the cycle C_n. See Figure 5 for an example.

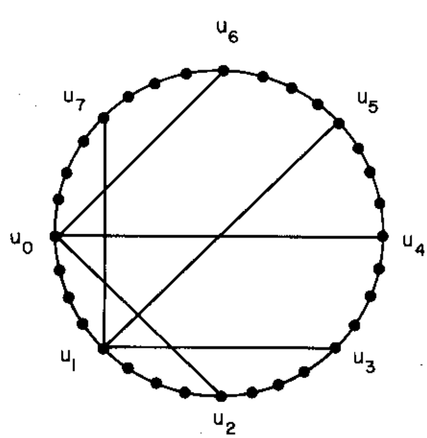

FIGURE 5. An example of the upper bound construction for $C(n,t)$ when t is even. Here $t = 6$, $n = 32$, and the resulting diameter is 6.

To bound the diameter of G, consider any two vertices v_1 and v_2 on the cycle C_n. Let T_0 denote the tree consisting of the $t/2$ additional edges incident on u_0, and let T_1 denote the corresponding tree containing u_1. The sum of the distances from v_1 to T_0 and to T_1 is at most x, since v_1 belongs to a segment of length at most x that has one end vertex in each tree. Similarly, the sum of the distances from v_2 to T_0 and to T_1 is at most x. Thus,

$$d(v_1,T_0) + d(v_1,T_1) + d(v_2,T_0) + d(v_2,T_1) \leqslant 2x,$$

and it must be the case that either $d(v_1,T_0) + d(v_2,T_0) \leqslant x$ or $d(v_1,T_1) + d(v_2,T_1) \leqslant x$. Without loss of generality, assume the former. Then the corresponding vertices of T_0 that are closest to v_1 and to v_2 are at most distance two apart in G, since T_0 is a star, so we have a path of length at most $x + 2$ joining v_1 and v_2. Therefore,

$$D(G) \leqslant x + 2 < \frac{n}{t+2} + 3$$

as required. ∎

Theorem 3. If t is odd,

$$\frac{n}{t+1} - 1 \leqslant C(n,t) \leqslant \frac{n}{t+1} + 3.$$

Proof. The upper bound follows immediately from the construction for $P(n,t)$. To prove the lower bound we shall use many of the ideas used in the proof of Theorem 1, although the proof in this situation is slightly more complicated.

Observe that now there are no tail segments, so all segments are regular segments. Furthermore, the number of such segments in this case is at most $t + r$, rather than $t + r - 1$. We construct the condensed graph G^* and the "shortest path tree" T exactly as in the proof of Theorem 1. Then Lemmas 2 and 3 also continue to hold. Let S denote the set of all segments, let L denote the set of all leaves of T other than the end vertices of s_0, and let R denote the set of all segments that are not contained in T. Since T contains exactly $r - 1$ segments, $|R| \leqslant t + 1$. We now divide into cases along somewhat different lines than done in the proof of Theorem 1.

Case 1. s_0 is not contained in T or $|R| \leqslant t$. Let s^* be any segment in R other than s_0, with c_1 and c_2 denoting the end vertices of s^* and with s^* empty if there is no such segment in R. Then

$$
\begin{aligned}
2 \cdot \sum_{s \in S} |E(s)| &= 2 \cdot \sum_{s \subseteq T} |E(s)| + 2 \cdot \sum_{s \in R} |E(s)| \\
&\leq 2 \cdot \sum_{c_i \in L} |E(p_i)| + 2 \cdot \sum_{s \in R - \{s_0\}} |E(s) + 2 \cdot |E(s_0)| \\
&\leq \sum_{s \in R - \{s_0, s^*\}} (|E(s)| + |E(p_i)| + |E(p_j)|) + |E(p_1)| + |E(p_2)| \\
&\quad + \sum_{s \in R - \{s_0, s^*\}} |E(s)| + 2 \cdot |E(s_0)| + 2 \cdot |E(s^*)|
\end{aligned}
$$

since every leaf in L has degree at least 2 in $G^* - T$. Observe that, by the properties defining this case, the number of segments in $R - \{s_0, s^*\}$ is at most $t - 1$. Applying Lemma 3, we obtain

$$
\begin{aligned}
2 \cdot \sum_{s \in S} |E(s)| &\leq (t - 1)(2D + 2 - |E(s_0)|) + (t - 1) \cdot |E(s_0)| \\
&\quad + 2 \cdot (2D + 2) \\
&= (t + 1) \cdot (2D + 2)
\end{aligned}
$$

and hence

$$
n = \sum_{s \in S} |E(s)| \leq (t + 1)(D + 1)
$$

from which the desired inequality is immediate.

Case 2. s_0 is contained in T and $|R| = t + 1$. Let T_1 and T_2 denote the subtrees of T formed when s_0 is deleted. We first claim that there exists at least one segment in R that has both its end vertices in the same subtree (either T_1 or T_2) of T. To see this, consider a path in G^* that traverses every segment exactly once and in the same sequence that they occur in the cycle C_n, say starting with some component vertex in T_1. Since this path starts and ends at the same component vertex, it must cross between T_1 and T_2 an even number of times, and hence G^* contains an even number of segments joining a vertex in T_1 to a vertex in T_2. Since all such segments belong to $R \cup \{s_0\}$ and since $|R \cup \{s_0\}|$ must be odd, it follows that some segment in R does not join a vertex in T_1 to a vertex in T_2, and hence that segment satisfies the claim.

Lemma 6. Let s^* be any segment in R that joins two vertices in the same subtree of T (say T_1), and let c_1 and c_2 denote its end vertices, where c_1 is the closer of the two to s_0. Let p^* be a shortest path from s^* to the end of s_0 that is in T_2, let s_1^* be the segment on this path that joins a vertex in T_1 and a vertex in T_2, and let c_3 denote the end vertex of s_1^* in T_2. Then

$$|E(s^*)| + |E(p_1)| + |E(s_1^*)| + |E(p_3)| + |E(s_0)| \leq 2D + 2.$$

Proof. Applying the same proof as for Lemma 3, with s^* now playing the role of s_0, we obtain

$$|E(s^*)| + |E(p_1)| + |E(p^*)| + |E(s_0)| \leq 2D + 2.$$

Since $|E(p^*)| \geq |E(s_1^*)| + |E(p_3)|$, the lemma follows immediately. ∎

Let s^* be any segment in R that joins two vertices in the same subtree of T, where s^* is chosen to be a loop if there are any loops in R. Without loss of generality, we may assume that both end vertices of s^* belong to T_1. Let c_1, c_2, s_1^*, and c_3 be as stated in Lemma 6. We now complete the proof of the lower bound by considering three subcases.

Subcase 2.1. $s_1^* \neq s_0$ and either s^* is a loop or c_2 is not an end vertex for s_1^*.

$$\begin{aligned} 2 \cdot \sum_{s\in S}|E(s)| &\leq 2 \cdot \sum_{c_i\in L}|E(p_i)| + 2 \cdot \sum_{s\in R}|E(s)| + 2 \cdot |E(s_0)| \\ &\leq \sum_{s\in R-\{s^*,s_1^*\}} (|E(s)| + |E(p_i)| + |E(p_j)|) + 2 \cdot |E(p_1)| \\ &\quad + \sum_{s\in R-\{s^*,s_1^*\}} |E(s)| + 2 \cdot |E(s^*)| + 2 \cdot |E(s_1^*)| + 2 \cdot |E(s_0)| \end{aligned}$$

since every leaf in $L - \{c_1\}$ has degree at least 2 in $G^* - (T \cup \{s^*,s_1^*\})$. Using the fact that $|R - \{s^*,s_1^*\}| = t - 1$ and applying Lemmas 3 and 6, we obtain

$$\begin{aligned} 2 \cdot \sum_{s\in S}|E(s)| &\leq (t - 1)(2D + 2 - |E(s_0)|) + (t - 1) \cdot |E(s_0)| \\ &\quad + 2 \cdot (2D + 2) \\ &= (t + 1)(2D + 2), \end{aligned}$$

from which the desire inequality follows as before.

Subcase 2.2. $s_1^* = s_0$. We first claim that in this case there exists a segment $s_2 \in R - \{s^*\}$ such that every leaf in $L - \{c_1\}$ has degree at least two in $G^* - (T \cup \{s^*,s_2\})$. We know from Lemma 2 that every leaf in L has degree at least three in $G^* - T$. If s^* is a loop, then any segment in $R - \{s^*\}$ suffices as s_2 (and $R - \{s^*\}$ is nonempty because $|R| = t + 1$ and $t \geq 1$). If s^* is not a loop, then, by the choice of s^*, all segments in R join two distinct component vertices. Consider any leaf of T that belongs to T_2 (including possibly the end of s_0 in T_2, even though it was excluded from L). That leaf must be an end vertex for at least three segments in R, none of which is s^*. If any one of those segments does not have c_2 as its other end vertex, it suffices as s_2 and we are done. On the other hand, if all three of those segments have c_2 as their other end vertex, we can choose any one of them as s_2, since c_2 will still have two segments incident on it in $G^* - (T \cup \{s^*, s_2\})$. The claim follows.

Letting s_2 denote any such segment, we have

$$\begin{aligned} 2 \cdot \sum_{s \in S} |E(s)| &\leq 2 \cdot \sum_{c_i \in L} |E(p_i)| + 2 \cdot \sum_{s \in R} |E(s)| + 2 \cdot |E(s_0)| \\ &\leq \sum_{s \in R - \{s^*, s_2\}} (|E(s)| + |E(p_i)| + |E(p_j)|) + 2 \cdot |E(p_1)| \\ &\quad + \sum_{s \in R - \{s^*, s_2\}} |E(s)| + 2 \cdot |E(s_2)| + 2 \cdot |E(s_0)|, \end{aligned}$$

since every leaf in L, except possibly c_1, has degree at least two in $G^* - (T \cup \{s^*, s_2\})$. Using the fact that $|R - \{s^*, s_2\}| = t - 1$ and applying Lemmas 3 and 6 (the latter with $s_1^* = s_0$), we obtain

$$\begin{aligned} 2 \cdot \sum_{s \in S} |E(s)| &\leq (t - 1)(2D + 2 - |E(s_0)|) + (t - 1) \cdot |E(s_0)| \\ &\quad + 2(2D + 2 - |E(s_0)|) + 2|E(s_2)| \\ &= (t + 1)(2D + 2) + 2(|E(s_2)| - |E(s_0)|) \\ &\leq (t + 1)(2D + 2), \end{aligned}$$

from which the desired inequality follows as before.

Subcase 2.3. $s_1^* \neq s_0$, s^* is not a loop, and c_2 is an end vertex for s_1^*. (Note that in this case there can be no loops in R, by choice of s^*.)

Let $L' = L - \{c_1, c_3\}$, and let M_1 be a maximum-sized collection of vertex-disjoint segments joining pairs of vertices in L', i.e., a "maximum matching" on L'. For each vertex in $L' - V(M_1)$, choose a single segment from $R - \{s^*, s_1^*\}$ that is incident on that vertex, and let M_2 denote the collection of all such chosen segments. Notice that we can always choose such a segment, since every vertex in L' has degree at least one in $G^* - (T \cup \{s^*, s_1^*\})$ and, by the maximality of M_1, no segment in $R - \{s^*, s_1^*\}$ can join two vertices in $L' - V(M_1)$. Let $M = M_1 \cup M_2$.

We now claim that there are at least $|M|$ segments in $R - (M \cup \{s^*, s_1^*\})$. First, consider the degrees of the vertices in L' in the graph formed by the segments in $R - M$. Every vertex in $L' - V(M_1)$ has degree at least two in this graph. Furthermore, at least one end vertex of each segment in M_1 has no segment from M_2 incident on it, for otherwise we could replace this segment in M_1 by two segments from M_2 (one incident on each of its end vertices), contradicting the maximality of M_1. Such an end vertex of a segment in M_1 has degree at least two in the subgraph formed by $R - M$. Thus, adding up these degrees for vertices in L', subtracting two for the contributions of s^* and s_1^*, and dividing by two, we have at least

$$\lceil (2 \cdot |M_1| + 2 \cdot |M_2| - 2)/2 \rceil = |M| - 1$$

segments in $R - (M \cup \{s^*, s_1^*\})$. In addition, if any segment in M_1 does not have two or more segments from M_2 incident on one of its end vertices, we can add one to the degree sum, thereby obtaining $|M|$ segments as required. Thus we may assume that every segment in M_1 has at least two segments from M_2 incident with one of its end vertices, and therefore its other end vertex has at least two segments from $R - M$ incident on it that are not also incident on vertices in $L' - V(M_1)$ (again by the maximality of M_1). This implies that there are at least $|M_1|$ segments in $R - M$ that do not involve vertices in $L' - V(M_1)$. Each vertex in $L' - V(M_1)$ is an end vertex for at least two segments in $R - M$, and these are all distinct since no segment joins to vertices in $L' - V(M_1)$. It follows that there are at least $|M_1| + 2 \cdot |M_2| - 2$ segments in $R - (M \cup \{s^*, s_1^*\})$. If $|M_2| \geq 2$, we are done. If $|M_2| = 1$, our assumption that every segment in M_1 has two segments from M_2 incident on one of its end vertices implies that we must have $|M_1| = 0$. Consider any leaf on T (possibly an end vertex of s_0) that is different from c_2. It has degree at least four in G^* and is an end vertex for one segment in T, at most one of s^* and s_1^*, and possibly the single segment in M. Therefore it must be an end vertex for some segment in $R - (M \cup \{s^*, s_1^*\})$ so

$$|R - (M \cup \{s^*, s_1^*\})| \geq 1 = |M|.$$

If $|M_2| = 0$, then $|M_1| \geq 1$, and both end vertices of each segment in M_1 have degree at least two in $R - M$. This implies that there are at least

$$\lceil (4 \cdot |M_1| - 2)/2 \rceil = 2 \cdot |M_1| - 1 \geq |M_1| = |M|$$

segments in $R - (M \cup \{s^*, s_1^*\})$, and the claim is proved.

Therefore, we have

$$\begin{aligned}\sum_{s\in S}|E(s)| &\leq \sum_{c_i\in L}|E(p_i)| + \sum_{s\in R}|E(s)| + |E(s_0)| \\ &\leq \sum_{s\in M}(|E(s)| + |E(p_i)| + |E(p_j)|) + |E(p_1)| + |E(p_3)| \\ &\quad + \sum_{s\in R-M}|E(s)| + |E(s_0)|,\end{aligned}$$

since every vertex in $L' = L - \{c_1,c_3\}$ has degree at least one in the graph formed by the segments in M. Applying Lemmas 3 and 6 and using the facts that $|E(s)| \leq D + 1$ for $s \neq s_0$ (which follows from Lemma 3) and

$$|R - (M \cup \{s^*,s_1^*\})| \geq |M|$$

we obtain

$$\begin{aligned}\sum_{s\in S}|E(s)| &\leq |M| \cdot (2D + 2 - |E(s_0)|) + (t - 1 - |M|) \cdot \min\{|E(s_0)|,D + 1\} \\ &\quad + |E(p_1)| + |E(p_3)| + |E(s^*)| + |E(s_1^*)| + |E(s_0)| \\ &\leq |M| \cdot (2D + 2) + (t - 1 - 2 \cdot |M|)(D + 1) + (2D + 2) \\ &\leq (t + 1)(D + 1),\end{aligned}$$

from which the desired inequality follows, completing the proof of the lower bound. ∎

Corollary 2. If t is even, $C(n,t) = n/(t + 2) + O(1)$, and if t is odd, $C(n,t) = n/(t + 1) + O(1)$.

3. REMOVING EDGES FROM A GRAPH

We now use Theorem 1 to bound the maximum increase in the diameter of a graph caused by deleting t of its edges. Let $f(t,D)$ denote the maximum possible diameter of a graph G' obtained by deleting t edges from a $t + 1$ edge-connected graph G satisfying $D(G) = D$.

Theorem 4. For $t \geq 1$, $f(t,D) = (t + 1) \cdot D + O(t)$.

Proof. We first show that $f(t,D) \leq (t + 1) \cdot D + O(t)$. Let G be any $t + 1$ edge-connected graph, and let G' be obtained by deleting any t

edges from G. Since G' is still connected, $D(G') = D'$ is finite. Let v and v' be two vertices of G' such that $d_{G'}(v,v') = D'$.

Partition the vertices in $V(G') = V(G)$ into sets V_i, $i = 0,1, \ldots, D'$, by setting $V_i = \{u{:}d_{G'}(v,u) = i\}$. Notice that, by the choice of v, each of these sets is nonempty. Let H and H' be the graphs obtained from G and G', respectively, by contracting each set V_i to a single vertex v_i and removing any loops and duplicate edges. Then we have $D(H) \leqslant D(G) = D$, since every path in G remains a path in its contracted version, and the deletion of loops and duplicate edges has no effect on the length of the shortest path joining any two vertices. Moreover, H' consists simply of the $D' + 1$ vertex path $v_0, v_1, \ldots, v_{D'}$, and H differs from H' only in that it has at most t additional edges, namely those of the t edges in $E(G) - E(G')$ that did not become loops or duplicate edges when the sets v_i were contracted. Thus H is the union of a path on $D' + 1$ vertices and at most t additional edges, so its diameter is at least $P(D' + 1,t)$. By Theorem 1, we then have

$$D \geqslant D(H) \geqslant P(D' + 1,t) \geqslant \frac{D' + 1}{t + 1} - 1.$$

It follows that

$$D' \leqslant (t + 1) \cdot D + t = (t + 1) \cdot D + O(t)$$

and hence $f(t,D)$ satisfies the required upper bound.

The following construction shows that $f(t,D) \geqslant (t + 1) \cdot D + O(t)$: Choose $x = \lfloor D/2 \rfloor - 1$. By the form of the desired result, we may assume that D is large and hence that x is positive. The vertices of G are partitioned into $(t + 1)(2x + 1)$ levels, numbered from 0 up to $(t + 1)(2x + 1) - 1$. Each level contains $t + 1$ vertices, and each vertex in the level is joined by an edge to every vertex in the preceding level and to every vertex in the succeeding level. Notice that this already ensures that G is $t + 1$ edge-connected. Finally, for $0 \leqslant k \leqslant t$, choose a designated vertex u_k from level $x + k(2x + 1)$, and add the edges $\{u_0,u_k\}$, $1 \leqslant k \leqslant t$, to G.

To see that $D(G) \leqslant D$, we need only observe that every vertex is within distance x of some designated vertex u_k (this requires $x \geqslant 1$) and that every pair of designated vertices is joined by a path of length at most two. Furthermore, when the t edges joining the designated vertices are deleted, the diameter of the resulting graph G' is

$$\begin{aligned}(t + 1)(2x + 1) - 1 &\geqslant (t + 1)(D - 2)\\ &\geqslant (t + 1) \cdot D + O(t)\end{aligned}$$

as required. ∎

Since $t + 1$ edge-connectivity was used in the upper bound proof only to ensure the connectivity of G', we have also proved

Corollary 3. Suppose that t edges are removed from a graph G. If the resulting graph G' is still connected, then it must satisfy $D(G') \leqslant (t + 1) \cdot D(G) + t$.

With respect to the extremal graph problem of Vijayan and Murty [18], we can immediately conclude that

Corollary 4. If $D' \geqslant (t + 1) \cdot D + t$, then $f(n,D,D',t) = f(n,D,D' + 1,t)$.

The bound of Theorem 4 can be made exact in the case of $t = 1$.

Theorem 5. $f(1,D) = 2D$.

Proof. By considering the cycle C_{2D+1} on $2D + 1$ vertices, we immediately have that $f(1,D) \geqslant 2D$. We use a proof by contradiction to show that $f(1,D) \leqslant 2D$.

Suppose that G is a two edge-connected graph with diameter D and that $\{u,u'\}$ is an edge of G whose removal results in a graph G' with diameter greater than $2D$. Let v and v' be two vertices of G such that $d_{G'}(v,v') \geqslant 2D + 1$, and let $a_0,a_1, \ldots, a_k$, with $a_0 = v$, $a_k = v'$ and $k \geqslant 2D + 1$, be the sequence of vertices on a shortest path p from v to v' in G'. Such a path exists since G' must still be connected.

Consider first the pair of vertices v and a_{D+1}. By the choice of p as a shortest path in G', it must be the case that $d_{G'}(v,a_{D+1}) = D + 1$, and hence any shortest path in G from v to a_{D+1} must use the edge $\{u,u'\}$. Consider such a shortest path in G, and suppose without loss of generality that it encounters u before u'. Since $D(G) = D$, it follows that

$$d_G(v,u) + 1 + d_G(u',a_{D+1}) \leqslant D$$

and that the shortest paths in G from v to u and from u' to a_{D+1} do not use the edge $\{u,u'\}$. Therefore, we have

$$d_{G'}(v,u) + d_{G'}(u',a_{D+1}) \leqslant D - 1.$$

Now consider the pair of vertices v' and a_D. Since $k \geqslant 2D + 1$, $d_{G'}(a_D,v') \geqslant D + 1$, and hence any shortest path in G from a_D to v' must use the edge $\{u,u'\}$. Applying the same argument used for v and a_{D+1} (with no assumption on which of u and u' is encountered first by the shortest path), we have that either

$$d_{G'}(a_D,u') + d_{G'}(u,v') \leq D - 1$$

or

$$d_{G'}(a_D,u) + d_{G'}(u',v') \leq D - 1.$$

If the first of these two inequalities holds, we then have $d_{G'}(v,u) \leq D - 1$ and $d_{G'}(u,v') \leq D - 1$, which implies $d_{G'}(v,v') \leq 2(D - 1)$, a contradiction. Thus we may assume the second holds. Consider the path from v to v' in G' that first goes from v to u, then from u to a_D, then along the edge $\{a_D,a_{D+1}\}$, then from a_{D+1} to u', and finally from u' to v' (see Fig. 6). The length of this path is

$$\begin{aligned} d_{G'}(v,u) + d_{G'}(u,a_D) + 1 + d_{G'}(a_{D+1},u') + d_{G'}(u',v') \\ = 1 + (d_{G'}(v,u) + d_{G'}(u',a_{D+1})) \\ + (d_{G'}(a_D,u) + d_{G'}(u',v')) \\ \leq 1 + 2(D - 1) < 2D + 1, \end{aligned}$$

which is again a contradiction. It follows that $f(1,D)$ can be no larger than $2D$. ▮

Thus the diameter of a two edge-connected graph can at most double when a single edge is removed, and this bound is tight. In fact, once again two edge-connectivity was used only to ensure that G' remains connected, so we have also proved:

Corollary 5. Suppose that a single edge is removed from a graph G. If the resulting graph G' is still connected, then it must satisfy $D(G') \leq 2 \cdot D(G)$.

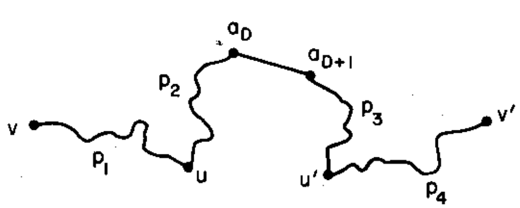

FIGURE 6. In the proof of Theorem 4, the sum of the lengths of the paths of p_1 and p_3 is at most $D - 1$ and the sum of the lengths of the paths p_2 and p_4 is at most $D - 1$, from which we obtain a path joining v and v' that has a length of at most $2 \cdot (D - 1) + 1$.

We have recently been informed that the factor of two result for removing a single edge has also been obtained independently by Plesnik [17].

4. REMOVING VERTICES FROM A GRAPH

In this section we deal with the analogous vertex removal problem, which turns out to be much simpler.

Theorem 6. Suppose G' is obtained by deleting t vertices from a λ vertex-connected graph G having n vertices, where $\lambda > t$. Then

$$D(G') \leq \left\lfloor \frac{n-t-2}{\lambda - t} \right\rfloor + 1.$$

Proof. Since G is λ vertex-connected, G' must be at least $\lambda - t$ vertex-connected. Consider a pair of vertices u and v in G' such that $d_{G'}(u,v) = D(G') = D'$. Since u and v are joined by $\lambda - t$ vertex-disjoint paths, all of which have length at least D' and hence contain at least $D' - 1$ vertices in addition to u and v, we then have

$$n - t \geq (\lambda - t)(D' - 1) + 2,$$

from which the desired inequality follows directly. ∎

Theorem 7. Suppose G' is obtained by deleting t vertices from a λ vertex-connected graph G having n vertices, where $\lambda > t$. Then

$$D(G') \leq \left\lceil \left(\left\lfloor \frac{n-t-2}{\lambda - t} \right\rfloor + 1 \right) \Big/ 2 \right\rceil \cdot D(G)$$

and this bound is achievable.

Proof. If $D(G) = 1$, then G is the complete graph K_n and G' is the complete graph K_{n-t}, so $D(G') = 1$. Thus we may assume $D(G) \geq 2$, and the upper bound follows immediately from Theorem 6.

To see that this bound is achievable, consider the following graph G: Its vertex set consists of t vertices u_i, $1 \leq i \leq t$, two vertices w_1 and w_2, and $n - t - 2$ additional vertices. These additional vertices are divided into $\lfloor (n - t - 2)/(\lambda - t) \rfloor$ levels, each of which contains at least $\lambda - t$ vertices. Each vertex in a particular level is joined by an edge to every vertex in the preceding level and to every vertex in the succeeding level. The vertex w_1 is joined by an edge to every vertex in the first level, and the vertex w_2 is joined by an edge to every vertex in the last level. Finally, each vertex u_i is joined to every other vertex in the graph by an edge.

It is straightforward to see that G is λ vertex-connected and has diameter $D(G) = 2$, since every pair of vertices is connected by a path of length two through u_1. Furthermore, when the vertices $\{u_i\}$ are removed from G, the resulting graph G' has diameter

$$D(G') = \left\lfloor \frac{n - t - 2}{\lambda - t} \right\rfloor + 1$$

as required. ∎

The key point to note here is that, unlike in the edge deletion case, the diameter of the graph G' obtained by deleting t *vertices* from G is in general unbounded in terms of t and $D(G)$.

5. SOME OPEN PROBLEMS

It is not difficult to see that many of our bounds, particularly those obtained by graph constructions, can be improved slightly, even though these improvements do not change the asymptotic behavior of the bounds. More ideas (or perhaps a large number of cases) would seem to be necessary to achieve significant improvements. In particular, it would be of great interest to determine the exact values of $P(n,t)$, $C(n,t)$, and $f(t,D)$ for general values of n, t, and D.

Beyond such obvious directions for improvement, there are two natural extensions of the problems we have considered that deserve attention. The first of these is to ask the same sorts of questions for directed graphs instead of undirected graphs; our techniques do not readily generalize to this situation, and it appears that the directed version of determining $f(t,D)$ may be more difficult. The second possible extension is to add a constraint on the maximum degree of any vertex in the graph, and then to ask similar questions to the ones in this paper with the degree bound being an additional parameter. This is particularly relevant for potential applications to network problems, since practical networks commonly satisfy such a degree constraint. More generally, one might seek refinements of our bounds that take into consideration a variety of additional graph parameters.

Finally, there are a number of related algorithmic questions that are of interest: Given t, D, and G, determine whether there exists a subgraph of G obtained by deleting t edges (vertices) that has diameter no more than D, and, if so, find such a subgraph. Given t, D, and G, determine whether there exists a supergraph of G obtained by adding t edges that has diameter no more than D, and, if so, find such a supergraph. What is the computational complexity of these problems? Are they NP-complete? Do good heuristics exist? There is much room here for further work.

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