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Unavoidable Stars in 3-Graphs

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Suppose $\mathscr{F}$ is a collection of 3-subsets of $\{1, 2,..., n\}$. The problem of determining the least integer $f(n, k)$ with the property that if $|\mathscr{F}| > f(n, k)$ then $\mathscr{F}$ contains a k-star (i.e., k 3-sets such that the intersection of any pair of them consists of exactly the same element) is studied. It is proved that, for k odd, $f(n, k) = k(k-1)\, n + \mathscr{O}(k^3)$ and, for k even, $f(n, k) = k(k-3/2)\, n + \mathscr{O}(n + k^3)$.

I. INTRODUCTION

Let H denote a 3-graph which is a collection $E = E(H)$ of 3-element subsets (called edges) of a set $V = V(H)$, the vertex set of H. A star of k edges or, in short, a k-star is a 3-graph S_k with k edges with the property that the intersection of each pair of the edges is equal to the intersection of all k edges and that this intersection has exactly one vertex. Let $f(n, k)$ denote the smallest integer m with the property that any 3-graph having n vertices and more than m edges must contain a star of k edges. Duke and Erdös [6] showed that

$$\begin{aligned} c_k n \geqslant f(n, k) &\geqslant k(k-1)(n-2k) && \text{for } k \text{ odd,} \\ &\geqslant (k-1)^2\,(n-2k+1) && \text{for } k \text{ even,} \end{aligned}$$

where c_k is some constant depending only on k.

Frankl [9] proved that

$$\begin{aligned} \tfrac{5}{3} k(k-1)\, n > f(n, k) &\geqslant k(k-1)(n-2k) && \text{for } k \text{ odd,} \\ &\geqslant k(k-3/2)(n-2k+1) && \text{for } k \text{ even.} \end{aligned}$$

In this paper we will prove that, for k odd, we have

$$k(k-1)\, n \geqslant f(n, k) \geqslant k(k-1)(n-(5k+2)/3)$$

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and, for k even, we have

$$(k-1)(k-1/2)\,n \geqslant f(n,k) \geqslant k(k-3/2)(n-k-2)+2k-3.$$

Thus, for k odd, $f(n,k)=k^2n-kn+\mathcal{O}(k^3)$ and, for k even,

$$f(n,k)=k^2n-\tfrac{3}{2}kn+\mathcal{O}(n)+\mathcal{O}(k^3).$$

II. A Weighting Function

The main idea of the proof rests on a weighting function which helps keep track of the number of edges. Roughly speaking, the weighting function distributes weights to pairs of vertices within each edge according to the frequency with which the pairs appear as subsets of edges.

We first partition the set F of all pairs of vertices in a 3-graph H into the three classes. We say a pair of vertices $\{u,v\}$ is in A if $z(u,v)=|\{w:\{u,v,w\}\in E(H)\}|\geqslant 2k-1$. We say $\{u,v\}$ is in B if $2k-1>z(u,v)\geqslant k$. A pair $\{u,v\}$ is said to be in C if it is not in A or B.

Now we define a weighting function $w: E(G)\times F\rightarrow R$, the set of nonnegative reals, as follows:

(i) $w(e,f)=0$ if f is not a subset of e.

(ii) Suppose f_1, f_2, f_3 are the three pairs in e, with

$$z(f_1)\geqslant z(f_3)\geqslant z(f_3).$$

(a) All f_i are in A or all f_i are in C,

$$w(e,f_i)=\tfrac{1}{3} \qquad \text{for all } i;$$

(b) $f_1, f_2\in A$ and $f_3\notin A$,

$$w(e,f_1)=w(e,f_2)=\tfrac{1}{2} \qquad \text{and} \qquad w(e,f_3)=0;$$

(c) $f_1\in A, f_2\in B$, and $f_3\in B\cup C$,

$$\begin{aligned} &w(e,f_1)=\tfrac{3}{4},\\ &w(e,f_i)=\tfrac{1}{4} \qquad \text{if } f_i\in B,\\ &w(e,f_i)=0 \qquad \text{if } f_i\in C,\end{aligned}$$

(d) $f_1\in A\cup B$ and $\{f_2,f_3\}\subset C$,

$$w(e,f_1)=1 \qquad \text{and} \qquad w(e,f_2)=w(e,f_3)=0,$$

(e) $f_1 \in B, f_2 \in B, f_3 \in B \cup C$,

$$w(e,f_1) = \frac{1}{2} + \frac{z(f_1) - z(f_2)}{4(k-1)},$$

$$w(e,f_2) = \frac{1}{2} + \frac{z(f_2) - z(f_1)}{4(k-1)},$$

$$w(e,f_3) = 0.$$

Lemma 1 follows immediately from the definition.

LEMMA 1. *For a fixed edge e in $E(H)$, we have*

$$\sum_f w(e,f) \geqslant 1.$$

Hence we have

LEMMA 2.

$$\sum_e \sum_f w(e,f) \geqslant |E(H)|.$$

III. THE NEIGHBORHOOD GRAPH

For a vertex V in $V(H)$, we define $N(v)$ to be the set of the pairs f with the property that the union of v and f is an edge in H. The 2-graph formed by $N(v)$ is called the neighborhood graph of v in H. The following observations can then be made:

LEMMA 3.

$$\sum_e \sum_f w(e,f) = \sum_v \sum_{f \in N(v)} w(\{v\} \cup f, f).$$

LEMMA 4. *Suppose H does not contain a k-star as a subgraph. Then for any vertex v, the neighborhood graph of v does not contain a matching of k edges. (A matching is a set of vertex-disjoint edges.)*

We will make use of the following theorem of Berge's [2] on maximum matchings in a 2-graph.

THEOREM A. *In a 2-graph G, the number of edges in a maximum matching is equal to*

$$\min_S (|S| + \tfrac{1}{2}(n - |S| - c(G-S))),$$

where S ranges over all subsets of $V(G)$ and $c(G-S)$ denotes the number of odd components in the induced subgraph on $V(G)-S$ in G.

Also we will use the following result on matchings and degrees [1, 5]:

THEOREM B. *A 2-graph G on n vertices can have at most* $\min\{n(k-1), (k-1)(2k-1)\}$ *edges if G does not contain a matching of k edges and the maximum degree in G is at most* $2k-2$.

IV. AUXILIARY PROPERTIES

Here we will mention several important facts about graphs which do not contain a k-start as a subgraph.

LEMMA 5. *Suppose H does not contain a k-star as a subgraph. Then for any vertex v there are fewer than k vertices u such that* $\{u, v\}$ *is in A.*

Proof. Suppose the contrary. We have u_i, $1 \leqslant i \leqslant k$, such that $\{u_i, v\}$ is in A. Then for each i the set $\{y : \{y, v, u_i\} \in E(H), y \notin \{u_1, ..., u_k\}\}$ has at least k elements. Using Hall's theorem on systems of distinct representatives [10], we can then construct a k-star with v as the center, which leads to a contradiction. Lemma 5 is proved.

LEMMA 6. *Suppose H does not contain a k-star as a subgraph. For a fixed vertex v, let* r_1 *denote the number of u with* $\{u, v\}$ *in A and* r_2 *denote the number of* u' *with* $\{u', v\}$ *in B. Then we have*

$$r_1 + r_2/2 \leqslant 2(k-1).$$

Proof. Suppose to the contrary that $r_1 + r_2/2 > 2(k-1)$. Since $r_1 \leqslant k-1$, we have $r_2 > 2(k-1) \geqslant 2$ (may assume $k > 1$). We consider the neighborhood graph G of v in H.

By Theorem A we know that there is a ubset S of $V(G)$ satisfying

$$|S| + \tfrac{1}{2}(n - |S| - c(G-S)) \leqslant k-1.$$

If two of the connected components of $G-S$ contain a vertex u with $\{u, v\}$ in $A \cup B$, then

$$|S| + \tfrac{1}{2}(n - |S| - c(G-S)) \geqslant |S| + \tfrac{1}{2}(2(k+1-|S|)),$$

which is impossible.

We may assume all u with $\{u, v\} \in A \cup B$ are in one connected component of $G - S$. Therefore, we have

$$|S) + \frac{1}{2}(n - |S| - c(G - S)) \geqslant |S| + \frac{1}{2}(r_1 + r_2 - |S| - 1)$$
$$\geqslant \frac{|S|}{2} + k - 1 + \frac{1}{2}\left(\frac{r_2}{2} - 1\right)$$
$$> k - 1.$$

This leads to a contradiction. Lemma 6 is proved.

LEMMA 7. *Suppose H does not contain a k-star as a subgraph. For any two vertices u and v, we have*

$$\sum_y w(\{y, u, v\}, \{y, u\}) \leqslant k - 1.$$

Proof. We consider the following possibilities:

Case 1.

$\{u, v\}$ is in C. There are fewer than k y's such that $w(\{y, u, v\}, \{y, u\}) > 0$. Thus we have

$$\sum_y w(\{y, u, v\}, \{y, u\}) \leqslant k - 1.$$

Case 2.

$\{u, v\}$ is in A. Let r_1 denote the number of y with $\{y, u\}$ in A and r_2 denote the number of y' with $\{y', u\}$ in B. Then from Lemma 6 we have

$$\sum_y w(\{y, u, v\}, \{y, u\}) \leqslant r_1/2 + r_2/4$$
$$\leqslant k - 1.$$

Case 3.

$\{u, v\}$ is in B. Let y_i, $1 \leqslant i \leqslant t$, denote vertices y satisfying $\{y, u, v\}$ in $E(H)$. In particular, we assume $z(y_i, u)$ is nonincreasing; say, $\{y_i, u\} \in A$ for $1 \leqslant i \leqslant a$, $\{y_i, u\} \in B$ for $a < i \leqslant a + b$, and $\{y_i, u\} \in C$ for $a + b < i \leqslant t \leqslant 2k - 2$.

Let m denote the largest integer j such that $|\{i : a < i \leqslant a + b, z(y_i, u) \geqslant k + j - 1\}| \geqslant j$. If $m \geqslant k$, by Hall's theorem we have a k-star in H with u as the center which is impossible. Thus we have $m \leqslant k - 1$ and $a \geqslant k - m - 1$.

Let z denote $z(u, v)$. We have

$$X=\sum_{y} w(\{y, u, v\}, \{y, u\})$$

$$\leqslant \frac{3a}{4}+\sum_{a<i\leqslant a+m} w(\{y_i, u, v\}, \{y_i, u\})+\sum_{a+m<i\leqslant a+b} w(\{y_i, u, v\}, \{y_i, u\})$$

$$\leqslant \frac{3a}{4}+m\left(\frac{1}{2}+\frac{2k-2-z}{4(k-1)}\right)+(b-m)\left(\frac{1}{2}+\frac{k+m-1-z}{4(k-1)}\right).$$

Since $a+b\leqslant z$ and $a\leqslant k-m-1$, we have

$$X\leqslant \frac{3(k-m-1)}{4}+m\left(\frac{1}{2}+\frac{2k-2-z}{4(k-1)}\right)$$

$$+(z-k+1)\left(\frac{1}{2}+\frac{k+m-1-z}{4(k-1)}\right)$$

$$=F(m, z).$$

Note that

$$\frac{\partial F}{\partial z}(m, z)=\frac{4(k-1)-2z}{4(k-1)}\geqslant 0.$$

Since $z\leqslant 2(k-2)$, we have

$$X\leqslant F(m, 2(k-1))\leqslant k-1.$$

This completes the proof of Lemma 7.

V. The Main Theorems

Now we are ready to prove the main theorems.

THEOREM 1. *A 3-graph H on n vertices contains a k-star if H has more than $k(k-1)\,n$ edges for k odd or has more than $(k-1)(k-1/2)\,n$ edges for k even.*

Proof. Suppose H is a 3-graph which does not contain a k-star. From Lemmas 2 and 3, we have

$$|E(H)|\leqslant \sum_{v}\sum_{f} w(\{v\}\cup f, f).$$

It suffices to show that for each fixed v we have

$$\sum_{f\in N(v)} w(\{v\}\cup f,f)\leqslant k(k-1) \qquad \text{if } k \text{ is odd,}$$

$$\leqslant (k-1)(k-1/2) \qquad \text{if } k \text{ is even.}$$

Let v be a fixed vertex and G denote its neighborhood graph. Now we use Lemma 4 and Theorem A, which imply the existence of a subset S of $V(G)$ such that

$$|S|+\tfrac{1}{2}(n-|S|-c(G-S))\leqslant k-1.$$

Suppose that $G-S$ contains j_i connected components of size i. Let $s=|S|$. We then have

$$\sum_{i\text{ even}} ij_i+\sum_{i\text{ odd}}(i-1)j_i\leqslant 2(k-s-1). \tag{$*$}$$

Let q_i denote the number of vertices in C_i. There are at most two largest connected components in $G-S$ containing more than k vertices.

We consider the following cases:

Case 1.

k is odd. Let G' denote $G-S-C_0-C_1$. We then have

$$\sum_{f\in E(G')} w(\{v\}\cup f,f)\leqslant \sum_{i\neq q_1,q_2}\frac{i(i-1)}{2}j_i$$

$$\leqslant (k-1)\sum_{i\neq q_1,q_2}\frac{(i-1)}{2}j_i.$$

$$\sum_{f\in E(C_0\cup C_1)} w(\{v\}\cup f,f)=\frac{1}{2}\sum_{u\in V(C_0\cup C_1)}\sum_{y} w(\{y,u,v\},\{y,u\})$$

$$\leqslant\frac{(k-1)(q_1+q_2)}{2} \qquad \text{(using Lemma 7).}$$

$$\sum_{f\cap S\neq\varnothing} w(\{v\}\cup f,f)\leqslant (k-1)\,s \qquad \text{(again by Lemma 7).}$$

Therefore we have, by $(*)$, that

$$\sum_{f\in N(v)} w(\{v\}\cup f,f)\leqslant (k-1)\sum_{i}\frac{(i-1)}{2}j_i+(k-1)\,s+k-1$$

$$\leqslant (k-1)\,k.$$

Case 2.

k is even. There is at most one connected component C_0 containing q, $q \geqslant k$, vertices because of $(*)$ and k being even. We then have

$$\sum_{f \in E(G-S-C_0)} w(\{v\} \cup f, f\} \leqslant \sum_{i<k} \frac{i(i-1)}{2} j_i \\ \leqslant (k-1) \sum_{i<q} \frac{(i-1)}{2} j_i.$$

Using Lemma 7, we have

$$\sum_{f \in E(C_0)} w(\{v\} \cup f, f) = \frac{1}{2} \sum_{u \in V(C_0)} \sum_{y} w(\{y, u, v\}, \{y, u\}) \\ \leqslant \frac{q(k-1)}{2},$$

$$\sum_{f \cap S \neq \varnothing} w(\{v\} \cup f, f) \leqslant (k-1)\, s.$$

Therefore,

$$\sum_{f \in N(v)} w(\{v\} \cup f, f) \leqslant (k-1) \sum_{i} \frac{(i-1)}{2} j_i + (k-1)\, s + \frac{k-1}{2} \\ \leqslant (k-1)(k-1/2).$$

This completes the proof of Theorem 1.

THEOREM 2. *There exist* 3-*graphs which contain no* k-*star and contain* $k(k-1)(n-(5k+2)/3)$ *edges for* k *odd or* $k(k-3/2)(n-k-2)+2k-3$ *edges for* k *even.*

Proof. First we consider a graph H_1 on n vertices for k odd described as follows:

Let G_1 denote the disjoint union of two copies of complete 2-graphs on k vertices, say K_1 and K_2. Let $V(H_1)$ contain $V(G_1)$ and $n-2k$ additional vertices, with

$$E(H_1) = \{f \cup \{v\} : f \in E(G_1) \text{ and } v \in V(H_1) - V(G_1)\} \\ \cup \{\{v_1, v_2, v_3\} : \{v_1, v_2, v_3\} \subset V(K_i) \text{ for } i = 1 \text{ or } 2\}.$$

Now, for $v \in V(H_1) - V(G_1)$, the neighborhood graph of v in H is G_1, which does not contain a matching of k edges. For $v \in V(K_i)$, $i = 1$ or 2, the neighborhood graph does not contain a matching of k edges since we have $|S| + \frac{1}{2}(n - |S| - c(G_1 - S)) = k - 1$ by taking $S = V(K_i) - \{v\}$. Therefore H_1 does not contain a k-star.

It is straightforward to check that

$$\begin{aligned}|E(H)| &= k(k-1)(n-2k) + 2\binom{k}{3} \\ &= k(k-1)\left(n - \frac{(5k+2)}{3}\right).\end{aligned}$$

Now, suppose k is even. We then construct H_2 as follows:

Let $V(H_2) = \{v_1, v_2, \ldots, v_n\}$. First consider a 2-graph G_2 with vertex set $\{v_1, v_2, \ldots, v_{2k-1}\}$ such that

$$\begin{aligned}E(G_2) = \{&\{v_i, v_j\} : 1 \leqslant i \leqslant k < j \leqslant 2k-1, j \neq k+i\} \\ &\cup \{\{v_{2i-1}, v_{2i}\} : i = 1, 2, \ldots, k/2 - 1\}.\end{aligned}$$

It is easily seen that the degree of v_i, $i \neq k-1$, is $k-1$ and the degree of v_{k-1} is $k-2$ in G_2. Now we define

$$\begin{aligned}E(H_2) = \{&f \cup \{v\} : f \in E(G_2) \text{ and } v \in V(H) - V(G_2)\} \\ &\cup \{\{v_i, v_j, v_m\} : (1 \leqslant i < j \leqslant k < m \leqslant 2k-1 \\ &\quad \text{or } 1 \leqslant i \leqslant k < j < m \leqslant 2k-1) \\ &\quad \text{and } \{v_p, v_{k+p}\} \not\subset \{v_i, v_j, v_m\} \text{ for} \\ &\quad \text{all } p = 1, 2, \ldots, k-1\}.\end{aligned}$$

For $v \in V(H_2) - V(G_2)$, the neighborhood graph is G_2, which does not contain a matching with k edges. For v_i, $1 \leqslant i \leqslant k-1$, all the edges in the neighborhood graph are incident to some vertex in $\{v_{k+1}, v_{k+2}, \ldots, v_{2k-1}\} \cup \{v_j\} - \{v_{k+i}\}$, where $j = i - 1 + 2(i - 2\lfloor i/2 \rfloor)$, and thus the neighborhood graph contains no matching of k edges (by applying Theorem A). For v_k, all the edges in the neighborhood graph are incident to $\{v_{k+1}, \ldots, v_{2k-1}\}$ and thus it contains no matching of k edges. For v_i, $k+1 \leqslant i \leqslant 2k-1$, all the edges in the neighborhood graph are incident to $\{v_1, \ldots, v_k\} - \{v_{i-k}\}$ and again it contains no matching of k edges. Thus H_2 does not contain a k-star.

It remains to check the number of edges in H_2.

$$|E(H_2)| = |E(G_2)|\,(n-2k+1) + k\binom{k-1}{2}$$

$$+ (k-1)\binom{k}{2} - (k-1)(2k-3)$$

$$= \frac{((2k-2)(k-1)+k-2)}{2}(n-2k+1) + (k-1)(k-3/2)(k-2)$$

$$= k(k-3/2)(n-k-2) + 2k-3.$$

This completes the proof of Theorem 2.

VI. Concluding Remarks

In this paper we obtained upper bounds and lower bounds for $f(n, k)$ which differ by $\mathcal{O}(n+k^3)$. It would be of interest to reduce this gap.

A generalized version of the problem we discussed can be stated as follows [6]:

Let $f(n, r, k, t)$ denote the smallest integer m with the property that any r-graph containing m edges must contain a strong Δ-system of type (r, k, t); i.e., k r-edges with the property that the intersection of any pair of these k edges is equal to the intersection of all k edges, and the intersection has t vertices [7, 8]. Note that $f(n, k)$ is just $f(n, 3, k, 1)$. The problem of determining $f(n, r, k, t)$ is one of the fundamental problems in extremal set theory. However, relatively little is known.

Erdös and the author [3, 4] recently investigated a related extremal graph problem: An r-graph is said to be (n, e)-unavoidable if it is contained in every r-graph with n vertices and e edges. The problem just considered can be viewed as determining the maximum unavoidable stars or strong Δ-systems. However, in many cases, the maximum unavoidable graphs are not strong Δ-systems. In [3, 4] some exact results and some sharp bounds have been obtained for the number of edges in a maximum unavoidable graph for the cases of $r = 2$ and 3. Numerous problems on this topic remain unsolved.

Note added in proof. P. Frankl and the author have recently improved the bounds for $f(n, k)$. In particular, it is proved that the value of $f(n, k)$, for k odd, is equal to the lower bound for $f(n, k)$ given in Section I.

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