5-9

A Note on Subtrees in Tournaments

F. R. K. Chung

Bell Laboratories
Murray Hill, New Jersey 07974

ABSTRACT

In a tournament every pair of distinct vertices is joined by exactly one directed edge. By a directed tree we mean a tree with its edges given some prescribed orientations (not just an arborescence). In this note we prove that every tournament on n vertices contains all directed trees on k vertices as subgraphs if $n \geq ck^{1+1/(\log k)^{1/2-\epsilon}}$ for any positive ϵ and some constant c depending on ϵ.

A Note on Subtrees in Tournaments

F. R. K. Chung

Bell Laboratories
Murray Hill, New Jersey 07974

I. Introduction

A tournament T_n consists of n vertices such that every pair of distinct vertices is joined by exactly one directed edge. By a directed tree we mean a tree with its edges given some prescribed orientations, not just an arborescence [2]. In this paper we will prove

Theorem: Every tournament on n vertices contains all directed trees on k vertices as subgraphs if $n \geq k^{1+1/(\log k)^{1/2-\epsilon}}$ for any positive ϵ and some constant c depending on ϵ.

This problem can be viewed as a generalized version of the following problem [8]. Does the tournament T_n contain all n-paths of all possible orientations?

Grünbaum [6], Rosenfeld [7] and Alspach [1] proved that certain special paths are contained in T_n. Forcade [5] proved that if n is a power of 2 then T_n contains all paths of all possible orientations. For general n, it is still not proved.

Burr [3] considers a variation of this problem by showing that any directed graph with its chromatic number no less than $(k-1)^2$ contains all directed trees on k vertices. The following problems [3] still remain unsettled:

Does every graph of chromatic number k or more contain all k paths of all possible orientations?

Does every graph of chromatic number $2k$ or more contain all directed trees with n vertices?

II. Preliminaries:

Let $f(k)$ denote the smallest integer m such that any tournament T_m contains all directed trees on k vertices. It is proved in [3] that $2k \leq f(k)$. Before we establish the upper bound of $f(k) \leq k^{1+1/(\log k)^{1/2-\epsilon}}$, we first require the following useful facts:

Lemma 1 [4]: Suppose t is a tree with more than k vertices. Then for some vertex v of t, there is a set S of subtrees which are connected components in $t - \{v\}$ (the forest obtained by removing v and the edges incident to v from t) so that

$$k < \sum_{t' \in S} |V(t')| \leq 2k \ .$$

where $|V(t')|$ denotes the number of vertices in the subtree t'.

Lemma 2: Suppose T_n is an arbitrary tournament. There are at most $2m$ vertices in T_n with indegree less than m.

Proof: Suppose there are $2m+1$ vertices in T_n with indegree less than m. Consider the subtournament on these $2m+1$ vertices. At least one vertex has indegree greater than or equal to the average indegree m of this subtournament. This is impossible. Lemma 2 is proved.

Symmetrically, we have the following:

Lemma 3: Suppose T_n is an arbitrary tournament. There are at most $2m$ vertices in T_n with outdegree less than m.

Lemma 4: For $1 \leq m < k$, we have

$$f(k) \leq 4k + 4f(2m) + f(k-m-1) .$$

Proof: Let T denote an arbitrary tournament on $4k + 4f(2m) + f(k-m-1)$ vertices. Let t denote an arbitrary tree on k vertices. It suffices to prove that T contains t. From Lemma 1, we can find v in t such that there is a set S of subtrees which are connected components in $t - \{v\}$ and

$$m < \sum_{t' \in S} |V(t')| \leq 2m .$$

Let $\bar{t}$ denote the tree which is formed by removing all t' in S from t. Then we know that $|V(\bar{t})| \leq k - m - 1$. From Lemma 2 and 3, we know that there at most $4k + 4f(2m)$ vertices in T with indegree or outdegree less than $k + f(2m)$. Now consider the subtournament T' of T on vertices with indegree and outdegree no less than $f(2m) + k$. Since T' contains at least $f(k-m-1)$ vertices, T' contains $\bar{t}$ and v is embedded into a vertex with indegree and out degree no less than $k + f(2m)$. Let F_1 denote the forest consisting of those trees in S joined to v by an edge from v and let F_2 denote the forest consisting of those trees in S joined to v by an edge to v. Now we know that there are at least $f(2m)$ vertices in $T - \bar{t}$ joined by an edge from v. ($T-\bar{t}$ denote the subtournament of T which does not contain any vertex in $\bar{t}$).

Thus the subtournament $T-\bar{t}$ must contain F_1. Similarly there are at least $f(2m)$ vertices in $T-\bar{t}-F_1$ joined by an edge to v and the subtournament on these $f(2m)$ vertices must contain F_2. Therefore T contains t and Lemma 4 is proved.

III. On the upper bound

We are now ready to prove the main theorem.

Theorem 1: $f(k) \le ck^{1+1/(\log k)^{1/2-\epsilon}}$ for any positive ϵ and some constant c.

Proof: Let m denote $k^{1-1/\sqrt{\log k}}$ and $g(x)$ denote $x^{1+1/(\log x)^{1/2-\epsilon}}$. We want to show that $f(k) \le cg(k)$. We will first prove the following.

Claim 1: $4k \le g(2m)$ for sufficiently large k.

Proof:

$$g(2m) \ge g(m) \ge \exp\left(\left[\log k - \sqrt{\log k}\right]\left[1+1/(\log k - \sqrt{\log k})^{1/2-\epsilon}\right]\right)$$

$$\ge \exp\left[\log k + 0.5(\log k)^{1/2+\epsilon}\right]$$

$$\ge 4k \quad \text{for } k \text{ sufficiently large}.$$

Claim 2: $4k + 4g(2m) + g(k-m) \le g(k)$ for sufficiently large k.

Proof: Since $4k \le g(2m)$ for large k, it is then enough to show that

$$5g(2m) + g(k-m) \le g(k)$$

In the following sequence of inequalities, each one is implied by the following one (always for sufficiently large x):

$$x^{1+1/(\log x)^{1/2-\epsilon}} - \left(x - x^{1-1/\sqrt{\log x}}\right)^{1+1/\left(\log x + \log\left(1-x^{-1/\sqrt{\log x}}\right)\right)^{1/2-\epsilon}}$$

$$\ge 5\left(2x^{1-1/\sqrt{\log x}}\right)^{1+1/\left(\left(1-1/\sqrt{\log x}\right)\log 2x\right)^{1/2-\epsilon}} .$$

$$\exp\left\{(\log x)^{1/2+\epsilon}\right\} - \left(1-x^{-1/\sqrt{\log x}}\right)\exp\left\{\left(\log x + \log\left(1-x^{-1/\sqrt{\log x}}\right)\right)^{1/2+\epsilon}\right\}$$

$$\ge 11\exp\left\{-\sqrt{\log x} + \left(\left(1-1/\sqrt{\log x}\right)\log 2x\right)^{1/2+\epsilon}\right\} .$$

$$\exp\left\{-\sqrt{\log x} + (\log x)^{1/2+\epsilon}\right\} \ge 11\exp\left\{-\sqrt{\log x} + (\log x)^{1/2+\epsilon}\left(1-0.5/\sqrt{\log x}\right)\right\} .$$

$$-\sqrt{\log x} \ge -\sqrt{\log x} - 0.5(\log x)^{\epsilon} + 20$$

which clearly holds for large x.

For a suitable fixed N (chosen large enough so that the preceding approximations are valid), we have $f(k) \le ck^{1+1/(\log k)^{1/2-\epsilon}}$ for all k satisfying $2 \le k \le N$ (by an appropriate choice of c). Thus, by using Lemma 4 we have

$$f(k) \le 4k + 4f(2m) + f(k-m-1) \le 4k + 4g(2m) + g(k-m-1)$$

$$\le ck^{1+1/(\log k)^{1/2-\epsilon}} .$$

This completes the proof of the main theorem.

References

1. B. Alspach and M. Rosenfeld, Realization of certain generalized paths in tournaments, Discrete Math 34, (1981), 199-202.
2. C. Berge, Graphs and Hypergraphs, (1973), Elsevier North-Holland Inc.
3. S. A. Burr, Subtrees of directed graphs and hypergraphs.
4. F. R. K. Chung and R. L. Graham, On graphs which contain all small trees, J. Comb. Theory (B), (1978), 14-23.
5. R. Forcade, Parity of paths and circuits in tournaments, Discrete Math. 6, (1973), 115-118.
6. B. Grünbaum, Antidirected hamiltonian paths in tournaments, J. Comb. Theory (B), 11(1971), 249-257.
7. M. Rosenfeld, Antidirected hamiltonian paths in tournaments, J. Comb. Theory (B) 12, (1972), 93-99.
8. M. Rosenfeld, Antidirected hamiltonian circuits in tournaments, J. Comb. Theory (B), 16, (1974), 234-242.