On a Ramsey-Type Problem

F. R. K. Chung
BELL LABORATORIES, MURRAY HILL, NJ 07974

ABSTRACT

Suppose a graph G has the property that if one colors the edges of G in r colors, there always exists a monochromatic triangle. Is it true that if one colors the edges of G in $r+1$ colors so that every vertex is incident to at most r colors then there must be a monochromatic triangle? This problem, which was first raised by P. Erdös, is answered in the negative here.

1. INTRODUCTION

Suppose G is a graph* satisfying the property that if one colors the edges of G in r colors there always is a monochromatic triangle. Is it true that if one colors the edges of G in $r+1$ colors so that every vertex is incident to at most r colors then there must be a monochromatic triangle?

In this paper, we solve the above problem, which was raised by Erdös, by showing that there is a graph G so that any 2-coloring of G contains a monochromatic triangle but there is a 3-coloring of G without a monochromatic triangle so that every vertex is incident to at most two colors. A similar solution was independently obtained by H. Enomoto [3].

This problem came up in connection with some work of Erdös, Hajnal, Szemerédi, and Sós [see 4]. In particular, the following interesting problem still remains unsolved:

Let n_r be the smallest integer for which if one colors the edges of $K(n_r)$ by r colors, there always is a monochromatic triangle. Is it true that if one colors the edges of $K(n_r)$ by $r+1$ colors so that every vertex is incident to at most r colors then there must be a monochromatic triangle?

Note that this is true for $r=2$ since it is easy to check that any 3-coloring of K_6 without a monochromatic triangle must contain a vertex incident to 3 colors.

*In general, we follow the terminology of [6].

Journal of Graph Theory, Vol. 7 (1983) 79–83
© 1983 by John Wiley & Sons, Inc. CCC 0364-9024/83/010079-05$01.50

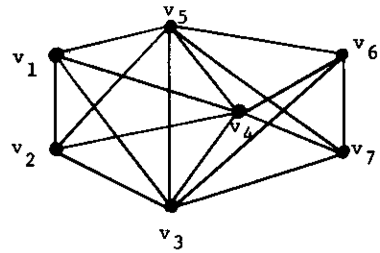

FIGURE 1

2. THE CONSTRUCTION OF G

First we construct several special subgraphs which will be used subsequently as parts of G.

(1) Graph F as shown in Figure 1.

F has the vertex set $\{v_i : i = 1, \ldots, 7\}$ and is such that the induced subgraphs on $\{v_i : 1 \leq i \leq 5\}$ and $\{v_i : 3 \leq i \leq 7\}$ are K_5's. F is called a signal sender by Burr [1] who proved the following:

Lemma 1. $\{v_1, v_2\}$ and $\{v_6, v_7\}$ in F always have the same color if F is 2-colored without a monochromatic triangle.

Proof. It is easy to see that if two of the three edges in the triangle $v_3v_4v_5$ are colored by color a then so are $\{v_1, v_2\}$ and $\{v_6, v_7\}$. ■

(2) Graph H as shown in Figure 2.

Roughly speaking, H contains two copies of F in such a way that if H is colored in two colors without a monochromatic triangle, then $\{v_1, v_2\}$ and $\{u_1, u_3\}$ have the same color. Also $\{v_1, v_2\}$ and $\{u_2, u_3\}$ have the same color. Thus we conclude:

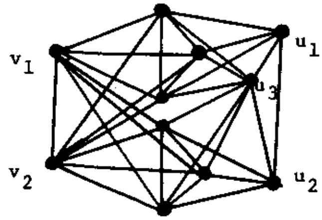

FIGURE 2

FIGURE 3

Lemma 2. If H is 2-colored without a monochromatic triangle, then $\{v_1,v_2\}$ and $\{u_1,u_2\}$ have different colors.

(3) Graph J as shown in Figure 3.
The vertex set $V(J)$ can be partitioned into X_J and Y_J in such a way that the induced subgraphs on X_J and Y_J are 5-cycles and there is a complete bipartite subgraph $K_{5,5}$ between X_J and Y_J.

Lemma 3. Suppose J is 2-colored. If the edges of the induced subgraph on X_J have color a and the edges of the induced subgraph on Y_J have color b, $a \neq b$, then there is a monochromatic triangle.

Proof. Of the 25 edges joining vertices in X_J to vertices in Y_J at least 13 edges are of one color, say a. There must be a vertex y in Y_J incident to three edges in color a connecting y to x_1,x_2,x_3, where $x_i \in X_J$. Since the independence number of C_5 is 2, there exists one edge in the induced subgraph on $\{x_1,x_2,x_3\}$. This forces a monochromatic triangle in color a. ■

Now we are ready to construct G by putting together 3 copies of J, 3 copies of H and 30 copies of F appropriately as follows (see Fig. 4, where copies of F are used to join copies of H to copies of X_J and Y_J as specified in (i), (ii) below):

(i) The edge $\{v_1,v_2\}$ of H_i is joined to every edge $\{y_j,y_k\}$ in $E(J_i)$, where

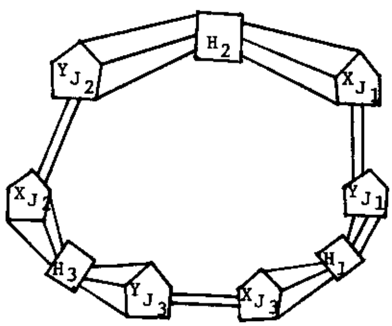

FIGURE 4

$y_j, y_k \in Y_{J_i}$ by a copy of F, i.e., $\{v_1, v_2\}$ of H_i is identified with $\{v_1, v_2\}$ of F and $\{v_6, v_7\}$ of F is identified with $\{y_j, y_k\}$ of J_i. We note that i ranges from 1 to 3.

(ii) The edge $\{u_2, u_2\}$ of H_i is joined to every edge $\{x_j, x_k\}$ in $E(J_{i-1})$, where $x_j, x_k \in X_{J_{i-1}}$ by a copy of F, $i = 1,2,3 \pmod 3$.

We want to show that G can be 3-colored without monochromatic triangles so that every vertex is incident to at most two colors. Now, we consider the following 3-coloring:

For $i = 1,2,3 \pmod 3$, $\{x, y\} \in E(J_i)$ is in color i; $\{x_j, x_{j'}\} \in E(J_i)$ is in color $i + 1$; $\{y_k y_{k'}\} \in E(J_i)$ is in color $i - 1$ where x's are in X_{J_i} and y's are in Y_{J_i}.

Also edges of H_i are in either color i or color $i - 1$ such that $\{u_1, u_2\}$ is in color i and $\{v_1, v_2\}$ is in color $i - 1$.

It is easy to see that all other edges can be colored in such a way that there does not exist a monochromatic triangle and every vertex is incident to two colors.

Theorem. There exists a graph G satisfying the following properties:

(i) If one colors the edges of G in two colors, there exists a monochromatic triangle.

(ii) G can be 3-colored so that there is no monochromatic triangle and every vertex is incident to two colors.

Proof. It suffices to prove (i) for the graph G constructed in this section. Suppose we 2-color the edges of G without forming a monochromatic triangle. From Lemma 1 we know that all edges within each X_{J_i} and within each Y_{J_i} are of the same color. Furthermore, from Lemma 3 we know that the edges of X_{J_i} have the same color as the edges of Y_{J_i}. Lemma 2 guarantees that the color of the edges of X_{J_i} differs from that of the edges of $Y_{J_{i+1}}$ for $i = 1,2,3 \pmod 3$. Since only two colors are available, this leads to a contradiction, and the theorem is proved. ■

3. CONCLUDING REMARKS

There are many interesting related problems, several of which we now mention.

(1) Let $f_r(H)$ be the smallest integer n for which if one colors the edges of K_n by r colors, there always is a monochromatic subgraph isomorphic to H. Is it true that if one colors the edges of K_n by $r + 1$ colors, where $n = f_r(H)$, such that every vertex is incident to at most r colors, then there always exists a monochromatic subgraph isomorphic to H? If so, let us say that H is r-admissible. Some graph H are not r-admissible; for example, let $H = P_5$, a path on five vertices. It is known [5] that $f_2(P_5) = 6$. However, K_6 can be 3-

FIGURE 5

colored such that every vertex is incident to two colors and there is no monochromatic P_5 (see Fig. 5).

The problem of Erdös, Hajnal, and Sós, mentioned in Sec. 1, is just determining whether K_3 is r-admissible. It would be of interest to characterize the class of all r-admissible graphs.

(2) A graph S is said to be a signal sender of type (r,H) if there exists a pair of edges in S, both of which have the same color when one colors edges of S in r colors without a monochromatic H. The graph in Figure 1 is a signal sender of the type $(2,K_3)$. It is not hard to construct a signal sender S of type $(3,K_3)$ as follows: Let $V(S) = \{1,2,\ldots,18\}$ and the induced subgraph S_1 on $\{1,2\} \cup \{5,6,\ldots,18\}$ is K_{16}, as is the induced subgraph S_2 on $\{3,4,\ldots,18\}$. Also every edge of S is either in S_1 or S_2. It is easy to check that $\{1,2\}$ has the same color as the color which occurs in more than a third of the edges of the induced subgraph on $\{5,6,\ldots,18\}$. The edge $\{3,4\}$ also has this property. Does there exist a signal sender of the type (r,K_3) for arbitrary r? In general, does there exist a signal sender of any type (r,H)? Burr et al. [2] have proved the existence of signal senders of type $(2,H)$ for all 3-connected graphs H. We remark that these signal senders, if they exist, can then be used to generalize our main theorem.

References

[1] S. A. Burr, private communication.

[2] S. A. Burr, J. Nešetil and V. Rödl, On the use of senders in generalized ramsey theory for graphs. Preprint.

[3] H. Enomoto, private communication.

[4] P. Erdös and V. T. Sós, Problems and results in Ramsey-Turán type theorems. *Proceeding of the West Coast Conference on Combinatorics, Graph Theory and Computing*, pp. 17–23 (1979).

[5] L. Gerencsér and A. Gyárfas, On Ramsey-type problems. *Ann. Univ. Sci. Budapest, Eötvös Sect. Math* 10, (1967) 167–170.

[6] F. Harary, *Graph Theory.* Addison Wesley, Reading, MA, 1972.