Some Problems and Results in Labelings of Graphs

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ABSTRACT

Suppose we label the vertices of a graph by distinct integers. The *weight* of an edge is defined to be the absolute value of the difference of the labels of its endpoints. We define the *cost* of the labeling to be the sum of weights of all the edges.

The problem we consider is that of determining for a given graph G the minimum cost labelings of G. In this paper we survey recent progress on this and related problems.

1.   <u>Introduction.</u>

For graph-theoretic terminology, the reader may consult [16]. Suppose G is a finite graph with vertex set V(G) and edge set E(G). A *labeling* π is a one-to-one mapping from V(G) to the positive integers. We define the λ-*weight* of an edge

$$e = \{u,v\} \in E(G)$$

to be $|\pi(u)-\pi(v)|^{\lambda}$. The *weight* of an edge is defined to be the 1-weight of the edge.

Many interesting problems in graph theory deal with weighted graphs. One of the most intriguing of such problems is the graceful labeling problem, i.e., determining the existence of a labeling of a graph G with weights of all edges being consecutive

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integers ranging from 1 to $|E(G)|$. The graceful labeling problem
has an extensive literature ([1,8,13,15,20,24,27]). However the
conjecture that all trees are graceful still remains open.

In this paper we will investigate several other graph
labeling problems. In each of the later sections, we will con-
sider one labeling problem, give a brief survey and mention re-
lated open problems.

2.   Optimal Linear Arrangements.

For a labeling π of a graph G, we define:

$$f_\pi(G) = \sum_{\{u,v\}\in E(G)} |\pi(u)-\pi(v)|.$$

The *cost* of G, denoted by $f(G)$, is the minimum value of $f_\pi(G)$
for π ranging over all labelings of G. A labeling π with
$f_\pi(G) = f(G)$ is called an *optimal linear arrangement*. It can be
easily checked that for a path P_n with n vertices, a star
$S_n = K_{1,n-1}$, a cycle C_n and the complete graph K_n, we have
respectively,

$$f(P_n) = n - 1,$$
$$f(C_n) = 2(n-1),$$
$$f(S_n) = \left\lceil \frac{n^2}{4} \right\rceil, \quad \text{and}$$
$$f(K_n) = n(n^2-1)/6.$$

Harper [17] determined the cost of the n-cube Q_n (which is
the product of n copies of P_2), namely,

$$f(Q_n) = 2^{n-1}(2^n-1).$$

In [2], I. Cahit conjectured that the value of the cost of a k-level
complete binary tree was $(k-1)2^{k-1}$. However, this was disproved
by the author, who showed [4] that the cost of a k-level complete
binary tree $T_{2,k}$ is:

$$f(T_{2,k}) = 2^k(k/3+5/18) + (-1)^k(2/9) - 2 .$$

The problem of determining the cost of a general graph is NP-complete as shown by Garey et al. [9]. (The reader is referred to [10] for an extensive survey of NP-completeness.) The problem of finding the cost of a tree can, however, be solved in polynomial time. Goldberg and Klipker [12] gave an n^3 algorithm and Shiloach [26] improved this with an $n^{2.2}$ algorithm. The author [5] recently obtained an algorithm which requires n^{λ} elementary operations where $\lambda = \log 3/\log 2 + \varepsilon \approx 1.58...$.

In [4] recurrence relations for the costs of k-level complete ternary trees were given. However, explicit expressions were not found. It would be of interest to find $f(T_{p,k})$ explicitly and also to improve the algorithm for determining the cost of a tree.

A. Meir first proposed the problem of determining the λ-cost $f^{(\lambda)}(T_{2,k})$ of a complete binary tree $T_{2,k}$ for general λ. In general, one can study the problem of determining the cost of an arbitrary graph. Relatively little is known for the $\lambda \neq 1$. In particular, it would be interesting to know what is the class of trees T such that there exists a labeling with

$$f_{\pi}(T) = f(T) \quad \text{and} \quad f_{\pi}^{(\lambda)}(T) = f^{(\lambda)}(T)$$

for any $\lambda > 1$?

3. Bandwidth.

For a labeling π of a graph G, we define:

$$b_{\pi}(G) = \underset{\{u,v\} \in E(G)}{\text{Max}} |\pi(u)-\pi(v)|.$$

The *bandwidth* of G, denoted by $b(G)$, is the minimum value of $b_{\pi}(G)$ as π ranges over all labelings of G. A labeling π with $b_{\pi}(G) = b(G)$ is called a *bandwidth labeling*. It can be easily checked (see [6]) that

$$b(P_n) = 1,$$

$$b(C_n) = 2,$$

$$b(S_n) = \lfloor n/2 \rfloor$$

$$b(K_n) = n - 1, \quad \text{and}$$

$$b(K_{m,n}) = \lfloor (m-1)/2 \rfloor + n \quad \text{where} \quad m \geq n > 0.$$

It is known [7] that $b(T_n) \leq \lfloor n/2 \rfloor$ where T_n is any tree on n vertices. Harper [18] determined $b(Q_n)$, namely,

$$b(Q_n) = \sum_{i=0}^{n-1} \binom{i}{\lfloor i/2 \rfloor} \quad .$$

Also, it can be easily shown that $b(T_{2,k}) = \left\lceil \dfrac{2^{k-1}-1}{k-1} \right\rceil$ and

$$b(T_{p,k}) = \left\lceil \frac{p(p^{k-1}-1)}{2(k-1)(p-1)} \right\rceil \quad .$$

Chvátalová [7] investigated the bandwidths of various classes by graphs and gave bounds for the bandwidth of a graph in terms of other invariants of the graph.

As to the algorithmic aspects, Papadimitriou [23] proved that the problem of determining the bandwidth of a graph is NP-complete. Garey et al. [11] showed that the bandwidth problem remains NP-complete even when restricted to trees with maximum degree 3. However, Saxe [25] proved that for fixed k, the problem "Is $b(G) \leq k$?" can be solved in polynomial time.

Chinn and Erdös conjectured that for a graph G and its complement $\overline{G}$, the following is true:

$$b(G) + b(\overline{G}) \geq n - 2 \quad .$$

Chinn, Chung, Erdös and Graham [3] proved this conjecture (which was also independently proved by Kahn and Kleitman [19]; Milner and Sauer [22]). Suppose K_n is partitioned into k

parts, say $G_1, G_2, \ldots, G_k$. The analogue of the above conjecture,

$$\sum_{i=1}^{k} b(G_i) \geq n - k,$$

it is not true, first pointed out by J. Shearer. It can be shown that there exists a partition $G_1, \ldots, G_k$ of K_n such that $\sum_{i=1}^{k} b(G_i)$ is approximately $n/2$.

4.   Folding Number.

For a labeling π of a graph G we define:

$$t_\pi(G) = \underset{i}{\text{Max}} \left| \{(u,v) \in E : \pi(u) \leq i < \pi(v)\} \right| .$$

The *folding number* of G, denoted by $t(G)$, is the minimum value of $t_\pi(G)$ as π ranges over all labelings of G. A labeling π with $t_\pi(G) = t(G)$ is called a *folding labeling*. A folding labeling is also occasionally called a minimum cut linear arrangement. It can be easily checked that

$$t(P_n) = 1,$$

$$t(C_n) = 2,$$

$$t(K_n) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil , \quad \text{and}$$

$$t(S_n) = \left\lfloor \frac{n}{2} \right\rfloor .$$

Stockmeyer [28] proved that the problem of determining the folding number of a graph is NP-complete. However, the problem of finding the folding number of a tree remains open. It can be shown that for a fixed number k, the problem "Is $t(G) \leq k?$" can be solved in polynomial time.

Here are some interesting properties of the folding labelings of trees. It can be easily shown that for any given tree there exists a folding labeling $\pi: V(T) \rightarrow \{1, \ldots, n\}$, where

$n = |V(T)|$, satisfying the following:

(i)   The vertices labeled by 1 and n are leaves.

(ii)   Let P denote the path connecting the two vertices labeled by 1 and n. Suppose P has vertices $v_0, \ldots, v_t$ with v_i adjacent to v_{i+1}. Then the labeling of the vertices of P is monotone, i.e., $\pi(v_i) < \pi(v_{i+1})$ for $i = 0, \ldots, t-1$; or $\pi(v_i) > \pi(v_{i+1})$ for $i = 0, \ldots, t-1$.

(iii)   If we remove all edges of P (but let the vertices stay), the remaining graph is a forest. Let $\bar{T}_i$ denote the subtree which contains v_i. Then, for each i, $\{\pi(u) : u \in \bar{T}_i\}$ is a set of consecutive integers.

(iv)   The labeling restricted to each $\bar{T}_i$ is a folding labeling for $\bar{T}_i$.

We note that for any tree there exists an optimal linear arrangement satisfying (i) ∿ (iv) and there exists a bandwidth labeling satisfying (i) ∿ (ii). These properties may be helpful in finding efficient algorithms for determining the folding number to a tree.

5.   Harmonious Labelings.

We call a connected graph with v vertices and $e \geq v$ edges *harmonious* if it is possible to label the vertices v with distinct elements $\pi(v)$ of $\mathbb{Z}_e$ (the integers modulo e) in such a way that, when each edge $\{u,v\}$ is labeled with $\pi(u) + \pi(v)$ (mod e), the resulting edge labels are distinct. If the graph is a tree we allow exactly one vertex label to be repeated. Such a labeling of the vertices and edges is called a *harmonious labeling* of the graph. Graham and Sloane [14] first investigated harmonious labelings for various graphs. They proved that almost all graphs are not harmonious. It is conjectured that all trees are harmonious (that is true for trees with no more than 9

vertices). It would be of interest, for example, to find harmo-
nious labelings for grid graphs (which are subgraphs of rectan-
gular grids).

6.   Edge Labelings of Trees.

Leech [21] first proposed the following interesting problem.
Does there exist a tree on n vertices with the edges labeled
by integers in such a way that, when the distance between two
vertices is defined to be the sum of edge labels on the (unique)
path between the vertices, the distances between the $\binom{n}{2}$ pairs
of vertices take the consecutive values $1, 2, \ldots, \binom{n}{2}$?

Such edge labeled trees exist [21] for n = 2, 3, 4, and 6
but not for 5. Taylor [29] proved that a necessary condition
for the existence of an edge-labeled tree of n vertices is that
either n is a square or n is a square plus 2. S. Lin proved
by computer that there is no such labeled tree with 9 vertices.
The problem remains unsolved for $n \geq 11$.

We can consider several more general problems:

(i)   For each integer n what is the greatest integer g(n)
         such that there exists a labeled tree with n vertices
         in which the distances between pairs of vertices include
         the consecutive values $1, 2, \ldots, g(n)$? (see [21])

(ii)   For each integer n what is the least integer h(n)
         such that there exists a labeled tree with n vertices
         in which the distances between pairs of vertices are
         distinct integers $\leq h(n)$?

7.   Concluding Remarks.

Many graph labeling problems we just mentioned originated
from various problems of practical interest, although they are
interesting on their own right. The optimal linear arrangement
problem arose in solving wiring problems or some placement problems.

The folding number problem came up in VLSI (Very Large Scale
Integration) design problems. The harmonious labelings are
closely related to problems in error-correcting codes. The edge-
labeling problem originated in the context of electrical networks
in which the vertices are the terminals and the distances are the
electrical resistances.

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