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ON UNIMODALITY FOR LINEAR EXTENSIONS OF PARTIAL ORDERS*

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Abstract. R. Rivest has recently proposed the following intriguing conjecture: Let x^* denote an arbitrary fixed element in an n-element partially ordered set P, and for each k in $\{1, 2, \cdots, n\}$ let N_k be the number of order-preserving maps from P onto $\{1, 2, \cdots, n\}$ that map x^* into k. Then the sequence $N_1, \cdots, N_n$ is unimodal. This note proves the conjecture for the special case in which P can be covered by two linear orders. It also generalizes this result for P that have disjoint components, one of which can be covered by two linear orders.

1. Introduction. Given a finite partially ordered set $(P, <)$, where $<$ is asymmetric, we say that an injection λ from P into the set Z of integers is a *linear extension* of P if, for all $x, y \in P$,

$$x < y \Rightarrow \lambda(x) < \lambda(y).$$

We shall presume that P has n elements and, in the main part of the paper, restrict ourselves to bijections $\lambda : P \to [n] \equiv \{1, 2, \cdots, n\}$. Generalizations are discussed later.

Let x^* be an arbitrary fixed element in P. For each $k \in [n]$, define N_k to be the number of linear extensions $\lambda : P \to [n]$ for which $\lambda(x^*) = k$. Rivest [2] has proposed the following tantalizing conjecture.

CONJECTURE. *The sequence N_k, $k \in [n]$, is unimodal.*

By unimodal we mean that, for all $1 \leqq i < j < k \leqq n$,

$$N_j \geqq \min \{N_i, N_k\}.$$

In this note we shall prove that the conjecture is valid for the important class of partially ordered sets that can be partitioned into two linearly ordered subsets, i.e., *chains*, with $<$-pairs allowed between the chains. In fact, we show that the N_k's in this case satisfy the stronger property of logarithmic concavity, i.e.,

$$N_k^2 \geqq N_{k-1} N_{k+1} \quad \text{for } 1 < k < n.$$

A similar proof provides an interesting result involving the unimodality of certain sequences of integers.

2. Lattice paths in Z^2. We shall say that the partially ordered set $(P, <)$ *can be covered by two chains* if there is a partition $\{A, B\}$ of P such that the restriction of $<$ on each of A and B is a linear order. To avoid the trivial case, we shall suppose that $<$ on P is not linear, and that $(P, <)$ can be covered by two chains, denoted as $A = \{a_1 < \cdots < a_r\}$ and $B = \{b_1 < \cdots < b_s\}$, with $r \geqq 1$, $s \geqq 1$ and $r + s = n$. There can be "cross-relations" like $a_i < b_j$ or $b_j < a_i$ from $(P, <)$, but in any event $<$ must be asymmetric ($x < y \Rightarrow$ not $(y < x)$) and transitive.

Let L denote the set of all ordered pairs of nonnegative integers. Each linear extension $\lambda : P \to [n]$ induces maps of A and B into $[n]$, with $\lambda(a_1) < \cdots < \lambda(a_r)$ and $\lambda(b_1) < \cdots < \lambda(b_s)$. To each such λ we will associate a lattice path $\pi(\lambda)$ in L as follows. The first point on $\pi(\lambda)$ is $(0, 0)$. If the kth point on $\pi(\lambda)$ is (x_k, y_k) and if $\lambda(p) = k + 1$, then the $(k+1)$st point on $\pi(\lambda)$ is $(x_k + 1, y_k)$ if $p \in A$, and $(x_k, y_k + 1)$ if $p \in B$. The terminal point on $\pi(\lambda)$ is (r, s). An example appears in Fig. 1.

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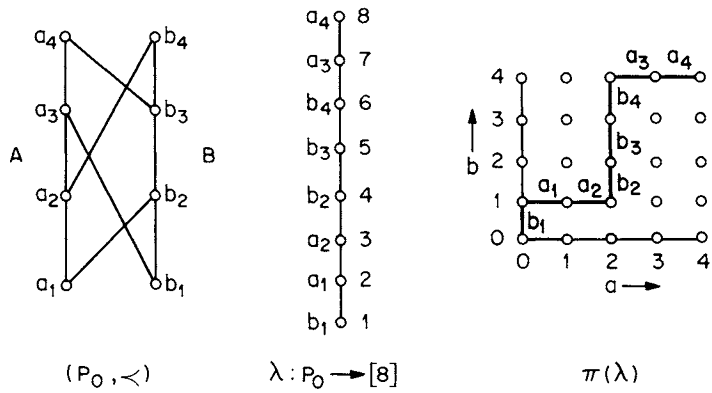

FIG. 1. *The correspondence between λ and $\pi(\lambda)$.*

The fact that λ preserves the linear orders on A and B is reflected in the fact that the indices of the a_i and b_j increase as we move along $\pi(\lambda)$ from (0, 0) to (r, s). But how do the *other* $\prec$-pairs show up in $\pi(\lambda)$? For Fig. 1, what constraint does $a_1 \prec b_2$ (which forces $\lambda(a_1) < \lambda(b_2)$) place on $\pi(\lambda)$? The answer is very simple. Each $a_i \prec b_j$ corresponds to a rectangular "barrier" which the path $\pi(\lambda)$ is not allowed to penetrate. This barrier is defined to be all lattice points (x, y) in L for which $x \leqq i$ and $y \geqq j-1$, as illustrated in Fig. 2.

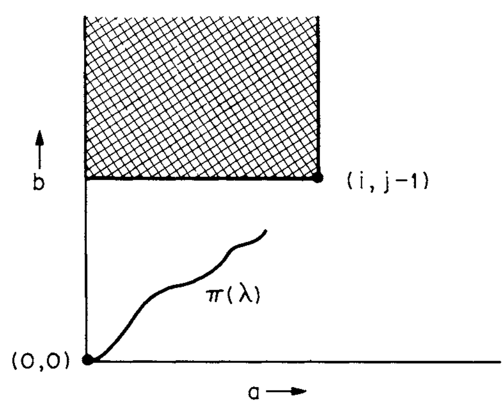

FIG. 2. *The barrier for $a_i \prec b_j$.*

The barrier for $a_i \prec b_j$ forces $\pi(\lambda)$ to reach a lattice point with x-coordinate i *before* it reaches one with y-coordinate j, i.e., a_i occurs before b_j on $\pi(\lambda)$. This is precisely what is needed for $\lambda(a_i) < \lambda(b_j)$.

In a similar manner, $b_j \prec a_i$ corresponds to a rectangular barrier consisting of all (x, y) in L for which $x \geqq i-1$ and $y \leqq j$. For λ to be a linear extension of P, $\pi(\lambda)$ must not penetrate *any* of the barriers formed from the cross-relations in $(P, \prec)$. Fig. 3 shows the union of the barriers for $(P_0, \prec)$ from Fig. 1.

The next point we consider is how $\lambda(x^*) = k$ is reflected in $\pi(\lambda)$. Without loss of generality, we assume that $x^* = a_i$, so that $x^* \in A$. Then it is easy to see that $\lambda(a_i) = k$ iff $\pi(\lambda)$ contains the two points $(i-1, k-i)$ and $(i, k-i)$. (Similarly, $\lambda(b_j) = k$ iff $\pi(\lambda)$ contains $(k-j, j-1)$ and $(k-j, j)$.)

Suppose N_{k-1} and N_{k+1} are both positive, and let λ^+ and λ^- be linear extensions of P such that $\lambda^+(a_i) = k+1$ and $\lambda^-(a_i) = k-1$. Thus, $\pi(\lambda^+)$ contains points $(i-1, k+1-$

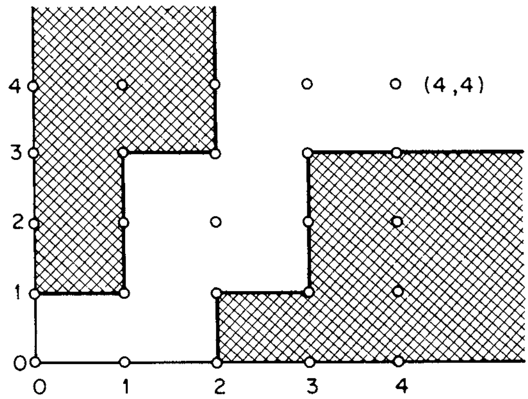

FIG. 3. *The union of barriers for* $(P_0, <)$.

$i)$ and $(i, k+1-i)$, and $\pi(\lambda^-)$ contains $(i-1, k-1-i)$ and $(i, k-1-i)$. Let x_0 be the largest integer that is $\leqq i-1$ such that, for some y, $(x_0, y+1)$ is on $\pi(\lambda^+)$ and (x_0, y) is on $\pi(\lambda^-)$, and let y_0, which cannot exceed $k-1-i$, be the largest integer such that (x_0, y_0+1) is on $\pi(\lambda^+)$ and (x_0, y_0) is on $\pi(\lambda^-)$. Similarly, let x_1 be the smallest integer $\geqq i$ such that, for some y, $(x_1, y+1)$ is on $\pi(\lambda^+)$ and (x_1, y) is on $\pi(\lambda^-)$, and let y_1, which cannot be less than $k-i$, be the smallest integer such that (x_1, y_1+1) is on $\pi(\lambda^+)$ and (x_1, y_1) is on $\pi(\lambda^-)$.

We now form two new lattice paths $\pi(\lambda_1)$ and $\pi(\lambda_2)$ as follows. Let $\pi(\lambda_1)$ consist of the points on $\pi(\lambda^-)$ from $(0, 0)$ to (x_0, y_0), plus the points on $\pi(\lambda^+)$ from (x_0, y_0+1) to (x_1, y_1+1) translated by -1 in the y-direction, plus the points on $\pi(\lambda^-)$ from (x_1, y_1) to (r, s). Let $\pi(\lambda_2)$ consist of the points on $\pi(\lambda^+)$ from $(0, 0)$ to (x_0, y_0+1), plus the points on $\pi(\lambda^-)$ from (x_0, y_0) to (x_1, y_1) translated by $+1$ in the y-direction, plus the points on $\pi(\lambda^+)$ from (x_1, y_1+1) to (r, s). It is of course possible to have $\pi(\lambda_1) = \pi(\lambda_2)$, or, equivalently, $\lambda_1 = \lambda_2$, but this will not affect our conclusions. We observe that:

(i) $\pi(\lambda_1)$ and $\pi(\lambda_2)$ are lattice paths from $(0, 0)$ to (r, s) which contain $(i, k-i)$ and $(i-1, k-i)$, and, therefore, $\lambda_1(a_i) = \lambda_2(a_i) = k$;

(ii) since $\pi(\lambda^+)$ lies strictly above $\pi(\lambda^-)$ in the region where the translations occur in the construction, neither $\pi(\lambda_1)$ nor $\pi(\lambda_2)$ penetrates any of the barriers formed by $(P, <)$. It follows that λ_1 and λ_2 are linear extensions of P;

(iii) if two ordered pairs of the form (λ^+, λ^-) are distinct, then their associated (λ_1, λ_2) pairs are distinct. This follows from the construction: if two $(\pi(\lambda^+), \pi(\lambda^-))$ differ prior to i on the abscissa, then their associated $(\pi(\lambda_1), \pi(\lambda_2))$ will differ before i; if two $(\pi(\lambda^+), \pi(\lambda^-))$ differ after $i-1$, then their associated $(\pi(\lambda_1), \pi(\lambda_2))$ will differ after $i-1$.

Thus, our construction provides an injection from the ordered pairs (λ^+, λ^-) into pairs (λ_1, λ_2), where λ^+ and λ^- are any linear extensions of P for which $\lambda^+(a_i) = k+1$ and $\lambda^-(a_i) = k-1$, and λ_1 and λ_2 are linear extensions of P that satisfy $\lambda_1(a_i) = \lambda_2(a_i) = k$. If α, β and γ are the number of linear extensions of P for which $\lambda(a_i) = k+1$, $\lambda(a_i) = k-1$, and $\lambda(a_i) = k$, respectively, then such an injection requires $\gamma^2 \geqq \alpha\beta$, for otherwise two (λ_1, λ_2) pairs associated with distinct (λ^+, λ^-) pairs would have to be identical.

The preceding argument applies analogously when $x^* = b_j$. Thus, we have proved the following result.

THEOREM 1. *Let x^* be a fixed element in a partially ordered set $(P, <)$ on n elements, and suppose $(P, <)$ can be covered by two chains. For $k \in \{1, 2, \cdots, n\}$, let N_k be the* *number of linear extensions* $\lambda: P \to \{1, 2, \cdots, n\}$ *for which* $\lambda(x^*) = k$. *Then*

$$N_k^2 \geqq N_{k-1}N_{k+1} \quad \text{for } k = 2, \cdots, n-1.$$

COROLLARY. *Given the hypotheses of Theorem* 1, *the sequence* $N_1, N_2, \cdots, N_n$ *is unimodal.*

The same basic argument for Theorem 1 can be used to prove the following result for sequences of integers. Let $A = (a_1 \geqq a_2 \geqq \cdots)$ be a nonincreasing sequence of nonnegative integers. Given A, let S_n be the number of nonincreasing sequences $x = (x_1 \geqq x_2 \geqq \cdots \geqq x_n)$ of integers for which $0 \leqq x_k \leqq a_k$, for $k = 1, \cdots, n$.

THEOREM 2. *The sequence* $S_1, S_2, \cdots$ *is logarithmically concave, i.e.,*

$$S_n^2 \geqq S_{n-1}S_{n+1} \quad \textit{for all } n \geqq 2.$$

When A is constant, say $A = (t, t, t, \cdots)$, Theorem 2 shows the (easily proved) logarithmic concavity of the binomial coefficients $\binom{t+k}{k}$ for $k = 1, 2, \cdots$.

3. A generalization. We now generalize our analysis of logarithmic concavity by considering disjoint partial orders along with linear extensions that map P into $[m] \equiv \{1, \cdots, m\}$ when m exceeds the cardinality of P. The following lemma provides a basis for the generalization.

LEMMA. *Let* $(P, <)$ *and* $(P \cup C, <)$ *be partially ordered sets on* n *and* $n + \alpha$ *elements, respectively, that have the same ordered pairs in their partial orders with* $C \cap P = \varnothing$. *Let* $x^* \in P$ *be fixed, and let* N_k *and* N'_k, *respectively, be the number of linear extensions* $\lambda: P \to [n]$ *and* $\lambda': P \cup C \to [n+\alpha]$ *that have* $\lambda(x^*) = k$ *and* $\lambda'(x^*) = k$. *If* $N_1, \ldots, N_n$ *is logarithmically concave, then so is* $N'_1, \cdots, N'_{n+\alpha}$.

If C is empty, there is nothing to prove; so suppose initially that $C = \{c\}$, with $\alpha = 1$. Since neither $c < x$ nor $x < c$ for each $x \in P$, each λ for P generates $n+1$ λ' for $P \cup \{c\}$ according to the $n+1$ placements of c. With $N_0 = N_{n+1} = 0$,

$$N'_k = (k-1)N_{k-1} + (n-k+1)N_k \quad \text{for } k = 1, \cdots, n+1.$$

Using this relationship, $(N'_k)^2 - N'_{k-1}N'_{k+1}$, for $2 \leqq k \leqq n$, reduces to

$$k(k-2)[N_{k-1}^2 - N_{k-2}N_k] + (n-k)(n-k+2)[N_k^2 - N_{k-1}N_{k+1}]$$
$$+ (k-2)(n-k)[N_{k-1}N_k - N_{k-2}N_{k+1}] + (N_{k-1} - N_k)^2,$$

which must be nonnegative if $\{N_k\}$ is logarithmically concave.

This completes the proof of the lemma if $\alpha \leqq 1$, so suppose in this paragraph that $\alpha \geqq 2$ with $C = \{c_1, \cdots, c_\alpha\}$. The $\lambda': P \cup C \to [n+\alpha]$ can be generated from the $\lambda: P \to [n]$ by adding one c_i at a time. For a given λ, we first add c_1 to obtain $n+1$ linear extensions from $P \cup \{c_1\}$ onto $[n+1]$; for each of these $n+1$, we then add c_2 to obtain $n+2$ linear extensions from $P \cup \{c_1, c_2\}$ onto $[n+2]$; and so forth. If $\{N_m\}$ is logarithmically concave, then successive applications of the result obtained in the preceding paragraph for each c_i addition show that $\{N'_k\}$ must be logarithmically concave. The lemma is thus proved.

We now state our generalization, discuss its features, and then conclude this section with its proof.

THEOREM 3. *Suppose* $(P_1, <_1)$, $(P_2, <_2)$ *and* $(P, <)$ *are partially ordered sets on* n_1, n_2 *and* n *elements respectively such that* $0 < n_1 \leqq n$, $P_1 \cup P_2 = P$, $P_1 \cap P_2 = \varnothing$ *and* $<_1 \cup <_2 = <$. *Let* $x^* \in P_1$ *be fixed, and let* N_k $(k = 1, \cdots, n_1)$ *be the number of linear extensions* $\lambda: P_1 \to [n_1]$ *for which* $\lambda(x^*) = k$. *In addition, given* $m \geqq n$, *let* M_k $(k =$

$1, \cdots, m$) *be the number of linear extensions* $\lambda^*: P \to [m]$ *for which* $\lambda^*(x^*) = k$. *If* $N_1, \cdots, N_{n_1}$ *is logarithmically concave, then so is* $M_1, \cdots, M_m$.

When $n_2 = 0$ and $m > n$, this shows that logarithmic concavity for $\lambda: P \to [n]$ carries over to $\lambda^*: P \to [m]$. When $n_2 > 0$ and $m = n$, Theorem 3 says that logarithmic concavity for the elements within a part of $(P, <)$, namely $(P_1, <_1)$, carries over to all of $(P, <)$ for those same elements, provided that the rest of $(P, <)$ is not connected to the first part. The combination of these two cases provides the generalization stated in the theorem.

Theorems 1 and 3 together yield the following result.

THEOREM 4. *If an n-element partially ordered set* $(P, <)$ *can be partitioned into partially ordered sets* $(P_1, <_1)$ *and* $(P_2, <_2)$ *with no* $<$*-connection between* P_1 *and* P_2, *if* $(P_1, <_1)$ *can be covered by two chains, and if* $x^* \in P_1$, $m \geqq n$, *and* M_k *is the number of linear extensions* $\lambda: P \to [m]$ *for which* $\lambda(x^*) = k$, *then* $M_1, \cdots, M_m$ *is logarithmically concave and unimodal.*

We now sketch the proof of Theorem 3 using the notation in its statement. In addition, let T_k be the number of linear extensions $\lambda_0: P \to [n]$ for which $\lambda_0(x^*) = k$, and if $n_2 > 0$, let β be the number of linear extensions $\lambda_2: P_2 \to [n_2]$, and let N'_k be the number of linear extensions $\lambda': P_1 \cup C \to [n]$ that have $\lambda'(x^*) = k$ when C is a completely unordered n_2-element set (see the lemma) that is disjoint from P_1.

If $n_2 = 0$ then $T_k = N_k$, so assume henceforth in this paragraph that $n_2 > 0$. We shall apply the lemma with $\alpha = n_2$. Consider a fixed $\lambda_2: P_2 \to [n_2]$ along with a generic $\lambda_1: P_1 \to [n]$. The n_2 numbers in $[n]$ that are not in $\lambda_1(P_1)$ can be bijectively assigned to the elements in P_2 in exactly one way that preserves the λ_2 order and yields a $\lambda_0: P \to [n]$—as compared to the $n_2!$ ways this could be done for the unordered set C. Since this is true for each such λ_1, it follows that the number of $\lambda_0: P \to [n]$ that have $\lambda_0(x^*) = k$ *and* have P_2 in its λ_2 order is $N'_k/n_2!$. Since there are β such λ_2, $T_k = \beta N'_k/N_2!$. If $N_1, \cdots, N_{n_1}$ is logarithmically concave, then the lemma says that $T_1, \cdots, T_n$ is too.

This proves Theorem 3 if $m = n$. If $m > n$, we reapply the lemma with $\alpha = m - n$. In this case let C' be a completely unordered $(m-n)$-element set disjoint from P and, with respect to $(P \cup C', <)$, let T'_k be the number of linear extensions $\lambda': P \cup C' \to [m]$ for which $\lambda'(x^*) = k$. By the lemma, if $\{T_k\}$ is logarithmically concave then so is $\{T'_k\}$. Since the $m - n$ numbers in $[m]$ that aren't in a $\lambda'(P)$ can be bijectively assigned to C' in $(m-n)!$ ways, it follows that M_k as defined in Theorem 3 equals $T'_k/(m-n)!$. When this is combined with preceding conclusions, we see that if $N_1, \cdots, N_{n_1}$ is logarithmically concave, then so is $M_1, \cdots, M_m$.

4. Concluding remarks. The preceding techniques can be used to prove other unimodality results for restricted lattice path problems. For example, consider lattice paths π that are not allowed to penetrate barriers of the type shown in Fig. 3, so that π is bounded between two increasing staircases. Let $D_{n,k}$ be the number of such paths that go through point $(k, n-k)$. Then, for each n, the sequence $D_{n,k}$, $0 \leqq k \leqq n$, is logarithmically concave and therefore unimodal. (Of course, here we are just looking at the intersections of lattice paths with the line $x + y = n$.) The reader is referred to the recent paper of Graham, Yao, and Yao [1] for similar applications of these ideas.

Finally, we note another open conjecture that is suggested by our analysis. Within the context used for the earlier conjecture, we propose:

CONJECTURE*. *The sequence* N_k, $k \in [n]$, *is logarithmically concave.*

Conjecture* is stronger than Rivest's Conjecture since unimodality follows from logarithmic concavity, but not conversely. Thus, a counterexample for Conjecture* need not disprove unimodality, while verification of Conjecture* would establish Rivest's Conjecture.

Note added in proof. R. Stanley has just proved Conjecture* using a very ingenious application of the Alexandroff-Fenchel theorem (which guarantees the logarithmic concavity of certain coefficients arising from the volume of weighted sums of n-dimensional polytopes).

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