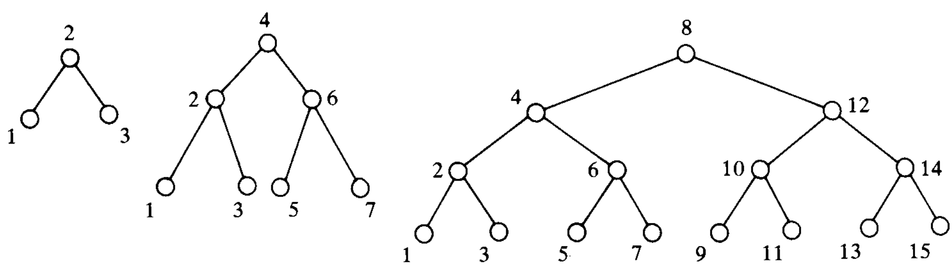
Problem 77-15*, *A Conjectured Minimum Valuation Tree*, by I. CAHIT (Turkish Telecommunications, Nicosia, Cyprus).

Let T denote a tree on n vertices. Each vertex of the tree is labeled with distinct integers from the set $1, 2, \cdots, n$. The weight of an edge of T is defined as the absolute value of the difference between the vertex numbers at its endpoints. If S denotes the sum of all the edge weights of T with respect to a given labeling, it is conjectured that for a k-level complete binary tree, the minimum sum is given by

$$S^k_{\min} = \min_{\text{all labelings}} \sum_{(i,j)\in T} |i-j| = (k-1)2^{(k-1)}, \qquad k>1.$$

Examples of minimum valuation trees for $k=2, 3, 4$ are given by



$S^{(2)}_{\min} = 2$ $\qquad$ $S^{(3)}_{\min} = 8$ $\qquad$ $S^{(4)}_{\min} = 24$

Editorial note. A. Meir suggested the related problem of determining $\min \sum (i-j)^2$. More generally, one can also consider max and $\min \sum |i-j|^m$. [M.S.K.]

Solution by F. R. K. CHUNG (Bell Laboratories).

The conjecture is not true for $k>4$. The following labeling for the 5-level complete binary tree shows that $S_{\min}^{(5)} \leqq 60 < 4.2^4$:

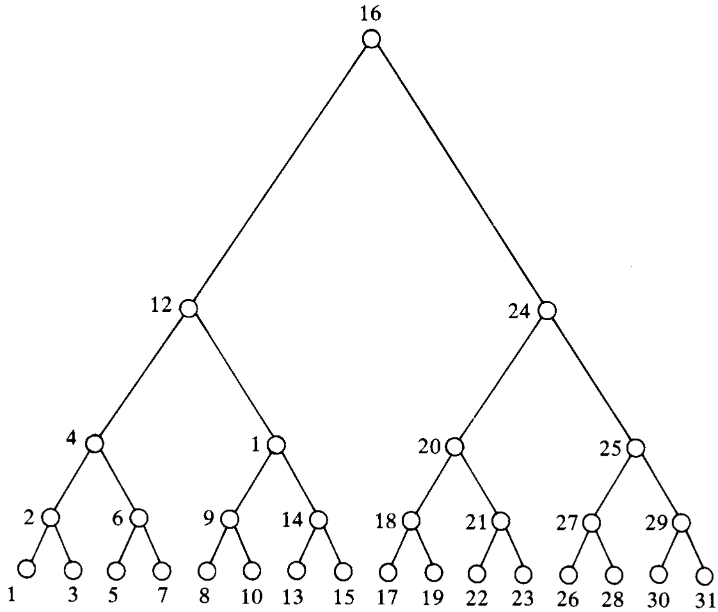


FIG. 1

We let $S_k = S_{\min}^{(k)}$. It can be shown that

$$S_k = 2^k(k/3+5/18)+(-1)^k(2/9)-2.$$

Proof. The optimal labeling L_k for the k-level binary tree T_k satisfies the following properties. The proof can be found in [1], [2] or can be easily verified.

Property 1. The vertex labeled by 1 or $n=2^k-1$ in L_k is a leaf (a *leaf* is a vertex of degree 1).

Property 2. Let P denote the path connecting the two vertices labeled by 1 and n in L_k. Let P have vertices $v_0, \cdots, v_t$. Then the labeling of the vertices of P is monotone, i.e.,

$$L(v_i) < L(v_{i+1}) \quad \text{for } i=0, \cdots, t-1$$

or

$$L(v_i) > L(v_{i+1}) \quad \text{for } i=0, \cdots, t-1.$$

Property 3. In T_k, we remove all edges of P. The resultant graph is a union of vertex disjoint subtrees. Let $\bar{T}_i$ denote the subtree which contains the vertex v_i, $i=1, \cdots, t-1$. Then for a fixed i, the set of labelings of vertices in $\bar{T}_i$ consists of consecutive integers. Moreover, the labeling on each $\bar{T}_i$ are optimal.

Property 4. Let $\bar{v}$ be the only vertex of T_k with degree 2. Then P passes $\bar{v}$.

Property 5. Let T'_k denote the tree which contains T_k as subtree and T'_k has one more vertex than T_k, which is a leaf adjacent to $\bar{v}$. Then $S(T'_k)=S_k+2$.

From Properties 1 to 5, the following recurrence relation holds:

$$S_k = 2^{k-1}+4+S_{k-1}+2S_{k-2} \quad \text{for } k \geqq 4$$

and

$$S_2 = 2, \qquad S_3 = 8.$$

It can be easily verified by induction that

$$S_k = 2^k(k/3 + 5/18) + (-1)^k(2/9) - 2. \quad \square$$

If we consider k-level complete p-nary trees T_k^p, some asymptotic estimates for S_k^p, minimum sum of all edge weights of T_k^p over all labelings, have been obtained in [1], [2]. We will briefly discuss the case that $p = 3$.

Let $T_p(k, i)$ denote a k-level tree which has the root connected to i copies of $(k-1)$-level p-nary tree. For example, the graph as shown in Fig. 2 is $T_3(3, 2)$. Let $S_p(k, i)$ denote the minimum value of all edge weights of $T_p(k, i)$ over all labelings of $T_p(k, i)$.

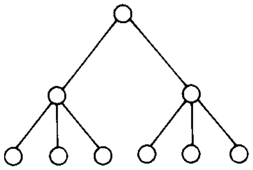


FIG. 2

It can be easily verified that

$$S_3(k, 2) = 3S_3(k-1, 2) + 2 \cdot 3^{k-2} \quad \text{for } k \geqq 3$$

and

$$S_3(2, 2) = 2.$$

Therefore,

$$S_3(k, 2) = 2(k-1)3^{k-2} \quad \text{for } k \geqq 2.$$

In general, it can be shown that, for p odd,

$$S_p(k, 2) = k(p+1)p^{k-2}/2 + (-2p^k + 3p^{k-1} + p^{k-2} + p - 3)/(2(p-1)).$$

and

$$S_p(k, p-1) = (k-1)(p^2-1)p^{k-2}/4.$$

The recurrence relation for S_k^3 is as follows: Let $f(k)$ be the integer l satisfying

$$(l-1)3^{l-2} + l + 1 \leqq k < l3^{l-1} + l, \qquad k \geqq 3.$$

Then we have

$$S_k^3 = 3^{k-2}(2k - 1/2) - 1/2 + k - f(k) + S_{k-1}^3 \quad \text{for } k \geqq 3,$$

and

$$S_2^3 = 4.$$

This complicated recurrence relation reveals the possible difficulty in getting explicit expression for S_k^p for general p.

REFERENCES

[1] M. A. IORDANSKII, *Minimal numberings of the vertices of trees*, Soviet Math. Dokl., 15(1974), no. 5, pp. 1311–1315.
[2] M. A. ŠEĬDVASSER, *The optimal numbering of the vertices of a tree*, Diskret. Analiz., 17(1970) pp. 56–74.

Also solved by W. F. SMYTH (Winnipeg, Manitoba) who sent a copy of a long paper, *A labelling algorithm for minimum edge weight sums of complete binary trees*, which had been submitted to Comm. ACM, whose interests were felt to be more directly related to the subject matter. An abstract of the paper is as follows:

Given a K-level complete binary tree $T_K = (V_K, E_K)$ on $2^K - 1$ vertices and a set $\mathcal{W}_K \equiv \{N+1, N+2, \cdots, N+2^K-1\}$ of integers, it is desired to label the vertices V_K from the set $\mathcal{W}_K$ without replacement, in such a way that the sum $S_K = \sum |n(u) - n(v)|$, taken over all edges $(u, v) \in E_K$, is a minimum, where $n(u)$ denotes the label assigned to vertex u. The labeled tree is called a valuation tree and, corresponding to a minimum labeling, a minimum valuation tree. An algorithm for this purpose is specified, with execution time $O(2^K - 1)$. An expression is derived for $S_K^{\min}$, and it is shown that in fact the algorithm achieves this minimum. Connections to the minimum bandwidth and minimum profile problems are outlined. Some open problems are stated.

Late solution. Additionally, P. A. VAN DER HELM (Twente University of Technology, Enschede, The Netherlands) gives a counterexample for $k = 5$.