# A Problem on Blocking Probabilities in Connecting Networks

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ABSTRACT

*We begin with a three-stage linear graph in which the first stage has a single node u and the third stage a single node v. The second stage has k independent nodes, each of which is connected by one link to u and to v. In general, we can form a (2n+1)-stage linear graph recursively by letting each node in the second stage of a three-stage linear graph be replaced by a copy of a (2n-1)-stage linear graph.*

*A link can either be in the busy state or the idle state. We assume that the states of each link are mutually independent and that any link between stage i and stage i + 1 has the probability* $I_i$ *of being idle. The nodes u and v are said to be connectable if there exists at least one path from u to v with no busy link. Let P(u,v) denote the probability of such a path existing. Further, let N(2n+1,k) denote the set of (2n+1)-stage linear graphs whose center stages have k nodes.*

*In this paper, we determine the size of N(2n+1,k). We also give the linear graph in N(2n+1,k) which has the largest P(u,v) and the one which has the smallest. We then show how our results apply to a recent problem in connecting networks.*

1. INTRODUCTION

We begin by forming a three-stage linear graph defined as follows: the first stage has a single node u and the third stage a single node v. The second stage has k nodes, each of which is connected by exactly one link to u and by one link to v.

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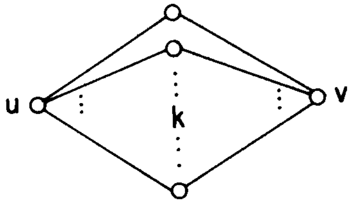


Fig. 1

A link can be either in the *busy* state or in the *idle* state. We assume that the probability of any link incident to u being idle is $I_1$ and the probability of any link incident to v being idle is $I_2$. Furthermore, the states of all links are assumed to be mutually independent. The nodes u and v are said to be *connectable* if there exists at least one path from u to v with both of its links idle. Let P(u,v) denote the probability that u and v are connectable. Then clearly

$$P(u,v) = 1 - (1-I_1 I_2)^k.$$

The three-stage linear graph becomes a five-stage linear graph if each node in the second stage is itself a copy of a three-stage linear graph (Figure 2).

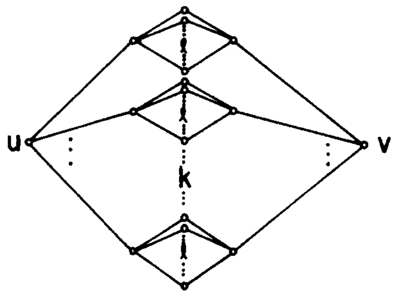


Fig. 2

In general, we can form a (2n+1)-stage linear graph by letting each node in the second stage of a three-stage linear graph be a copy of a fixed (2n-1)-stage linear graph. In the general case for each i = 1, 2, ..., n we let $I_i$ be the probability that a link between stage i and i + 1 is idle. Again all links are assumed to be independent.

Let $N(2n+1,k)$ denote the set of $(2n+1)$-stage linear graphs which have $k$ nodes in the $(n+1)^{st}$, i.e., the center stage. In a $(2n+1)$-stage linear graph, let $k_i$ be the number of links from each node in the $i^{th}$ stage to nodes in the $(i+1)^{st}$ stage. Notice that the number of nodes in the $(i+1)^{st}$ stage is equal to the number of links from the $i^{th}$ stage $(i=1,\ldots,n)$. Thus, by induction there are $\prod_{j=1}^{i} k_j$ links between the $i^{th}$ stage and the $(i+1)^{st}$ stage, $i = 1, \ldots, n$. In particular, the number of nodes in the center stage is $\prod_{i=1}^{n} k_i = k$. Let $P(k_1,\ldots,k_n;n,k)$ denote $P(u,v)$ for such a $(2n+1)$-stage linear graph.

In this paper we determine the size of $N(2n+1,k)$. We also give the linear graph in $N(2n+1,k)$ which has the largest $P(u,v)$ and the one which has the smallest. We then show how our results apply to a recent problem in connecting networks.

## 2. THE SIZE OF N(2n+1,k)

Suppose $k = \prod_{i=1}^{h} p_i^{\alpha_i}$ where the $p_i$ are distinct primes and the $\alpha_i$ are positive integers. Let $f(2n+1,k)$ denote the size of $N(2n+1,k)$.

*Lemma 2.1:* $f(2n+1,p^{\alpha}) = \binom{n+\alpha-1}{\alpha}$ *for* $n \geq 1$.

*Proof:* $f(2n+1,p^{\alpha})$ is the number of partitions of $\alpha$ identical objects into $n$ ordered compartments. It is well known [2] that this number is $\binom{n+\alpha-1}{\alpha}$.

Note that $f(2n+1,p^{\alpha})$ is independent of $p$.

*Lemma 2.2: If* $p$ *and* $q$ *are relatively prime, then*

$$f(2n+1,pq) = f(2n+1,p)f(2n+1,q).$$

*(i.e.,* $f$ *is multiplicative).*

*Proof:* Lemma 2.2 is trivially true for $n=1$ since $f(3,k) = 1$ for all k. For $n \geq 2$, we prove Lemma 2.2 by induction.

Let $D(m)$ denote the set of divisors of m, including 1 and m. Every $d \in D(pq)$ can be uniquely written as the product $d_p d_q$ where $d_p \in D(p)$ and $d_q \in D(q)$. Thus,

$$f(2n+1,pq) = \sum_{d \in D(pq)} f(2n-1,pq/d)$$

$$= \sum_{d \in D(pq)} f(2n-1,p/d_p) f(2n-1,q/d_q) \quad \text{by induction}$$

$$= \sum_{d_p \in D(p)} \sum_{d_q \in D(q)} f(2n-1,p/d_p) f(2n-1,q/d_q)$$

$$= f(2n+1,p) f(2n+1,q).$$

*Theorem 2.3:* $f(2n+1,k) = \prod_{i=1}^{h} \binom{n+\alpha_i-1}{\alpha_i}$ where $k = \prod_{i=1}^{h} p_i^{\alpha_i}$.

*Proof:* This follows immediately from Lemmas 2.1 and 2.2.

## 3. THE BEST LINEAR GRAPH AND THE WORST LINEAR GRAPH

In this section we give the linear graph in $N(2n+1,k)$ which has the largest $P(u,v)$ and the one which has the smallest. We first need some lemmas.

*Lemma 3.1:* $P(k_1,\ldots,k_n;n,k) = 1-[1-P(d_1,k_2,\ldots,k_n;n,kd_1/k_1)]^{k_1/d_1}$ *if* $d_1$ *divides* $k_1$.

*Proof:* Obvious from the independence assumption.

*Lemma 3.2:* $P(k_1,\ldots,k_n;n,k) \geq P(1,k_1k_2,k_3,\ldots,k_n;n,k)$.

*Proof:* $P(k_1,\ldots,k_n;n,k)$

$$= 1 - [1-P(1,k_2,\ldots,k_n;n,k/k_1)]^{k_1} \qquad \text{by Lemma 3.1}$$

$$= 1 - [1-I_1 I_{2n} P(k_2,\ldots,k_n;n-1,k/k_1)]^{k_1}$$

$$= I_1 I_{2n} P(k_2,\ldots,k_n;n-1,k/k_1) \sum_{i=0}^{k_1-1} [1-I_1 I_{2n} P(k_2,\ldots,k_n;n-1,k/k_1)]^i.$$

Also, $P(1,k_1k_2,k_3,\ldots,k_n;n,k)$

$$= I_1I_{2n}P(k_1k_2,k_3,\ldots,k_n;n-1,k)$$

$$= I_1I_{2n}\left\{1 - [1-P(k_2,\ldots,k_n;n-1,k/k_1)]^{k_1}\right\} \qquad \text{by Lemma 3.1}$$

$$= I_1I_{2n}P(k_2,\ldots,k_n;n-1,k/k_1) \sum_{i=0}^{k_1-1} [1-P(k_2,\ldots,k_n;n-1,k/k_1)]^i.$$

But

$$[1-I_1I_{2n}P(k_2,\ldots,k_n;n-1,k/k_1]^i \geq [1-P(k_2,\ldots,k_n;n-1,k/k_1)]^i$$

for every i. Lemma 3.2 is proved.

*Theorem 3.3:* $P(k,1,\ldots,1;n,k) \geq P(k_1,\ldots,k_n;n,k)$

$$\geq P(1,\ldots,1,k;n,k).$$

*Proof:* Proof is by induction on n. For n = 1, N(3,k) = 1 by Theorem 2.3, hence Theorem 3.3 is trivially true. We proceed to prove the theorem for general n.

$$P(k_1,\ldots,k_n;n,k)$$

$$= 1 - [1-P(1,k_2,\ldots,k_n;n,k)]^{k_1} \qquad \text{by Lemma 3.1}$$

$$= 1 - [1-I_1I_{2n}P(k_2,\ldots,k_n;n-1,k/k_1)]^{k_1}$$

$$\leq 1 - [1-I_1I_{2n}P(k/k_1,1,\ldots,1;n-1,k/k_1)]^{k_1} \qquad \text{by induction}$$

$$= 1 - [1-P(1,k/k_1,1,\ldots,1;n,k/k_1)]^{k_1}$$

$$\leq 1 - [1-P(k/k_1,1,\ldots,1;n,k/k_1)]^{k_1} \qquad \text{by Lemma 3.2}$$

$$= P(k,1,\ldots,1;n,k). \qquad \text{by Lemma 3.1}$$

$P(k_1,\ldots,k_n;n,k)$

$\geq P(1,k_1k_2,k_3,\ldots,k_n;n,k)$ by Lemma 3.2

$= I_1I_{2n}P(k_1k_2,k_3,\ldots,k_n;n-1,k)$

$\geq I_1I_{2n}P(1,k_1k_2k_3,k_4,\ldots,k_n;n-1,k)$ by Lemma 3.2

.....

$\geq I_1I_{2n}I_2I_{2n-1} \cdots I_{n-2}I_{n+3}P(1,k;2,k)$

$= P(1,\ldots,1,k;n,k)$.

Hence the best linear graph is the one which has all the branching in the first and last stages, and the worst linear graph is the one which has all the branching in the inner-most stages (see Figure 5, (a) is the best and (b) is the worst).

## 4. AN APPLICATION

Our problem was motivated by a study of blocking probabilities in symmetric *multistage connecting networks*. A symmetric multistage connecting network can be described by the following:

(i) It has (2n+1) ordered stages. The network is symmetric with respect to the center stage. The $i^{th}$ stage, hence the $(2n+2-i)^{th}$ stage, has $r_i$ copies of a switch $\nu_i$, $i = 1, \ldots, n+1$.

(ii) Each $\nu_i$ has $x_i$ input links connecting to $x_i$ copies of $\nu_{i-1}$ and $y_i$ output links connecting to $y_i$ copies of $\nu_{i+1}$. $\nu_1$ are called *input switches* and their input links are connected to the input terminals of the network. Similarly $\nu_{2n+1}$ are called *output switches* and their output links are connected to the output terminals of the network.

The network is to provide simultaneous connections for pairs of input terminals and output terminals, or equivalently, for pairs of input switches and output switches. In some cases, input switches and output switches can be partitioned into groups such that most of the connection attempts and paths are between switches of the same group.

Thus it would be desirable to design the linking in such a way that any intra-group pair of switches has $\lambda_1$ paths (treating a switch as a node in a graph) for connection, but any inter-group pair has only $\lambda_2 < \lambda_1$ paths. For example, let $\nu$ be a five-stage connecting network where each stage has 6 $2 \times 2$ switches. The 6 input (output) switches are divided into 3 groups each of which contains 2 switches. There are several ways to link up the switches such that each pair of intra-group switches has 4 paths (the maximum possible) while each inter-group has only 2 paths (see Figure 3 for two such examples).

(a) (b)

Fig. 3

Define a linear graph between an input switch u and an output switch v as the union of the set of paths between u and v. Note that for either network in Figure 3, the linear graph for each intra-group (inter-group) pair is isomorphic. Hence, we can talk about the *intra-group* (inter-group) *linear graphs* for the two networks. Furthermore, the intra-group linear graphs for the two networks are the same (see Figure 4).

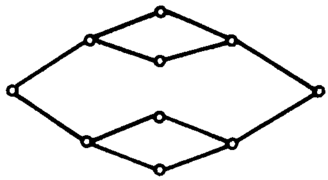

Fig. 4

Hence, as far as blocking probabilities are concerned, the two networks can be compared by their respective inter-group linear graphs which are shown in Figure 5.

Fig. 5

By Theorem 3.3, the linear graph in Figure 5(a), which is derived from the network in Figure 3(a), is better. Note that the two networks in Figure 3 have the same numbers of crosspoints and links.

For a general construction of multistage connecting networks, each of which has isomorphic intra-group and inter-group linear graphs respectively, see [1,3].

The type of linear graph we have studied in the paper is the *series-parallel* type as defined in [4].

For networks with more complex linear graphs, e.g., those sometimes referred to as "meshed" or "spiderweb" graphs, similar analyses may prove much more difficult because of the way in which blocking probability formulas must be constructed.

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