A Note on Constructive Methods for Ramsey Numbers

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ABSTRACT

Let $r(k)$ denote the least integer n such that for any graph G on n vertices either G or its complement $\bar{G}$ contains a complete graph K_k on k vertices. In this paper, we prove the following lower bound for the Ramsey number $r(k)$ by explicit construction: $r(k) \geqslant \exp(c(\log k)^{4/3}/[(\log\log k)^{1/3}]$ for some constant $c > 0$.

For an integer k, the Ramsey number $r(k)$ is defined to be the least integer n such that for any graph G on n vertices, either G or its complement $\bar{G}$ contains a complete graph K_k on k vertices. The theory of Ramsey numbers has been extensively studied in the past. However, relatively few results for $r(k)$ have yet been found. With respect to exact values, we only know $r(3) = 6$ and $r(4) = 18$ (see [1, 8]). A lower bound 42 for $r(5)$ was proved (but unpublished) by S. Lin and, independently, by J. P. Burling. An upper bound 55 for $r(5)$ was given in [11]. Thus we have

$$42 \leqslant r(5) \leqslant 55.$$

For general k, the following upper bound for $r(k)$ is still the best known so far.*

$$r(k) \leqslant c\binom{2k-2}{k-1}$$

for a suitable constant c.

P. Erdös [3] has proved the following lower bound by probabilistic

* The widely quoted upper bound $c\frac{\log\log k}{\log k}\binom{2k-2}{k-1}$ by J. Yackel [12] seems now in question [13].

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arguments:

$$r(k) \geqslant k2^{k/2}\left(\frac{1}{e\sqrt{2}} + o(1)\right).$$

J. Spencer [10] improved the above bound by a factor of 2 also by nonconstructive methods. Erdös [4] has asked whether one can find an explicit construction for a graph G on $2^{k/2}$ vertices such that neither G nor its complement $\bar{G}$ contains K_k. This problem, however, falls into an interesting category of problems which have the property that for any large n the existence of a "good configuration" is assured by probabilistic methods, (in fact most of the configurations are good), but we cannot explicitly find even one "good configuration."

H. L. Abbott [1] gives a recursive construction which shows that $r(k) \geqslant ck^{c'}$, where $c' = \log 41/\log 4 = 2.679\ldots$. Nagy [9] gives a construction which shows that $r(k) \geqslant ck^3$. P. Frankl [7] shows constructively that

$$r(k) \geqslant ck^m$$

for any m and some constant c.

In this note, we will give the constructive lower bound:

$$r(k) \geqslant \exp\left[c(\log k)^{4/3}/\log\log k)^{1/3}\right], \qquad \text{for some constant } c.$$

In other words, we present an explicit construction of a graph G on n vertices such that neither G nor its complement $\bar{G}$ contains a complete graph on $\exp\left[c(\log n)^{3/4}(\log\log n)^{1/4}\right]$ vertices.

The basic ideas of this lower bound are due to P. Frankl [7]. We will tighten up some loose ends in [7] and give a self-contained proof of the following theorem on intersecting families (except for the use of a result of P. Erdös and R. Rado).

Theorem 1. For integers x, y, z, w, p, u, with $x \geqslant 2z > 0$, $p \leqslant xy + z + w$, $p \leqslant xu + w$ we define

$$L(x, y, z, w) = \{xy' + z' + w : 0 \leqslant y' < y, 0 \leqslant z' < z\}.$$

Let $\mathbf{F}$ be a family of distinct p-subsets of $X = \{1, \ldots, n\}$ such that for any two sets $F_1, F_2 \in \mathbf{F}$ we have $|F_1 \cap F_2| \in L(x, y, z, w)$. Then we have

$$|\mathbf{F}| \leqslant n^{u+y+z} p^{2p(u+y)}.$$

The proof of Theorem 1 is based on a recursive argument using Δ-systems. First we need some definitions.

A family of sets, $S_1, \ldots, S_n$, is said to be a Δ-system if $S_i \cap S_j = S_{i'} \cap S_{j'}$

for any $i \neq j$, $i' \neq j'$. In this case, $D = S_i \cap S_j$, $i \neq j$, is called the *kernel* of this Δ-system.

Theorem (Erdös and Rado [6]). For any t sets, $S_1, \ldots, S_t$, with $|S_i| \leqslant s$ for $1 \leqslant i \leqslant t$, there exists a Δ-system consisting of $r+1$ S_i's provided

$$t > s!\, r^{s+1}\left(1 - \frac{1}{2!\, r} - \frac{2}{3!\, r^2} \cdots - \frac{s-1}{s!\, r^{s-1}}\right).$$

Let $\mathbf{F}$ be a family of distinct p-subsets of X such that for any $F_1, F_2 \in \mathbf{F}$ we have $|F_1 \cap F_2| \in L(x, y, z, w)$. If $|\mathbf{F}| \geqslant p^{2p}$, then $\mathbf{F}$ contains a Δ-system with $p+1$ subsets. Let D_1 be a subset of X with maximal cardinality such that D_1 is the kernel of a Δ-system consisting of $p+1$ sets in $\mathbf{F}$. We define $\mathbf{F}_1 = \{F \in \mathbf{F}: D_1 \subset F\}$. If $|\mathbf{F} - \mathbf{F}_1| \geqslant p^{2p}$, then in a similar manner we consider D_2 which is the kernel of maximal cardinality of a Δ-system consisting of $p+1$ sets in $\mathbf{F} - \mathbf{F}_1$ and we define $\mathbf{F}_2 = \{F \in \mathbf{F} - \mathbf{F}_1: D_2 \subset F\}$. After a finite number of steps, we have found $D_1, \ldots, D_t$, $\mathbf{F}_1, \ldots, \mathbf{F}_t$ and $\mathbf{F} - \bigcup_{i \leqslant t} \mathbf{F}_i = \mathbf{F}_{t+1}$ contains fewer than p^{2p} sets.

We note that $\mathbf{F}_i$, $i = 1, \ldots, t$, contains no more than np^{2p-1} sets (otherwise there are at least $p^{2(p-|D_i|)}$ sets in $\mathbf{F}_i$ all of which contain a common element in $X - D_i$; this would contradict the maximality of D_i). We also note that for a Δ-system $F_1, \ldots, F_{p+1}$ with kernel D_i, we know that $F_{i'} - D_i$, $i' = 1, \ldots, p+1$, are pairwise disjoint. Thus, for any p'-subset $X' \subset X$ $p' \leqslant p$ we have $X' \cap D_i = X' \cap F_{i'}$ for some i'. Thus $D_i \cap D_j = F_{i'} \cap F_{j'}$ and $|D_i \cap D_j| \in L(x, y, z, w)$.

We will prove Theorem 1 by induction on p. It is easy to see that Theorem 1 holds for $p = 0$. For $p > 0$, we let $\mathbf{E}_i = \{F: F \in \mathbf{F}_j$ and $|D_j| = p - i\}$ for $0 < i \leqslant p$. Let $\mathbf{E}_{i_0} = \mathbf{E}$ have the property that $|\mathbf{E}_{i_0}| \geqslant |\mathbf{E}_i|$ for any i. Therefore we have

$$p\,|\mathbf{E}| \geqslant |\mathbf{F}| - p^{2p}.$$

Suppose $i_0 < z$. We define $\mathbf{X}_A = \{D_i: D_i \bigcup A \in \mathbf{E}$ and $D_i \cap A = \phi\}$ for $A \subset X$ and $|A| = i_0$. It is easy to see that for D_i, $D_j \in \mathbf{X}_A$ we have $|D_i \cap D_j| \in L(x, y, z - i_0, w)$. Therefore by the induction assumptions, we have

$$|\mathbf{X}_A| \leqslant n^{u+y+z-i_0}(p - i_0)^{2(p-i_0)(u+y)}$$

and

$$|\mathbf{F}| \leqslant p^{2p} + p \sum_A |\mathbf{X}_A| \leqslant p^{2p} + \binom{n}{i_0} n^{u+y+z-i_0} p^{(2p-1)(u+y)} \leqslant n^{u+y+z} p^{2p(u+y)}.$$

So, we may assume $i_0 \geqslant z$. We consider $\mathbf{D} = \{D_i: |D_i| = p - i_0, 1 \leqslant i \leqslant t\}$.

We have

$$|\mathbf{E}| \leqslant \sum_{|D_i|=p-i_0} |\mathbf{F}_j| \leqslant np^{2p-1}|\mathbf{D}|.$$

It suffices to show that

for any $D_i, D_j \in \mathbf{D}$ we have $|D_i \cap D_j| \in L(x, \bar{y}, z, w)$,

$$\bar{p} = p - i_0 \leqslant x\bar{y} + z + w, \qquad \bar{p} \leqslant x\bar{u} + w \quad \text{and} \quad \bar{u} + \bar{y} \leqslant u + y - 1, \qquad (1)$$

since by the induction assumptions we have

$$|\mathbf{D}| \leqslant n^{u+y+z-1}(p-i_0)^{2(p-i_0)(u+y-1)}$$

and

$$|\mathbf{F}| \leqslant p^{2p} + np^{2p}n^{u+y+z-1}(p-z)^{2(p-z)(u+y-1)} \leqslant n^{u+y+z}p^{2p(u+y)}.$$

It is straightforward to verify (1) by considering the following two cases.

Case 1. $x(y-1)+2z+w < p \leqslant xy+z+w$, we can choose $\bar{u}$, $\bar{y}$ satisfying $\bar{u} \leqslant y \leqslant u-1$, $\bar{y} \leqslant y$.

Case 2. $p \leqslant x(y-1)+2z+w$. We can choose $\bar{u}$, $\bar{y}$ satisfying $\bar{u} \leqslant u$, $\bar{y} \leqslant y-1$.

This completes the proof of Theorem 1.

Now, for any integer k, we construct a graph G such that the vertex set $V(G)$ consists of all p-subsets of $X = \{1, \ldots, n\}$ and for $v_1, v_2 \in V(G)$, v_1 is adjacent to v_2 iff $|v_1 \cap v_2| \in \{2xx' + x'' : 0 \leqslant x', x'' < x\}$ where we choose

$$\begin{aligned} n &= \lceil \exp[(\log k)^{2/3}(\log\log k)^{1/3}/3^{1/3}] \rceil, \\ x &= \left\lceil \frac{(3\log k)^{1/3}}{4(\log\log k)^{1/3}} \right\rceil, \\ p &= 2x^2, \end{aligned} \qquad (2)$$

with $\lceil y \rceil$ denoting the least integer greater than or equal to y. It follows Theorem 1 that neither G nor its complement $\bar{G}$ contains any complete graph on k vertices and G has at least $\exp[(\log k)^{4/3}/8(\log\log k)^{1/3}]$ vertices. Therefore, we have the following result.

Theorem 2. The graph G constructed as above has at least $\exp[c(\log k)^{4/3}/(\log\log k)^{1/3}]$ vertices, for some constant c, and has the property that neither G nor its complement $\bar{G}$ contains K_k. Therefore we have the constructive lower bound:

$$r(k) \geqslant \exp[c(\log k)^{4/3}/(\log\log k)^{1/3}].$$

REMARK 1. We note that the graph G we constructed by (2) contains a complete subgraph on $\exp[c'(\log k)^{4/3}/(\log\log k)^{1/3}]$ vertices for large k.

Therefore by using the above construction, the bound in Theorem 2 can not be improved asymptotically. Some new ideas will be needed in order to give a constructive lower bound of $(1+\epsilon)^k$ for Ramsey number $r(k)$ for a fixed $\epsilon > 0$.

REMARK 2. Let $r_t(k)$ denote the least integer n such that if every edge of K_n is colored by one of t colors, then there exists a monochromatic K_k. It has been shown in [2] that $r_t(3) \geqslant c(3+\delta)^t$ where $\delta = 0.103\ldots$ is the positive root of $x^3+6x^2+9x-1=0$ and $c=50\delta^2$. By a recursive construction used in [4], it can be shown that $r_t(k) \geqslant \exp[c't(\log k)^{4/3}/(\log\log k)^{1/3}]$ for sufficiently large k and some positive constant c'.

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