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ON THE COVERINGS OF GRAPHS

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Let $\rho(n)$ denote the smallest integer with the property that any graph with n vertices can be covered by $\rho(n)$ complete bipartite subgraphs. We prove a conjecture of J.-C. Bermond by showing $\rho(n) = n + o(n^{11/14+\epsilon})$ for any positive ϵ.

1. Introduction

Suppose G is a connected graph[1] with vertex set $V(G)$ and edge set $E(G)$. A *covering* of G is a family of subgraphs, say $G_1, G_2, \ldots, G_t$, having the property that each edge of G is contained in at least one graph G_i, for some i. If all G_i, $1 \leq i \leq t$, belong to a specified class of graphs **H**, such a covering is called an **H**-covering of G. If we require all subgraphs in the covering to be edge-disjoint, the covering is also called a *decomposition* of G.

One of the fundamental topics in graph theory is to study the coverings and the decompositions of graphs. Much work has been done on **H**- covering and **H**-decompositions for various classes **H** (see [3]). In this note, we prove a conjecture of J.-C. Bermond [1] on **B**-coverings of graphs, where **B** is the set of complete bipartite graphs, as follows:

Let $\rho(n)$ be the smallest number with the property that any graph on n vertices has a **B**-covering consisting of no more than $\rho(n)$ subgraphs. It was conjectured by J.-C. Bermond that

$$\lim_{n\to\infty} \rho(n)/n = 1$$

We will show that this conjecture is true.

2. Preliminaries

In the remaining part of the paper, a covering usually means a **B**-covering. We note that the complete graph K_n has a covering of $\lceil \log_2 n \rceil$ complete bipartite

[1] We only consider graphs without loops or multiple edges. The reader is referred to [9] for undefined terminology.

graphs, where $\lceil x \rceil$ denotes the least integer greater than or equal to x. A path on n vertices has a covering of $\lfloor \frac{1}{2}n \rfloor$ complete bipartite subgraphs where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. It is easy to see that $\lim_{n\to\infty} \rho(n)/n \geq \frac{2}{3}$ by considering the graph which is the vertex-disjoint union of $\lfloor \frac{1}{3}n \rfloor$ copies of K_3. It was first suspected that $\frac{2}{3}$ might be the value of $\lim_{n\to\infty} \rho(n)/n$. However, $\rho(n)$ can be shown to be much greater than $\frac{2}{3}n$ for large n. In fact, $\rho(n)$ is fairly close to its upper bound $n-1$. We note that a graph on n vertices can be covered by $n-1$ stars i.e., complete bipartite graphs $K_{1,t}$. Therefore we have

$$\rho(n)/n < 1.$$

In the next section, we will show

$$\lim_{n\to\infty} \rho(n)/n = 1$$

by proving a lower bound $n - n^{11/14+\epsilon}$ for any $\epsilon > 0$.

3. A lower bound

We will show the following.

Main Theorem. (i) *For infinitely many n, we have $\rho(n) > n - n^{\frac{3}{4}}$.*

(ii) *For any positive ϵ, we have*

$$\rho(n) > n - n^{11/14+\epsilon} \quad \textit{for sufficiently large } n.$$

Proof. Let us consider a graph G on $n = q^2+q+1$ vertices, where q is a prime power.

It is well known [11] that there exists a difference set $\{d_1, \ldots, d_{q+1}\} \subset \{1, \ldots, q^2+q+1\}$ such that for any $x \equiv 0 \pmod{q^2+q+1}$, there is exactly one ordered pair (d_i, d_j), such that $d_i - d_j \equiv x \pmod{q^2+q+1}$. Now in the graph G, v_i is adjacent to v_j if and only if $i \neq j$ and $i+j \equiv d_k \pmod{q^2+q+1}$ for some k.

Suppose there is a four-cycle on distinct vertices v_x, v_y, v_z, v_w. Then clearly we have

$$x+y \equiv d_{i'}, \qquad y+z \equiv d_{j'}$$

where all congruences are to modulus q^2+q+1. Therefore, $x - z \equiv d_{i'} - d_{j'}$.

Similarly we have

$$x+w \equiv d_{i''}, \qquad z+w \equiv d_{j''}$$

where

$$i' \neq i'', \qquad j' \neq j''.$$

Then $d_{i'} - d_{j'} \equiv d_{i''} - d_{j''}$.

This contradicts the definition of a difference set. We conclude that this graph G does not contain any four-cycle as a subgraph. Thus any complete bipartite subgraph of G must be a star. We note that this graph G has also been used in [2, 4, 8].

Now, we consider a covering of G consisting of stars $S_1, \ldots, S_s$. Let u_i, $1 \leq i \leq s$, be a vertex of S_i such that u_i is adjacent to every other vertex in S_i.

The following observations are immediate.

Fact 1. Let v be a vertex in $V(G) - U$ where $U = \{u_i : 1 \leq i \leq s\}$. Any vertex adjacent to v must belong to U.

Fact 2. We define $N(v) = \{u \in U : \{u, v\} \in E(G)\}$. Then $N(v)$ has q or $q+1$ elements and the union of $N(v)$ over all $v \in V(G) - U$ is U.

Fact 3. For $u, v \in \mathrm{P}$, $u \neq v$, we have

$$|N(u) \cap N(v)| \leq 1.$$

We now consider a (0, 1)-matrix $A = \{A_{i,j} : 1 \leq i \leq t = n - s,\ 1 \leq j \leq s\}$ defined by

$$A_{ij} = \begin{cases} 1 & \text{if } u_j \geqslant N(p_i), \\ 0 & \textit{otherwise} \end{cases}$$

where

$$V(G) - U = \{p_1, p_2, \ldots, p_t\}.$$

We note that every row of A has sum q or $q+1$. We evaluate in two ways the sum of the inner products of the rows:

$$\sum_{i=1}^{t} \sum_{\substack{j=1 \\ j \neq i}}^{t} \sum_{k=1}^{s} A_{ik} A_{jk} \leq t(t-1). \tag{1}$$

We note that (1) follows from Fact 3 and the left-hand side of (1) is equal to

$$\sum_{k=1}^{s} \sum_{i=1}^{t} \sum_{\substack{j=1 \\ j \neq i}}^{t} A_{ik} A_{jk}. \tag{2}$$

Let q_i denote the column sum of the ith column, $1 \leq i \leq s$. Then (2) is equal to

$$\sum_{k=1}^{s} q_k (q_k - 1).$$

Note that

$$t(q+1) \geqslant \sum_{k=1}^{s} q_k \geqslant tq.$$

Therefore

$$\sum_{k=1}^{s} q_k^2 \geqslant t^2 q^2 / s.$$

From (1) we obtain

$$t(t-1) \geqslant t^2 q^2 / s - t(q+1).$$

By substituting $s=n-t=q^2+q+1-t$, we have

$$t^2-t-q(q^2+q-1)\leqslant 0.$$

Thus

$$t<\sqrt{q(q^2+q+1)+\tfrac{1}{4}}+\tfrac{1}{2}<n^{\frac{3}{4}} \quad \text{and} \quad s>n-n^{\frac{3}{4}}.$$

Therefore, we have shown that for $n=q^2+q+1$, q a prime power, we have

$$\rho(n)\geqslant n-n^{\frac{3}{4}}.$$

Now suppose n is an *arbitrary* integer. It is recently shown [10] that there exists a prime q between $\sqrt{n}-1$ and $\sqrt{n}-n^{\frac{2}{7}+\sigma}$ for any given $\sigma>0$ for large enough n. It is easy to see that $n\geqslant q^2+q+1$. We also note that $\rho(n)\geqslant\rho(n')$ for any n' with $n\geqslant n'$ since any graph on n' vertices can be viewed as a graph on n vertices.

Therefore we have

$$\begin{aligned}\rho(n)&\geqslant\rho(q^2+q+1)\\&\geqslant(q^2+q+1)-(q^2+q+1)^{\frac{3}{4}}\geqslant n-n^{11/14+\epsilon}\end{aligned}$$

for any given $\epsilon>0$.

Thus, the main theorem is proved.

Professor Erdős [6] pointed out that a graph G on n vertices has either an independent set of size $(c\log n)$ or it contains a complete subgraph on at least $(c\log n)$ vertices for $c=\log\frac{1}{2}n\log 2$ since the Ramsey number

$$r(a,b)<\binom{a+b-2}{a-1}.$$

In either case, G can always be covered by $n-c'\log n$ complete bipartite subgraphs for some consta c'.

Concluding remarks

The preceding results suggest a number of related problems, several of which we now mention:

(1) Consider $\rho'(n)=n-\rho(n)$. We know that $\rho'(n)$ is between $c_1\log n$ and $c_2 n^{11/14+\epsilon}$ for any $\epsilon>0$ and some constants c_1, c_2. What is the asymptotic behavior of ρ'?

(2) For a given graph G and a specified family of graphs $\mathbf{H}$ we define $\rho(G;\mathbf{H})$ to be the minimum number of subgraphs from $\mathbf{H}$ needed to cover G. We also define $\rho(n,\mathbf{H})$ to be the maximum value of $\rho(G;\mathbf{H})$ over all graphs G with n vertices.

Let $\mathbf{P}$ denote the set of all simple paths. We can then ask whether any graph with n vertices can always be covered by $\lceil\frac{1}{2}n\rceil$ simple paths, i.e., is the following true?

Conjecture. $\rho(n, \mathbf{P}) = \lceil \frac{1}{2}n \rceil$.

We note that this is an analogue of the Gallai conjecture on the decomposition of graphs.

(3) Let **C** denote the set of all simple cycles. We let $\rho(G; \mathbf{C}) = 0$ if G has a vertex of odd degree. It seems reasonable to conjecture that any graph with n vertices can be covered by $\lfloor \frac{1}{2}n \rfloor$ simple cycles, i.e.,

Conjecture. $\rho(n, \mathbf{C}) = \lfloor \frac{1}{2}n \rfloor$.

(We note that this a weaker version of the Hajos conjecture on the decomposition of graphs.)

(4) We can ask the question of determining $\rho(n, \mathbf{H})$ for **H** being a class of graphs with certain specified properties, e.g., each graph has diameter $\leqslant x$, has chromatic number $\leqslant y$, has connectivity $\leqslant z$, etc.

We remark that V. Chvátal [5] has also proved the conjecture $\lim_{n\to\infty} \rho(n)/n = 1$ by showing $\rho(n) \geqslant n - n^{\frac{7}{8}+\epsilon}$, based on a probablistic result of P. Erdös [7].

References

[1] J.-C. Bermond, Converture des artes d'un graphes bipartis complets, preprint.
[2] F.R.K. Chung, Ramsey numbers in multi-colors, Dissertation, University of Pennsylvania, 1974.
[3] F.R.K. Chung, On the decompositions of graphs, preprint.
[4] F.R.K. Chung and R.L. Graham, On multicolor Ramsey numbers for complete bipartite graphs J. Combinatorial Theory 18 (1975) 164–169.
[5] V. Chvátal, personal communication.
[6] P. Erdös, personal communication.
[7] P. Erdös, Graph theory and probability, Canad. J. Math. 11 (1959) 34–38.
[8] P. Erdös, A. Rényi and V.T. Sós, On a problem of graph theory, Studia Sci. Math. Hung. 1 (1966) 215–235.
[9] F. Harary, Graph Theory (Addison-Wesley, New York, 1969).
[10] H. Iwaniec, personal communication.
[11] J. Singer, A theorem in finite projective geometry and some applications to number theory, Trans. Amer. Math. Soc. 43 (1935) 377–385.