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ON THE DECOMPOSITION OF GRAPHS*

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Abstract. In this paper, we study the decompositions of a graph G into edge-disjoint subgraphs all of which belong to a specified class of graphs $\mathcal{H}$. Let $\alpha(G; \mathcal{H})$ denote the minimum value of the total sum of the sizes of subgraphs in $\mathcal{H}$ into which G can be decomposed, taken over all such decompositions of G. Let $\alpha(n; \mathcal{H})$ denote the maximum value of $\alpha(G; \mathcal{H})$ over all graphs G with n vertices.

In this paper, we settle a conjecture of Katona and Tarján by showing

$$\alpha(n; \mathcal{K}) = \lfloor n^2/2 \rfloor,$$

where $\mathcal{K}$ denotes the set of all complete graphs. Moreover, we show that the complete bipartite graph G on $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ vertices is the only graph with $\alpha(G; \mathcal{K}) = \alpha(n; \mathcal{K})$.

I. Introduction. Many interesting problems in graph theory[1] can be described in the following general framework.

Suppose G is a finite connected graph with vertex set $V(G)$ and edge set $E(G)$. Consider a decomposition of G into subgraphs $G_1, G_2, \cdots, G_t$, such that any edge in G is an edge of exactly one of the G_i's, and all G_i's belong to a specified class of graphs $\mathcal{H}$. Such a decomposition will be called an *$\mathcal{H}$-decomposition* of G.

Let f be a *cost function* of graphs which assigns certain nonnegative values to all graphs. It is often of interest to consider the $\mathcal{H}$-decomposition of a given graph so that the total "cost" (i.e., the sum of the f values of all subgraphs in the $\mathcal{H}$-decomposition) is minimized. We define

$$\alpha_f(G; \mathcal{H}) = \min_P \sum_i f(G_i),$$

where $P = \{G_1, \cdots, G_t\}$ ranges over all $\mathcal{H}$-decompositions of G.

Typical questions one asks are to find $\alpha_f(G; \mathcal{H})$ or to determine

$$\alpha_f(n; \mathcal{H}) = \min_G \alpha_f(G; \mathcal{H}),$$

where G ranges over all graphs on n vertices.

Before proceeding to our main results, we shall first survey some of the known related results in this area. We abbreviate $\alpha(G; \mathcal{H}) = \alpha_{f_1}(G; \mathcal{H})$ and $\alpha(n; \mathcal{H}) = \alpha_{f_1}(n; \mathcal{H})$ where $f_1(G) = |V(G)|$. We also write $\alpha_*(G; \mathcal{H}) = \alpha_{f_0}(G; \mathcal{H})$ and $\alpha_*(n; \mathcal{H}) = \alpha_{f_0}(n; \mathcal{H})$ where $f_0(G) = 1$ for any graph G.

Let $\mathcal{B}$ denote the set of all complete biparite graphs. The problem of determining $\alpha(n; \mathcal{B})$ arises in the study of the networks of contacts realizing certain symmetric monotone Boolean functions (see [9], [13] and [16]). For this problem G. Hensel [9] obtained the estimate

$$n \log_2 n \leqq \alpha(K_n; \mathcal{B}) \leqq n \log_2 n + (1 - \log_2 e + \log_2 \log_2 e)n,$$

where e is the base of the natural logarithm. This question was also investigated by P. Erdös, A. Rényi and V. T. Sós [5], and the first part of the preceding inequality was also proved independently by G. Katona and E. Szemerédi [11].

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[1] The reader is referred to [8] for undefined terminology.

The bounds for $\alpha(n; \mathscr{B})$ were found by F. R. K. Chung, P Erdös and J. Spencer [2]; namely,

$$(1-\varepsilon)\frac{n^2}{2e \log n} < \alpha(n; \mathscr{B}) < (1+\varepsilon)\frac{n^2}{\log n},$$

for a given ε and large n, where $e = 2.718 \cdots$.

A theorem of R. L. Graham and H. O. Pollak [6] asserts that for the complete graph K_n on n vertices,

$$\alpha_*(K_n; \mathscr{B}) = n-1.$$

It is easily seen that a graph on n vertices can be decomposed into $n-1$ stars. Thus, we have

$$\alpha_*(n; \mathscr{B}) = n-1.$$

Let $\mathscr{B}^*$ be the set of all bipartite graphs. It can be easily verified that

$$\alpha(n; \mathscr{B}^*) = \alpha(K_n; \mathscr{B}^*) = \alpha(K_n; \mathscr{B}) = \alpha(n; \mathscr{B}),$$

and

$$\alpha_*(n; \mathscr{B}^*) = \alpha_*(K_n; \mathscr{B}^*) = \lceil \log_2 n \rceil,$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x.

Let $\mathscr{F}$ denote the class of all forests, (i.e., acyclic graphs). In this case $\alpha_*(G, \mathscr{F})$ is usually called the *arboricity* of G (see [8]). Nash-Williams [17] gives the following expression for $\alpha(G; \mathscr{F})$:

$$\alpha_*(G; \mathscr{F}) = \max_S \left\lceil \frac{|E(S)|}{|V(S)|-1} \right\rceil,$$

when S ranges over all nontrivial induced subgraphs of G.

It is immediate that

$$\alpha_*(n; \mathscr{F}) = \alpha_*(K_n; \mathscr{F}) = \lceil n/2 \rceil.$$

Let $\mathscr{T}$ denote the class of all trees. F. R. K. Chung [1] showed that

$$\alpha_*(n; \mathscr{T}) = \lceil n/2 \rceil.$$

It can be easily seen that

$$\alpha(G; \mathscr{T}) = |E(G)| + \alpha_*(G; \mathscr{T})$$

and

$$\alpha(G; \mathscr{T}) \geqq \alpha(G; \mathscr{F}).$$

Thus we have

$$\alpha(n; \mathscr{T}) = \alpha(n; \mathscr{F}) = \lceil n^2/2 \rceil.$$

Finally, we should mention the striking work of R. M. Wilson [18] who investigated the decomposition of the complete graph into subgraphs which are all isomorphic to a specified graph, i.e., the case $G = K_n$ and $\mathscr{H} = \{H\}$. If such an $\mathscr{H}$-decomposition of G exists, then it follows immediately that: (a) the number of edges in H divides the number of edges in K_n;(b) the greatest common divisor of the degrees of vertices in H

divides $n-1$. Wilson showed that these two necessary conditions are sufficient for n sufficiently large (as a function of H).

A 15 year-old conjecture of T. Gallai asserts that for $\mathscr{P}$, the set of all paths, the following equality holds:

CONJECTURE (Gallai). $\alpha_*(n; \mathscr{P}) = \lceil n/2 \rceil$.

Let $\mathscr{C}$ denote the set of all simple cycles. G. Hajós conjectured that any graph on n vertices having all degrees even can always be decomposed into $\lfloor n/2 \rfloor$ or fewer simple cycles. For a graph G containing vertices having odd degree, we set $\alpha_*(G; \mathscr{C}) = 0$. We can write Hajós's conjecture as follows:

CONJECTURE (Hajós). $\alpha_*(n; \mathscr{C}) = \lfloor n/2 \rfloor$, *where* $\lfloor x \rfloor$ *denotes the greatest integer less than or equal to x.*

L. Lovász [15] proved a variation of the above conjecture by showing

$$\alpha_*(n; \mathscr{H}) = \lfloor n/2 \rfloor,$$

where $\mathscr{H}$ is the class of all paths and cycles. P. Erdös, A. W. Goodman and L. Pósa showed in [4] that a graph on n vertices can always be decomposed into $\lfloor n^2/4 \rfloor$ complete subgraphs, i.e., for $\mathscr{K}$, the set of all complete subgraphs,

$$\alpha_*(n; \mathscr{K}) = \lfloor n^2/4 \rfloor.$$

In fact, they sharpened the above result by showing

$$\alpha_*(n; \{K_2, K_3\}) = \lfloor n^2/4 \rfloor.$$

Finally G. Katona and T. Tarján [12] conjectured that

$$\alpha(n; \mathscr{K}) = \lfloor n^2/2 \rfloor.$$

In this paper, we prove this conjecture. Moreover, we show that the complete bipartite graph on $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ vertices, denoted by B_n, is the only graph with $\alpha(G; \mathscr{K}) = \alpha(n; \mathscr{K})$.

2. On $\alpha(n; \mathscr{K})$ and $\alpha_*(n; \mathscr{K})$. First we remark that $\alpha(G; \mathscr{K})$ can still be defined (in the obvious way) if G is not connected. In the remaining part of the paper, the graphs we consider are not necessarily connected.

The main theorem of the paper will be the following:

THEOREM. *Any graph on n vertices can be decomposed into complete subgraphs so that the sum of the sizes of all subgraphs in this decomposition does not exceed* $\lfloor n^2/2 \rfloor$. *That is,*

(1)
$$\alpha(n; \mathscr{K}) = \max_{|V(G)|=n} \alpha(G; \mathscr{K}) = \lfloor n^2/2 \rfloor.$$

The only graph on n vertices satisfying $\alpha(G; \mathscr{K}) = \lfloor n^2/2 \rfloor$ *is the complete bipartite graph* B_n.

Proof. It is easy to see that the complete bipartite graph B_n has the property that

$$\alpha(B_n; \mathscr{K}) = \lfloor n^2/2 \rfloor.$$

To prove $\alpha(n; \mathscr{K}) = \lfloor n^2/2 \rfloor$, it suffices to show that for any graph on n vertices, we have

(2)
$$\alpha(G; \mathscr{K}) \leqq \lfloor n^2/2 \rfloor.$$

Let $G_1, \cdots, G_t$ be a decomposition of G into complete subgraphs. Let p_j denote the number of G_i's which are isomorphic to the complete graph on j vertices (denoted by K_j).

Thus,

$$\alpha(G;\mathcal{K}) \leqq \sum_{i=2}^{n} ip_i = 2e - \sum_{i=3}^{n} p_i i(i-2),$$

since

$$\sum_{i=2}^{n} p_i \binom{i}{2} = |E(G)| = e.$$

We note that inequality (2) holds if and only if there exists a $\mathcal{K}$-decomposition of G which satisfies the following:

$$2e - \sum_{i=3}^{n} p_i i(i-2) \leqq \lfloor n^2/2 \rfloor. \tag{3}$$

It is easily seen that the theorem holds for $n = 1$ or 2. We may assume $n \geqq 3$, and for any graph H on m vertices, $m < n$, we have

$$\alpha(H;\mathcal{K}) \leqq \lfloor m^2/2 \rfloor,$$

with equality if and only if H is B_m.

Let v^* be a vertex of G such that the degree of v^* does not exceed the degree of any other vertex in G; i.e.,

$$\deg v^* = \delta = \min_{v \in V(G)} \deg v.$$

Let L denote the induced subgraph on the set of vertices of G which are adjacent to v^*. A *vertex decomposition* of L is defined to be a set of vertex-disjoint complete subgraphs of L, say $M_1, M_2, \cdots, M_r$, such that $\sum_i |V(M_i)| = |V(L)|$. A vertex decomposition $M_1, \cdots, M_r$, where $|V(M_1)| \geqq |V(M_2)| \geqq \cdots \geqq |V(M_r)|$, of L is said to be *maximal* if for any vertex decomposition $N_1, \cdots, N_t$ either we have $|V(M_i)| = |V(N_i)|$ for $i = 1, \cdots, r$ where $r = t$, or there exists k such that $|V(M_k)| > |V(N_k)|$ and $|V(M_j)| = |V(N_j)|$ for $j < k$. Lst X_i denote the set of all complete subgraphs on i vertices in a fixed maximal vertex decomposition $P = \{M_1, \cdots, M_r\}$ of L.

We consider the graph G' with vertex set $V(G) - \{v^*\}$ and edge set $\{\{u, v\} \varepsilon E(G): v^* \notin \{u, v\}$, and $\{u, v\}$ is not an edge of any M_i, $i = 1, \cdots, r\}$.

By the induction assumptions and (3), there exists a decomposition of G' into complete subgraphs, p_i' of which are isomorphic to K_i, $i = 2, \cdots, n-1$, such that

$$\sum_{i=3}^{n-1} p_i' \, i(i-2) \geqq 2e' - \lfloor (n-1)^2/2 \rfloor, \tag{4}$$

where

$$e' = |E(G')| = e - \delta - \sum_{i=2}^{\delta} x_i \binom{i}{2}$$

and x_i denotes $|X_i|$.

We consider a decomposition P^* of G consisting of the union of the preceding decomposition of G' and x_i complete subgraphs isomorphic to K_{i+1} with v^* as one of the vertices, $1 \leqq i \leqq n$.

The number of subgraphs in P^* which are isomorphic to K_i is just

$$p_i = p_i' + x_{i-1} \quad \text{for } 2 \leqq i \leqq n.$$

We want to show that this choice of p_i satisfies (3). We have

$$\begin{aligned}\sum_{i=3}^{n} p_i\, i(i-2) &= \sum_{i=3}^{n} (p'_i + x_{i-1}) i(i-2)\\ &\geqq 2e' - \lfloor (n-1)^2/2 \rfloor + \sum_{i\geqq 1} x_i(i+1)(i-1)\\ &= 2\Big(e - \delta - \sum_{i\geqq 2} x_i \binom{i}{2}\Big) - \lfloor (n-1)^2/2 \rfloor + \sum_{i\geqq 1} x_i(i^2-1)\\ &= 2e - \lfloor n^2/2 \rfloor + \Big(\lfloor n^2/2 \rfloor - \lfloor (n-1)^2/2 \rfloor - 2\delta + \sum_{i\geqq 1} (i-1)x_i\Big).\end{aligned} \tag{5}$$

In order to establish (3) it suffices to show that

$$\begin{aligned}\sum_{i\geqq 1} (i-1)x_i &\geqq 2\delta - \lfloor n^2/2 \rfloor + \lfloor (n-1)^2/2 \rfloor\\ &= 2\delta - n + \varepsilon_n,\end{aligned} \tag{6}$$

where

$$\varepsilon_n = \begin{cases} 0 & \text{if } n \text{ is even},\\ 1 & \text{if } n \text{ is odd}.\end{cases}$$

We note that (6) is obviously true if $2\delta - n + \varepsilon_n < 0$. We may assume that $2\delta - n + \varepsilon_n \geqq 0$.

We consider the subgraph L. By the minimality of δ it is easy to see that any vertex in L has degree at least $2\delta - n$ in L. Let u^* be an arbitrary vertex of M_r. By the maximality of P, u^* is adjacent to at most $j-1$ vertices of any subgraph in X_j. Therefore, we have

$$2\delta - n \leqq \deg_L u^* \leqq \sum_{i\geqq 1} (i-1)x_i. \tag{7}$$

Consider first the case where n is even. Then (6) follows from (7) so that (3) is established and (2) is valid. Now, suppose G is a graph with $\alpha(G;\mathscr{K}) = \lfloor n^2/2 \rfloor$. We consider the $\mathscr{K}$-decomposition P^* of G. It is easily seen that

$$2e = \sum_{i\geqq 3} p_i i(i-2) = \sum_{i\geqq 2} ip_i \geqq \alpha(G;\mathscr{K}) = \lfloor n^2/2 \rfloor.$$

Thus, equality in (4), (5) and (6) holds. By the induction assumptions, G' is isomorphic to B_{n-1}. Hence δ is at most $\lceil (n-1)/2 \rceil$. From the equality in (6), δ is at least $n/2$. Therefore we have

$$\delta = n/2 \quad \text{and} \quad \sum_{i\geqq 1} (i-1)x_i = 0.$$

Thus $x_i = 0$ for all $i > 1$ and L is the trivial graph on $n/2$ vertices. Therefore G is B_n.

Next, consider the case where n is odd. From (6) and (7), we note that (3) holds unless

$$2\delta - n = \deg_L u^* = \sum_{i\geqq 1} (i-1)x_i,$$

and therefore we will assume the equality in (7).

Suppose equality in (4) holds. By the induction assumption, G' is, in fact, isomorphic to B_{n-1}. Since δ is the minimum degree in G, δ is at most $(n-1)/2$. However, we are assuming that $2\delta - n + \varepsilon_n \geqq 0$. Thus $2\delta - n + 1 = 0$ and inequality (3) holds. We may assume equality in (4) does not hold. This implies that the equality in (5) also becomes strict. Therefore we have

$$\sum_{i=3}^{n} p_i i(i-2) \geqq 2e - \lfloor n^2/2 \rfloor + 1 + \sum_{i \geqq 1} (i-1)x_i - 2\delta + n - 1 \tag{8}$$
$$= 2e - \lfloor n^2/2 \rfloor.$$

Thus (3) is valid for n odd.

Suppose G is a graph satisfying $\alpha(G; \mathcal{K}) = \lfloor n^2/2 \rfloor$ and n is odd. We consider the following two possibilities:

(a) $\alpha(G'; \mathcal{K}) = \lfloor (n-1)^2/2 \rfloor$. It follows that equality in (4) holds. Thus G' is isomorphic to B_{n-1} and δ is equal to $(n-1)/2$. Equality in (5) and (6) also holds. Therefore

$$\sum_{i \geqq 1} (i-1)x_i = 0.$$

Thus, $x_i = 0$ for $i > 1$ and G is isomorphic to B_n.

(b) $\alpha(G'; \mathcal{K}) < \lfloor (n-1)^2/2 \rfloor$. Therefore equality in (4) does not hold. However, it follows from $\alpha(G; \mathcal{K}) = \lfloor n^2/2 \rfloor$ that equality in (8) holds. Thus

$$\sum_{i \geqq 1} (i-1)x_i = 2\delta - n = \deg_L u^*, \tag{9}$$

$$\alpha(G'; \mathcal{K}) = \lfloor (n-1)^2/2 \rfloor - 1. \tag{10}$$

In Case (b) we will prove a sequence of claims to establish that G is a complete t-partite graph, $t \geqq 3$. This will then show by Lemmas 1 to 4 that G does not satisfy $\alpha(G; \mathcal{K}) = \lfloor n^2/2 \rfloor$.

Since we assume the equality in (7), it follows immediately that any vertex in M_r is then adjacent to exactly $j-1$ vertices of any graph in X_j. Moreover, based on the fact that $\sum_{i \geqq 1} ix_i = \delta$, we have $\sum_{i \geqq 1} x_i = n - \delta = r$.

CLAIM 1. *Any vertex in L is adjacent to at least $|V(M_r)| - 1$ vertices of M_r.*

Proof. Suppose to the contrary that a vertex w in L is not adjacent to u and u' in M_r. Assume w is a vertex of M_i for some i. Thus u and u' are adjacent to all vertices of M_i except for w. Then the induced graph of L on $(V(M_i) - \{w\}) \cup \{u, u'\}$ is a complete graph with more vertices than M_i. This contradicts the maximality of P. □

CLAIM 2. *Let u_i be a vertex of M_i such that u_i is not adjacent to $u^* = u_r$ for $i = 1, \cdots, r-1$. Then u_i, $1 \leqq i \leqq r$, is adjacent to exactly $j-1$ vertices of any graph in X_j.*

Proof. Suppose u_i is adjacent to all vertices in M_t. It follows that $r > t > i$. We consider another vertex decomposition of L, called $P' = \{N_1, \cdots, N_r\}$, such that $N_j = M_j$ if $j \neq i, t, r$; N_i is the induced subgraph of L on $(V(M_i) - \{u_i\}) \cup \{u^*\}$; N_t is the induced subgraph of L on $V(M_t) \cup \{u_i\}$; and N_r is the induced subgraph of L on $V(M_r) - \{u^*\}$. Thus P is not maximal. This is impossible. Therefore u_i, $1 \leqq i \leqq r$ is adjacent to at most $j-1$ vertices of any graph in X_j. Since

$$\deg_L u_i \geqq 2\delta - n = \sum_{j \geqq 1} (j-1)x_j,$$

we conclude that u_i is adjacent to exactly $j-1$ vertices of any graph in X_j. □

CLAIM 3. *u_i, $i=1,\cdots,r$, is adjacent to every vertex in $V(G)-V(L)$ and the degree of u_i in G is δ.*

Proof. It follows from Claim 2 that the degree of u_i is at most

$$r+\sum_{j\geqq 1}(j-1)x_j=(n-\delta)+(2\delta-n)=\delta.$$

On the other hand, δ is the minimum degree in G. Thus the degree of u_i in G is δ and u_i is adjacent to any vertex in $V(G)-V(L)$. □

CLAIM 4. *Let w be a vertex in L which is not adjacent to u_i for some i. Then w is of degree δ.*

Proof. Suppose $i=r$. It follows from Claim 3 that w has degree δ. We may assume $i\neq r$. We can also assume $w\neq u_i$ and w is not a vertex of M_i. Let w be a vertex of M_j. Suppose $j=r$. Then w has degree δ. We consider the case $j\neq r$. Suppose w is adjacent to all vertices in M_t. If $t\neq r$, we consider the following vertex decomposition $P''=\{L_1,\cdots,L_r\}$ such that $L_k=M_k$ if $k\neq i,j,t,r$; L_i is the induced subgraph of L on $(V(M_i)-\{u_i\})\cup\{u^*\}$; L_j is the induced subgraph of L on $(V(M_j)-\{w\})\cup\{u_i\}$; L_t is the induced subgraph of L on $V(M_t)\cup\{w\}$; and L_r is the induced subgraph of L on $V(M_r)-\{u^*\}$. This contradicts the maximality of P. Thus, w is adjacent to at most $j-1$ vertices of any graph in X_j. If $t=r$, we consider the following vertex decomposition $\bar{P}''=\{L'_1,\cdots,L'_r\}$ such that $L'_k=M_k$ if $k\neq i,j,r$; L'_i is the induced subgraph of L on $(V(M_i)-\{u_i\})\cup\{u^*,w\}$; L'_j is the induced subgraph of L on $(V(M_j)-\{u_j\})\cup\{u_i\}$; and L'_r is the induced subgraph of L on $V(M_r)-\{u^*\}$. This again contradicts the maximality of P. Since the degree of w is at least δ in G and $2\delta-n$ in L, we conclude that the degree of w is δ in G and w is adjacent to exactly $j-1$ vertices of any graph in X_j. □

CLAIM 5. *$|V(M_i)|=|V(M_1)|$ for $i=1,\cdots,r$.*

Proof. We choose w in G' with minimum degree δ' in G'. Let L' be the induced subgraph on the set of vertices of G' which are adjacent to w. Let $M'_1, M'_2,\cdots,M'_s$ be a maximal vertex decomposition of L'. We consider G'' with vertex set $V(G')-\{w\}$ and edge set $\{(u,v)\in E(G')\colon w\notin\{u,v\}$ and (u,v) is not an edge of any M'_i, $i=1,\cdots,s\}$.

By the induction assumptions, there exists a decomposition of G'' into complete subgraphs, p''_i of which are isomorphic to K_i, $i=2,\cdots,n-2$, such that

$$\sum_{i=3}^{n-2}p''_i i(i-2)=2e''-\alpha(G'';\mathcal{K})\geqq 2e''-\lfloor(n-2)^2/2\rfloor, \tag{11}$$

where

$$e''=|E(G'')|=e'-\delta'-\sum_{i=2}^{\delta'}x'_i\binom{i}{2},$$

and x'_i denotes the cardinality of the set X'_i which consists of all complete subgraphs on i vertices in the maximal vertex decomposition $P'=\{M'_1,\cdots,M'_s\}$ of L'.

We consider a decomposition of G' consisting of the union of P'' and x'_i complete subgraphs isomorphic to K_{i+1} with w as one of its vertices, $1\leqq i\leqq\delta'$. Let q_i denote the number of subgraphs in this decomposition of G' which are isomorphic to K_i. From (10) we have

$$2e'-\lfloor(n-1)^2/2\rfloor+1\geqq\sum_{i=3}^{n-2}q_i i(i-2).$$

From (11) and $\sum_{i\geqq 1} ix'_i = \delta'$, we also have

$$\sum_{i=3}^{n-2} q_i i(i-2) = \sum_{i=3}^{n-2} (p''_i + x'_{i-1}) i(i-2)$$

$$= 2e' - \alpha(G''; \mathcal{K}) - \delta' - \sum_{i\geqq 1} x'_i.$$

By the maximality of P', we have

$$(12) \qquad 2\delta' - (n-1) \leqq \text{minimum degree in } L' \leqq \sum_{i\geqq 1} (i-1)x'_i.$$

Therefore we have

$$(13) \qquad n-1-\delta' \geqq \sum_{i\geqq 1} x'_i \geqq \lfloor (n-1)^2/2 \rfloor - \alpha(G''; \mathcal{K}) - \delta' - 1,$$

i.e., $\alpha(G''; \mathcal{K}) \geqq \lfloor (n-2)^2/2 \rfloor - 1$.

Suppose $\alpha(G''; \mathcal{K}) = \lfloor (n-2)^2/2 \rfloor$. Then by the induction assumptions, G'' is B_{n-2} and δ' is at most $(n-1)/2$. On the other hand, δ' is at least $(n-1)/2$. (Suppose $\delta' \leqq (n-1)/2 - 1$. Since $\sum_{i\geqq 1} ix'_i = \delta'$, we have $n-3-\delta' \geqq \sum_{i\geqq 1} x'_i$. From (13), we will then have $\alpha(G''; \mathcal{K}) > \lfloor (n-2)^2/2 \rfloor$, which contradicts the induction assumptions.) Therefore $\delta' = (n-1)/2$. From (13), we also have $\sum_{i\geqq 1} x'_i \geqq n - \delta' - 2$, i.e., $\sum_{i\geqq 1} (i-1)x'_i \leqq 1$. Suppose $\sum_{i\geqq 1} (i-1)x'_i = 0$. Then G' is B_{n-1}, which contradicts (10). Therefore we have $x'_2 = 1$ and $x'_1 = \delta' - 2$. It can be easily verified that G' has a $\mathcal{K}$-decomposition which contains one K_3 and $(n-1)^2/4 - 3\ K_2$. This contradicts (10). Thus we may assume $\alpha(G''; \mathcal{K}) = \lfloor (n-2)^2/2 \rfloor - 1$, and

$$(14) \qquad \sum_{i\geqq 1} (i-1)x'_i = 2\delta' - n + 1 = \text{minimum degree in } L'.$$

We note that Claim 1 to Claim 4 all follow from (9) and the maximality of P. Therefore we can show in a similar manner that for any vertex w with degree δ' there exists a vertex w^* with degree δ' in G', so that the $n-1-\delta'$ vertices which are not adjacent to w^* are of degree δ' in G' and w adjacent to w^* in G'.

Let k be the size of $V(M_1)$. Then it can be easily seen that $\delta' = \delta - k$. We also note that any vertex in M_i has degree at least $\delta - |V(M_i)|$ in G'.

Since u_1 has degree δ' in G', there is a vertex $\bar{w}$ which is adjacent to u_1 in G' such that all the $n-1-\delta'$ vertices which are not adjacent to $\bar{w}$ are of degree δ' in G. Since any vertex in $G'-L$ had degree at least δ which is greater than δ', we may assume $\bar{w}$ is a vertex of $\bar{M} = M_i$, where $|V(M_i)| = k$. Since all vertices of $\bar{M}$ are not adjacent to $\bar{w}$ in G', all vertices of $\bar{M}$ have degree δ' in G'. Without loss of generality, we may assume that all vertices in M_1 have degrees δ' in G'.

Now, let M^* be the set of vertices in M_1 which are not adjacent to some vertex in M_r. Since every vertex in M_r is adjacent to exactly $k-1$ vertices in M_1 and any vertex in M_1 is adjacent to at least $k'-1$ vertices in M_r, where $|V(M_r)| = k'$, we have $|M^*| = |V(M_r)|$. Suppose M^* is a proper subset of $V(M_1)$. We may choose w to be a vertex in $V(M_1) - M^*$. Thus u^* is in L'. From Claims 1 to 4, there exists a vertex w^* with degree δ' in G' so that the $n-1-\delta'$ vertices which are not adjacent to w^* are of degree δ' in G'. We may assume, without loss of generality, that w^* is a vertex of M_2. Either u^* is nonadjacent to w^* or u^* is nonadjacent to a vertex in M_2 which is nonadjacent to w^* in L'. Thus, by Claims 3 and 4, the degree of u^* is δ' in G'. This implies $|V(M_r)| = k = |M^*|$.

This contradicts our assumption that M^* is a proper subset of $V(M_1)$. Therefore, we have shown that $|V(M_i)|=|V(M_1)|=k$.

CLAIM 6. *Each vertex in L is adjacent to exactly $j-1$ vertices of a graph in X_j and has degree δ.*

Proof. This follows from Claims 2 and 5. □

CLAIM 7. *For any i, j, $1 \leqq i, j \leqq r$, u_i and u_j are not adjacent to each other.*

Proof. Suppose u_i and u_j are adjacent. u_j is adjacent to exactly $k-1$ vertices of M_i. Let w denote the vertex in M_i which is not adjacent to u_j. From Claim 6, we have that w is adjacent to all vertices in M_j except for u_j. Now, we consider a vertex decomposition $\bar{P}=\{R_1, \cdots, R_r\}$ of L, where $R_t = M_t$ if $t \neq i, j, r$; M_j is the induced subgraph of L on $V(M_j)-\{u_j\}\cup\{u^*, w\}$; M_i is the induced subgraph of L on $V(M_i)-\{u_i\}$; M_r is the induced subgraph of L on $V(M_r)-\{u^*\}\cup\{u_i\}$. This contradicts the maximality of P. □

CLAIM 8. *Any vertex v in L has the property that the $n-\delta-1$ vertices which are not adjacent to v are mutually nonadjacent.*

Proof. This follows from Claims 5 and 7. □

CLAIM 9. *Any vertex v in G has the property that the degree of v is δ and the $n-\delta-1$ vertices which are not adjacent to v are mutually nonadjacent.*

Proof. This follows from Claim 8 and the fact that the choice of v^* is arbitrary. □

CLAIM 10. *G is a complete t-partite graph, where $t=n/(n-\delta)\geqq 3$. $V(G)$ is a disjoint union of t sets of cardinality $n-\delta$, and two vertices in G are adjacent if and only if they do not belong to the same set.*

Proof. Let I_1 be the set of vertices of G each of which is v_1 or is not adjacent to v_1. It follows from Claim 9 that any vertex in I_1 is adjacent to all vertices not in I_1 and not adjacent to any vertex in I_1 in G. If I_1 is a proper subset of $V(G)$, we choose a vertex v_2 in $V(G)-I_1$. Let I_2 be the set of vertices of G which is v_2 or is not adjacent to v_2. After a finite number of steps, we have sets $I_1, \cdots, I_t$ and G is a t-partite graph and $|I_i|=n-\delta$. Since $n \geqq 3$, we have $\alpha(G;\mathcal{K})=[n^2/2]>0$. Therefore G is not the trivial graph; i.e., $t>1$. Since n is odd, t is not 2. We have $t \geqq 3$. □

We have shown that G is a complete t-partite graph where $t \geqq 3$. In the following there are some auxiliary lemmas dealing with the value $\alpha(G;\mathcal{K})$ for complete t-partite graphs G.

LEMMA 1. *Let Q be a complete t-partite graph on $V(Q)=I_1\cup\cdots\cup I_t$ and $|I_i|=t$ for $i, \cdots, t$. Suppose t is a prime number. Then $\alpha(Q;\mathcal{K})=t^3$.*

Proof. Let v_{ij}, $1\leqq j\leqq t$, be vertices in I_i. Let ${}_jQ_k$ denote the complete graph on $v_{i,z}$, $i=1,\cdots,t$, where $z\equiv(i-1)(k-1)+j \pmod t$ and $1\leqq z\leqq t$. We note that $\{{}_jQ_k: 1\leqq j, k\leqq t\}$ is a $\mathcal{K}$-decomposition of Q. Thus

$$\alpha(Q;\mathcal{K})\leqq t^3.$$

On the other hand, the maximal complete subgraph contained in Q is K_t. Let P be a $\mathcal{K}$-decomposition of Q. For any edge e in Q, define w as follows:

$$w(e)=\frac{2}{f(e)-1},$$

where $f(e)$ is the number of vertices of the graph in P which contains e.

It is easy to see that

$$w(e)\geqq\frac{2}{t-1}.$$

We note that

$$\alpha(Q;\mathcal{K}) = \min_P \sum_e w(e) \geqq |E(Q)| \cdot \frac{2}{t-1} = \binom{t}{2} t^2 \cdot \frac{2}{t-1} = t^3.$$

Therefore, $\alpha(Q;\mathcal{K}) = t^3$. □

LEMMA 2. *Let Q be a complete 3-partite graph on $I_1 \cup I_2 \cup I_3$ and $|I_i| = t$ for $i = 1, 2, 3$. Then $\alpha(Q;\mathcal{K}) = 3t^2$.*

Proof. Let v_{ij}, $1 \leqq j \leqq t$, be vertices in I_i. We consider a $\mathcal{K}$-decomposition of Q which consists of the following graphs: $Q_{j,k}$, $1 \leqq j,\ k \leqq t$, where $Q_{j,k}$ is the complete graph on $v_{1,j}$, $v_{2,s}$, $v_{3,k}$ where $s \equiv k + j \pmod t$ and $1 \leqq s \leqq t$. Thus $\alpha(Q;\mathcal{K}) \leqq 3t^2$. By a similar proof to that in Lemma 1 we have $\alpha(Q;\mathcal{K}) = 3t^2$. □

LEMMA 3. *Let Q be a complete t-partite graph on $I_1 \cup I_2 \cup \cdots \cup I_t$ and $|I_i| = 3$ for $i = 1, 2, \cdots, t$. Then $\alpha(Q;\mathcal{K}) \leqq 3t^2$.*

Proof. We consider the following two possibilities:

Case 1. $t \neq 0 \pmod 2$. We consider a $\mathcal{K}$-decomposition consisting of

(a) Q^j, $1 \leqq j \leqq 3$, where Q^j is the complete graph on $v_{i,j}$, $i = 1, \cdots, t$; and

(b) Q_{jk}, $1 \leqq j \leqq t$, $1 \leqq k \leqq t$, $j \neq k$, where Q_{jk} is the complete graph on $v_{j,1}$, $v_{s,2}$, $v_{k,3}$ and $s \equiv \sigma^{-1}(j+k) \pmod t$ and σ is the permutation $\sigma(i) = 2i$.

Case 2. $t \equiv 0 \pmod 2$. We consider a $\mathcal{K}$-decomposition consisting of

(a) Q^j, $1 \leqq j \leqq 3$; and

(b) Q'_{jk}, $1 \leqq j \leqq t$, $1 \leqq k \leqq t$, $j \neq k$, where Q'_{jk} is the complete graph on $v_{j,i}$, $v_{s,2}$, $v_{k,3}$ and $s \equiv \zeta^{-1} f(j+k)$ where

$$f(x) = \begin{cases} x & \text{if } x \leqq t, \\ x - t - 1 & \text{if } x > t+1, \\ t & \text{if } x = t+1, \end{cases}$$

and ζ is the permutation $f(i) = f(2i)$.

It is easy to check that in both cases we have $\mathcal{K}$-decompositions and

$$\alpha(Q;\mathcal{K}) \leqq 3t^2. \qquad \square$$

LEMMA 4. *Let G be a complete t-partite graph on $V(G) = I_1 \cup \cdots \cup I_t$ and $|I_i| = s$ for $1 \leqq i \leqq t$. Let $q = \max(s, t)$. We have*

$$\alpha(G;\mathcal{K}) \leqq n(2q-2),$$

where

$$|V(G)| = n = st.$$

Proof. By Bertrand's postulate (see [10]), there exists a prime p between q and $2q-2$. We will show that $\alpha(G;\mathcal{K}) \leqq np$.

It is easy to see that G is a subgraph of the complete p-partite graph Q on $I'_1 \cup \cdots \cup I'_p$ where $|I'_i| = p$. From the proof of Lemma 1 there is a $\mathcal{K}$-decomposition P of Q which consists of subgraphs isomorphic to K_p. We consider a $\mathcal{K}$-decomposition P' of G which is defined to be $\{G_i \cap G : G_i \in P\}$. Therefore, we have

$$\alpha(G;\mathcal{K}) \leqq \sum_{G_i \in P} |V(G_i \cap G)| \leqq np \leqq n(2q-2),$$

since every vertex is in at most p of the K_p's. □

Now, from Lemmas 2 and 3 and the fact that $3t^2 < \lfloor (3t)^2/2 \rfloor$ for t positive, we may assume $r = n - \delta > 3$, $t = n/(n-\delta) > 3$. Since $rt = n$, we have $q \leqq n/4$. From Lemma 4, we have

$$\alpha(G; \mathcal{H}) \leqq n(2q-2) \leqq n(n/2-2).$$

This contradicts the assumption that $\alpha(G; \mathcal{H}) = \lfloor n^2/2 \rfloor$. Thus we have shown that Case (b) is impossible.

We also note that

$$2\alpha_*(G; \mathcal{H})) \leqq \alpha(G; \mathcal{H}).$$

Any graph G with $\alpha_*(G; \mathcal{H}) = \alpha_*(n, \mathcal{H}) = [n^2/4]$ must then have $\alpha(G; \mathcal{H}) = \lfloor n^2/2 \rfloor = \alpha(n; \mathcal{H})$. Therefore, a graph G on n vertices having $\alpha(G; \mathcal{H}) = \alpha(n; \mathcal{H})$ or $\alpha_*(G; \mathcal{H}) = \alpha_*(n; \mathcal{H})$ is the complete bipartite graph on $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ vertices. This completes the proof of the main theorem. □

3. Concluding remarks. Problems of the type we have discussed in this paper are not only interesting in their own right, but also have potential applications in communication and switching networks. Sometimes it is desirable to decompose a communication or switching network into parts of certain specified types. The problem of determining $\alpha(G; \mathcal{H})$ or $\alpha_*(G; \mathcal{H})$ is equivalent to the problem of minimizing the "cost" of building the network (with the corresponding graph G) by using certain types of parts $\mathcal{H}$. In fact, the study of $\alpha(n, \mathcal{B})$ was first motivated by consideration of contact networks.

There are many interesting problems left open in this area. For example, the Gallai conjecture on $\alpha(n; \mathcal{P})$ still remains unsolved. We can ask the question of determining $\alpha(n; \mathcal{H})$ for $\mathcal{H}$ a class of graphs with certain specified properties, e.g., each graph has connectivity $\leqq \rho$, has chromatic number $\leqq \tau$, etc.

We can consider a variation of the above problems. For a graph G and a class of graphs $\mathcal{H}$, we define an $\mathcal{H}$-covering of G to be a family of subgraphs, $G'_1, \cdots, G'_{t'}$, such that every edge is in at least one of the G'_i and every G'_i is in $\mathcal{H}$. For a cost function f, we define

$$\beta_f(G; \mathcal{H}) = \min_{P'} \sum_{1} f(G'_i),$$

where $P' = \{G'_1, \cdots, G'_{t'}\}$ ranges over all $\mathcal{H}$-coverings of G, and

$$\beta_f(n; \mathcal{H}) = \max_{G} \beta_f(G; \mathcal{H}),$$

where G ranges over all graphs on n vertices.

We note that an $\mathcal{H}$-decomposition of G is also an $\mathcal{H}$-covering of G. Therefore

$$\beta_f(G; \mathcal{H}) \leqq \alpha_f(G; \mathcal{H}),$$

and

$$\beta_f(n; \mathcal{H}) \leqq \alpha_f(n; \mathcal{H}).$$

The preceding equalities sometimes hold and sometimes do not. For example, from the main result of this paper it is easily seen that

$$\alpha_{f_1}(n; \mathcal{H}) = \beta_{f_1}(n; \mathcal{H}) = \lfloor n^2/2 \rfloor.$$

However,

$$\beta_{f_0}(K_n;\beta)=\lfloor \log_2 n\rfloor \quad \text{and} \quad \alpha_{f_0}(K_n;\beta)=n-1.$$

Also from [3] we have

$$n-n^{11/14+\varepsilon}<\alpha_{f_0}(n;\beta)<n-c\log n,$$

for a given $\varepsilon>0$ and some constant c if n is sufficiently large.

We can ask the corresponding question of determining $\beta_f(G;\mathcal{H})$ or $\beta_f(n;\mathcal{H})$ for various classes of graph $\mathcal{H}$ and cost functions f.

We could also consider another kind of variation of this problem in which we wish to decompose a graph G into *induced* subgraphs of some certain type $\mathcal{H}$. We can then ask the corresponding question of determining $\alpha'_f(G;\mathcal{H})$, the minimum cost of subgraphs over all possible decompositions of G, and $\alpha'_f(n;\mathcal{H})$, the maximum value of $\alpha'_f(G;\mathcal{H})$ over all graphs G on n vertices.

Remarks. J. Kahn [7] proved that $\beta_{f_1}(n;\mathcal{K})=\lfloor n^2/2\rfloor$. E. Györi and A. V. Kostočka [10] have recently proved the main result in this paper by a completely different method.

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