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On Blocking Probabilities for Switching Networks

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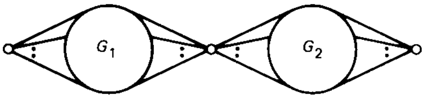
We study the blocking probabilities of multistage switching networks through their linear graphs using Lee's model. We give results which allow us to compare the blocking probabilities of various classes of linear graphs. In particular, we derive techniques for deciding when the blocking probability of one linear graph does not exceed the blocking probability of another linear graph under all possible traffic loads. This allows us to compare the blocking performances of corresponding switching networks containing these linear graphs. Our results apply not only to series-parallel linear graphs, but also to the more general "spider-web" linear graphs, which have recently attracted substantial interest in the theory of switching networks.

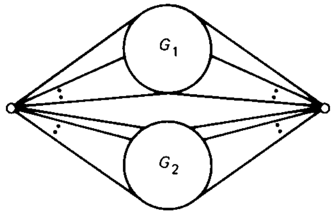
I. INTRODUCTION

A network N consists of a set of switches, a set of links, and two sets of terminals denoted by I and Ω, and called, respectively, the set of input terminals and output terminals. The union of all paths that can be used to connect one call between an input terminal u and an output terminal v is called the *linear graph* determined by u and v, and is denoted by $G(u,v)$. (A linear graph is also called a channel graph.[10]) Let P^* be the union of all paths connecting input terminals to output terminals. A *state* of N is a subset S of P^* such that no two paths in S have a common link. For a given state S, a link is *busy* if it is on a path in S. Otherwise it is *idle*.

Many existing switching networks consist of several stages. We say that N is an *n-stage network* if the set of switches of N can be partitioned into n sets, called *stages*, and links exist only between a switch in stage i and a switch in stage $i + 1$, for $1 \leq i \leq n - 1$. All input terminals are connected to switches in the first stage and all output terminals are connected to switches in the last stage.



(a)



(b)

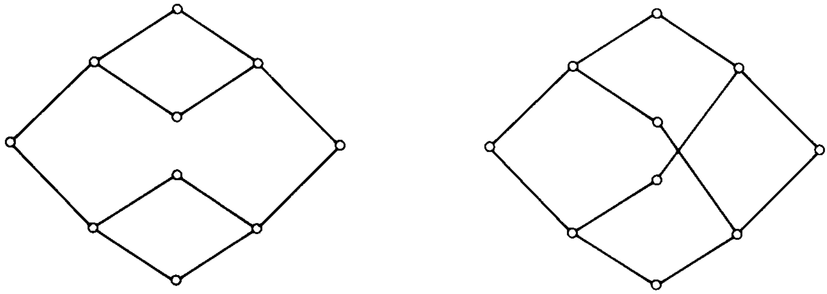
Fig. 1—(a) Series combination. (b) Parallel combination.

In order to simplify the analysis of the switching networks under consideration we will employ Lee's model in Ref. 8. We will also use Lee's independence assumption, namely, that the probabilities of being busy for links in successive stages are independent. Thus, we will assume all links between stage i and stage $i + 1$ have some probability p_i of being busy and some probability $q_i = 1 - p_i$ of being idle, for any i, $1 \leq i \leq k - 1$. Let $P(u,v)$, $u \in I$, $v \in \Omega$, denote the probability that there does not exist a path connecting u and v which consists of idle links. $P(u,v)$ is called the blocking probability for u and v. Note that because of the independence assumption, $P(u,v)$ actually only depends on the linear graph $G(u,v)$ between u and v. Furthermore, we will assume all switches in the same stage are of the same size (i.e., for any switch in stage i, there are r_i inlet lines and r_i' outlet lines).

A network is said to be *balanced* if all the linear graphs $G(u,v)$, $u \in I$, $v \in \Omega$, are isomorphic.[4] It is said to be *partially balanced* if there are only relatively few nonisomorphic linear graphs. We can then compare the blocking probabilities of two partially balanced switching networks by comparing the blocking probabilities of the corresponding linear graphs.

A linear graph is said to be a *series-parallel* linear graph if it is either a series combination or a parallel combination of two series-parallel linear graphs of smaller sizes (see Fig. 1a,b). A linear graph is said to be a *spider-web* linear graph if it is not series-parallel. In Fig. 2 we give examples of a series-parallel linear graph (Fig. 2a) and a spider-web linear graph (Fig. 2b). A linear graph $G(u,v)$ is said to be a *multilink* linear graph if there exist two switches in $G(u,v)$ which are connected by more than one link. Any linear graph which is not a multilink graph is said to be a *simple-link* linear graph.

In this paper, we present several general methods for comparing blocking probabilities of various classes of switching networks. These methods generalize and improve previous results in this area.[2,7] These results can be applied not only to series-parallel linear graphs but also to more general spider-web linear graphs. We also consider the general case in which two switches can be connected by more than one link.



(a) (b)

Fig. 2—(a) A series-parallel linear graph. (b) A spider-web linear graph.

II. LINEAR GRAPHS IN THREE-STAGE NETWORKS

We denote an n-stage network by the following:

(i) The switch set

$$\bigcup_{i=1}^{n} \{s(i,j): 1 \le j \le t_i\}$$

where the stage i consists of t_i switches which are labeled by $s(i,j)$, $j = 1,2,\ldots,t_i$;

(ii) The link set

$$\bigcup_{i=1}^{n} \{L(i,j,k): 1 \le j \le t_i,\ 1 \le k \le t_{i+1}\}$$

where $L(i,j,k)$ denotes the set of links connecting $s(i,j)$ and $s(i+1,k)$;

(iii) I and Ω, the input and output terminals, respectively. We note that for fixed i we have

$$\sum_{k=1}^{t_{i-1}} \ell(i-1,k,j) = \sum_{k=1}^{t_{i-1}} \ell(i-1,k,j') = r_i$$

$$\sum_{k=1}^{t_{i+1}} \ell(i,j,k) = \sum_{k=1}^{t_{i+1}} \ell(i,j',k) = r_i'$$

for any j,j', $1 \le j,\ j' \le t_i$, where $\ell(i,j,k)$ denotes the cardinality of $L(i,j,k)$.

An n-stage linear graph $G(u,v)$ can then be characterized by the following:

(i) The switch set is

$$\bigcup_{i=1}^{n} s_i'$$

where s_i' is a subset of the switch set in stage i and $s_1' = \{u\}$, $s_n' = \{v\}$. (We

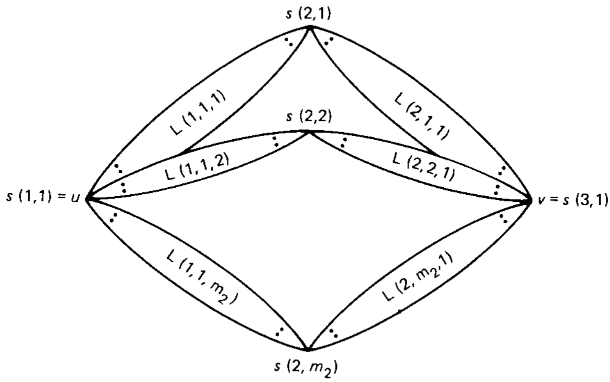


Fig. 3—Three-stage linear graphs.

relabel switches if necessary so that $s_i' = \{s(i,j)\colon 1 \le j \le m_i\}$ for some $m_i \le t_i$, $m_1 = m_n = 1$);

(*ii*) The link set is $\{L(i,j,k)\colon s(i,j)$ and $s(i+1,k)$ are in the switch set of $G(u,v)\}$.

Let $G'(u',v')$ be an n-stage linear graph with the set of switches

$$\bigcup_{i=1}^{n} \{s'(i,j)\colon 1 \le j \le m_i'\}$$

and the set of links $\{L'(i,j,k)\}$. We say $G(u,v)$ and $G'(u',v')$ are *isomorphic* if the following conditions are satisfied.

(*i*) $m_i = m_i'$ for $1 \le i \le n$;

(*ii*) The set of switches in each stage can be properly relabeled such that the following holds:

$$\ell(i,j,k) = \ell'(i,j,k).$$

Now, we consider a three-stage linear graph as shown in Fig. 3 (where switches in middle stages are labeled $s(2,1), \ldots, s(2,m_2)$).

Theorem 1: Let $G(u,v)$ be the linear graph with the set of switches

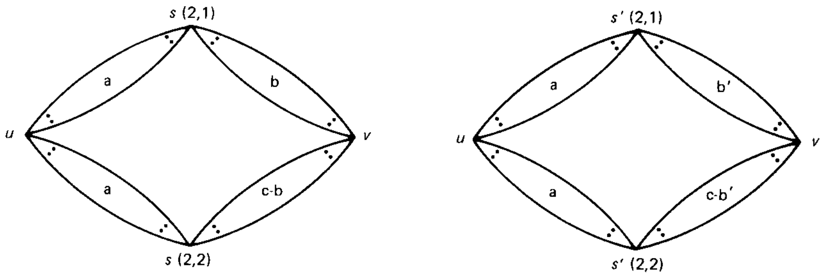
$$\bigcup_{i=1}^{3} \{s(i,j)\colon 1 \le j \le m_i\}$$

and the set of links $\{L(i,j,k)\}$.

Let $G'(u',v')$ be the linear graph with the set of switches

$$\bigcup_{i=1}^{3} \{s'(i,j)\colon I \le j \le m_i'\}$$

and the set of links $\{L'(i,j,k)\}$. Moreover, suppose $G(u,v)$ and $G'(u',v')$ satisfy the following conditions (see Fig. 4a,b):



(a) (b)

Fig. 4—Graphs for Theorem 1.

(*i*) $m_2 = m_2' = 2$
(*ii*) $\ell(1,1,1) = \ell(1,1,2) = \ell'(1,1,1) = \ell'(1,1,2)$
(*iii*) $\ell(2,1,1) + \ell(2,2,1) = \ell'(2,1,1) + \ell'(2,2,1)$
(*iv*) $|\ell(2,1,1) - \ell(2,2,1)| \le |\ell'(2,1,1) - \ell'(2,2,1)|$

where $\ell(i,j,k)$, $\ell'(i,j,k)$ denote the cardinalities of $L(i,j,k)$, $L'(i,j,k)$, respectively. Then we have $P(u,v) \le P(u',v')$.

Proof: Let p_i denote the probability of a link being busy between stage i and stage $i + 1$, $i = 1,2$. Let

$$a = \ell(1,1,1) = \ell(1,1,2) = \ell'(1,1,1) = \ell'(1,1,2)$$

and

$$c = \ell(2,1,1) + \ell(2,2,1) = \ell'(2,1,1) + \ell'(2,2,1).$$

We may assume without loss of generality that

$$b = \ell(2,1,1) \le \ell(2,2,1),$$

$$b' = \ell'(2,1,1) \le \ell'(2,2,1).$$

It is easy to verify that $b' \le b \le c/2$. Define the function $f(x)$ as follows:

$$f(x) = [1 - (1 - p_1^a)(1 - p_2^x)][1 - (1 - p_1^a)(1 - p_2^{c-x})]$$

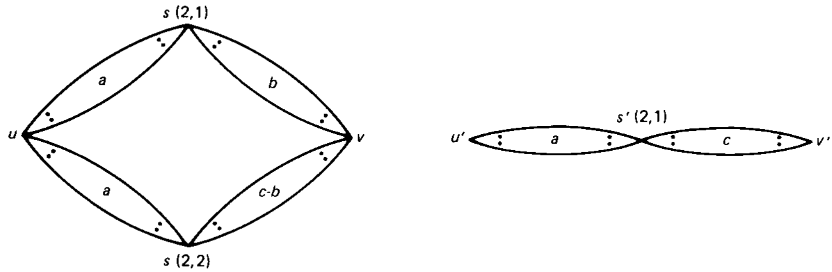
We note that $P(u,v) = f(b)$ and $P(u',v') = f(b')$. Furthermore, f attains its minimum at $x = c/2$ and f is a convex function. Thus we have

$$f(b) \le f(b')$$

and

$$P(u,v) \le P(u',v').$$

We note that the number of paths connecting u and v in $G(u,v)$ is ac, which is also equal to the number of paths connecting u' and v' in $G'(u',v')$.



(a) (b)

Fig. 5—Graphs for Theorem 2.

The following theorem can be viewed as a special case of Theorem 1. Because it is very useful in comparing linear graphs, we will state it here.

Theorem 2: Let $G(u,v)$ be the linear graph with the set of switches

$$\bigcup_{i=1}^{3} \{s(i,j): 1 \leq j \leq m_i\}$$

and the set of links $\{L(i,j,k)\}$, and let $G'(u',v')$ be the linear graph with the set of switches

$$\bigcup_{i=1}^{3} \{s'(i,j): 1 \leq j \leq m_i'\}$$

and the set of links $\{L'(i,j,k)\}$.

Suppose $G(u,v)$ and $G'(u',v')$ satisfy the following conditions (see Fig. 5a,b):

(*i*) $m_2 = 2, m_2' = 1,$
(*ii*) $\ell(1,1,1) = \ell(1,1,2) = \ell'(1,1,1),$
(*iii*) $\ell(2,1,1) + \ell(2,2,1) = \ell'(2,1,1).$

where $\ell(i,j,k)$, $\ell'(i,j,k)$ denote the cardinalities of $L(i,j,k)$, $L'(i,j,k)$, respectively.
Then we have

$$P(u,v) \leq P(u',v')$$

Theorem 2 can be proved by taking $b' = 0$ in Theorem 1.

In the following corollary, we give a short proof for the main theorem in Ref. 4, which asserts that a multilink linear graph can always be replaced by a simple-link linear graph having smaller blocking probability whereas the total numbers of paths in the two linear graphs are the same.

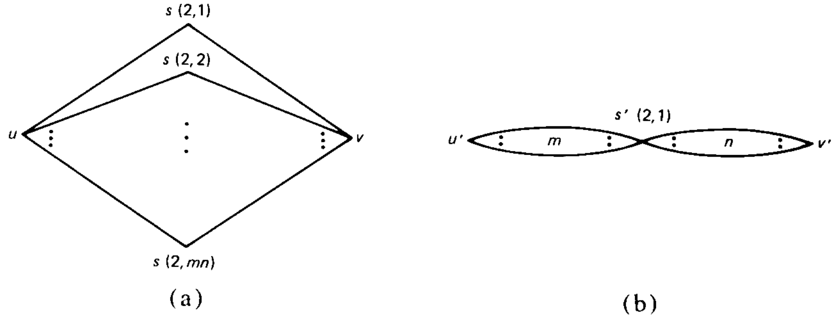


Fig. 6—(a) A single-link linear graph. (b) A multilink linear graph.

Corollary: Let $G'(u,v)$ be a three-stage linear graph with the set of switches $\{u,v\} \cup \{s(2,i): i = 1, \ldots, mn\}$ and $\ell(1,1,i) = \ell(2,i,1) = 1$ for $1 \le i \le m$ (see Fig. 6a). Let $G(u',v')$ be a three-stage linear graph with the set of switches $\{u,v,s(2,1)\}$ and satisfying $\ell(1,1,1) = m$, $\ell(2,1,1) = n$, (see Fig. 6b). Then we have

$$P(u,v) \le P(u',v').$$

Proof: We let $G''(u'',v'')$ have the set of switches $\{u'',v''\} \cup \{s''(2,i): 1 \le i \le m\}$ and satisfying $\ell''(1,1,i) = 1$ for $1 \le i \le m$, $\ell''(2,i,1) = n$ for $1 \le i \le m$ (see Fig. 7).

By using Theorem 2 (repeatedly), we have

$$P(u'',v'') \le P(u',v').$$

Similarly, we have

$$P(u,v) \le P(u'',v'').$$

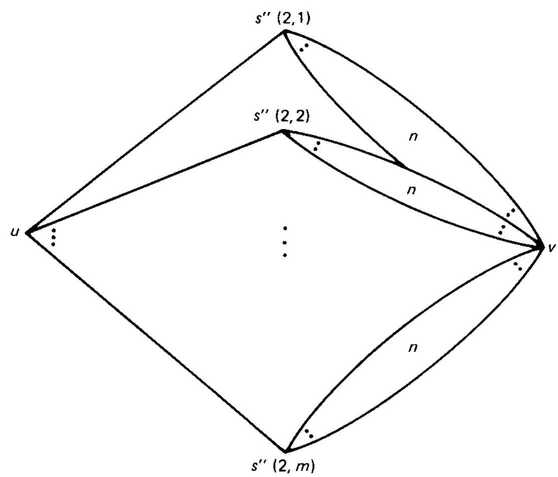


Fig. 7—An intermediate linear graph.

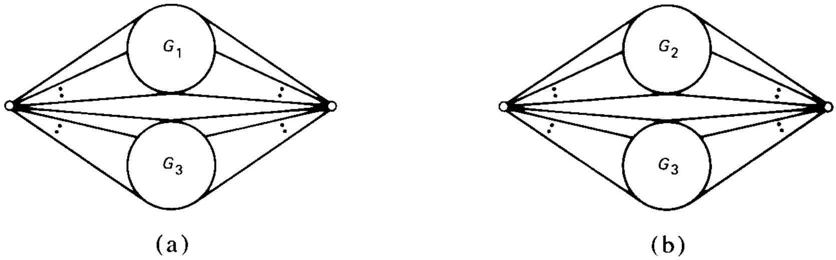


Fig. 8—Parallel combinations for n-stage linear graphs.

Thus, we have

$$P(u,v) \leq P(u',v')$$

and the corollary is proved.

III. LINEAR GRAPHS IN MULTISTAGE NETWORKS

In Section II, we presented several methods to compare blocking probabilities of small linear graphs. In fact, large linear graphs can be compared in very much the same way. The following two theorems show how to extend these methods to multistage linear graphs with a comparatively large set of switches.

Theorem 3: Let $G_1(u_1,v_1)$, $G_2(u_2,v_2)$, $G_3(u_3,v_3)$ be three n-stage linear graphs. We suppose the blocking probability $P(u_1,v_1)$ is smaller than or equal to the blocking probability $P(u_2,v_2)$. Let $G(u,v)$ be an n-stage linear graph obtained by a parallel combination of $G_1(u_1,v_1)$ and $G_3(u_3,v_3)$ (see Fig. 8a). Let $G'(u',v')$ be an n-stage linear graph obtained by a parallel combination of $G_2(u_2,v_2)$ and $G_3(u_3,v_3)$ (see Fig. 8b). Then we have

$$P(u,v) \leq P(u',v').$$

Similarly, if $\overline{G}(u_*,v_*)$ is a $(2n-1)$-stage linear graph obtained by a series combination of $G_1(u_1,v_1)$ and $G_3(u_3,v_3)$ and $\overline{G}'(u'_*,v'_*)$ is a $(2n-1)$-stage linear graph obtained by a series combination of $G_2(u_2,v_2)$ and $G_3(u_3,v_3)$, then we have

$$P(u_*,v_*) \leq P(u'_*,v'_*).$$

Proof: It is easy to see that

$$P(u,v) = P(u_1,v_1)P(u_3,v_3)$$

$$P(u_*,v_*) = 1 - [1 - P(u_1,v_1)]\,[1 - P(u_3,v_3)],$$

and

$$P(u',v') = P(u_2,v_2)P(u_3,v_3)$$

$$P(u'_*,v'_*) = 1 - [1 - P(u_2,v_2)]\,[1 - P(u_3,v_3)].$$

Thus we have

$$P(u,v) \le P(u',v'),\ P(u_*,v_*) \le P(u'_*,v'_*).$$

The following theorem is a generalized version of Theorem 2. Theorem 1 and Corollary 1 can be generalized similarly but will not be stated here.

Theorem 4: Let $G(u,v)$ be an n-stage linear graph with the set of switches

$$\bigcup_{i=1}^{n} \{s(i,j): 1 \le j \le m_i\}$$

and the set of links $\{L(i,j,k)\}$. Let $G'(u',v')$ be an n-stage linear graph with the set of switches

$$\bigcup_{i=1}^{n} \{s'(i,j): 1 \le j \le m'_j\}$$

and the set of links $\{L'(i,j,k)\}$.

Suppose $G(u,v)$ and $G'(u',v')$ satisfy the following conditions.

(*i*) $m_i = m'_i$ for any $i \ne w$, $1 \le i \le n$ (for a fixed w).
(*ii*) There exist k_1,k_2,k_3 such that the linear graph $G(u,v) - \{s(w,k_1),s(w,k_2)\}$ is isomorphic to the linear graph $G'(u',v') - \{s'(w,k_3)\}$.
(*iii*) $s(w,k_1)$, $s(w,k_2)$ and $s'(w,k_3)$ are connected to other switches so that the following conditions hold:

$$\ell(w-1,j,k_1) = \ell(w-1,j,k_2) = \ell'(w-1,j,k_3) \text{ for } 1 \le j \le m_{w-1},$$

$$\ell(w,k_1,k) + \ell(w,k_2,k) = \ell'(w,k_2,k) \text{ for } 1 < k < m_{w+1}.$$

where $\ell(i,j,k)$, $\ell'(i,j,k)$ denote the cardinalities of $L(i,j,k)$, $L'(i,j,k)$, respectively.

Then we have

$$P(u,v) \le P(u',v').$$

We note that (*iii*) could be replaced by (*iii'*) because of symmetry:

(*iii'*) $\ell(w-1,j,k_1) + \ell(w-1,j,k_2) = \ell'(w-1,j,k_3)$ for $1 \le j \le m_{w-1}$, $\ell(w,k_1,k) = \ell(w,k_2,k) = \ell'(w,k_3,k)$ for $1 \le k \le m_{w+1}$.

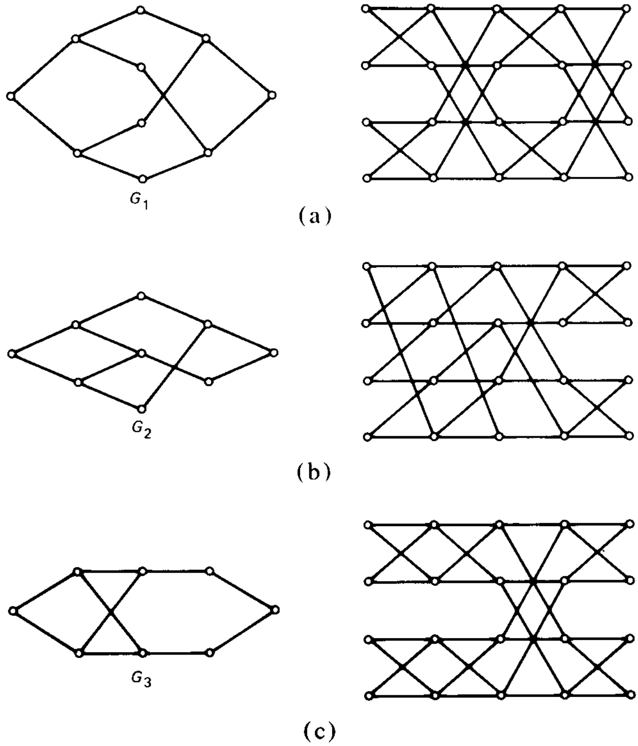


Fig. 9—Examples of linear graphs and corresponding balanced switching networks.

Proof: We may assume $n \geq 4$ because of Theorem 2. Thus, we may assume without loss of generality that $w \neq n - 1$. Therefore $m_i = m_i'$ for $i \neq w$, and in particular, $m_{n-1} = m_{n-1}'$. Let A be a subset of $\{j: 1 \leq j \leq m_{n-1}\}$. Let $G_A(u,v_A)$ be an $(n - 1)$-stage linear graph which can be viewed as the union of all paths in G which connect u and a switch $s(n - 1,j)$, where $j \in A$ and all switches in A have been identified. (It can be viewed that all switches in A are condensed into one switch.) In other words, $G(u,v_A)$ has the set of switches $\{v_A = s_A\,(n - 1,1)\} \cup \{s_A(i,j): i \neq n - 1$ and $s(i,j)$ is on a path which passes through a switch $s(n - 1,j)$ where $j \in A\}$. G_A has the set of links $\{L_A(i,j,k)\}$ where

$$\ell_A(n - 2,j,1) = \sum_{k \in A} \ell(n - 2,j,k)$$

and $\ell_A(i,j,k) = \ell(i,j,k)$ for $i \neq n - 2$. Let $G_A'(u',v_A')$ be the linear graph similarly obtained from G' by identifying all switches in A. By the induction assumption, we have

$$P(u,v_A) \leq P(u',v_A').$$

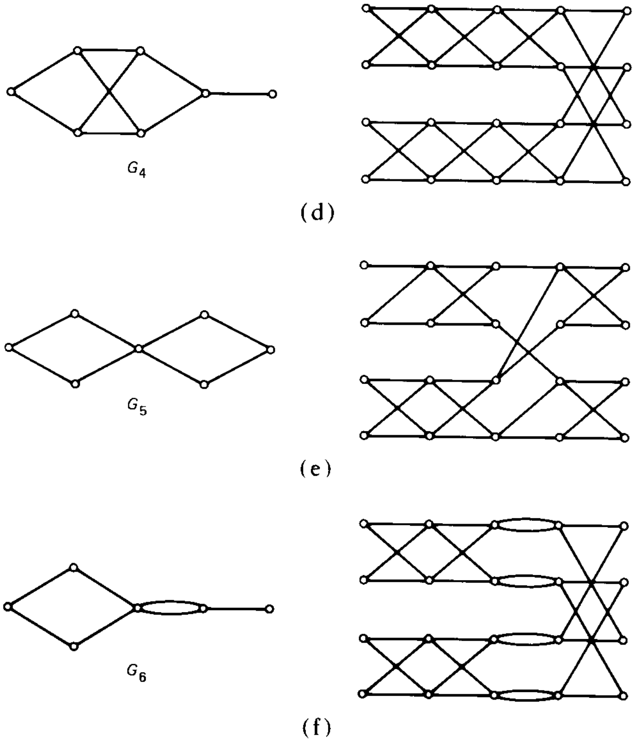


Fig. 9 (*continued*)

Moreover, $P(u,v)$ can be written as follows:

$$P(u,v) = \sum_{A} p_{n-1}^{|A|} (1 - p_{n-1})^{m_{n-1}-|A|} P(u,v_A)$$

where A ranges over all subsets of $\{j: 1 \le j \le m_{n-1}\}$.

Since $P(u',v')$ has the similar expression

$$P(u',v') = \sum_{A} p_{n-1}^{|A|} (1 - p_{n-1})^{m_{n-1}-|A|} P(u',v'_A),$$

then we have

$$P(u,v) \le P(u',v')$$

In Ref. 2, the present authors consider a special class of linear graphs $G(u,v)$ with $m_{n-i} = m_i$, n odd and m_i dividing m_{i+1} for $i = 1,2, \ldots, [n/2]$. It can be easily seen that the linear graphs in the class can be compared by using Theorem 4.

In Fig. 9a to f, we give several examples of linear graphs together with their corresponding balanced switching networks.

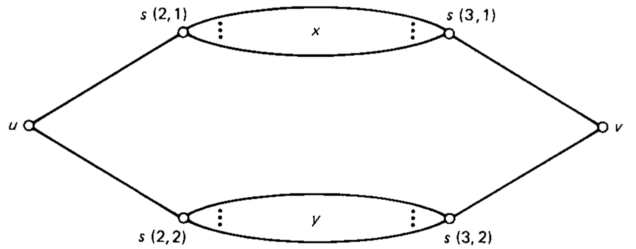


Fig. 10—A four-stage linear graph.

Let P_{G_i} denote the blocking probability for the balanced network N_i with linear graph G_i. It is easy to verify that $P_{G_1} \le P_{G_2}$ by taking $w = 3$, $k_1 = 1$, $k_2 = 3$, $k_3 = 2$. Similarly, it is easy to see that

$$P_{G_1} \le P_{G_2} \le P_{G_3} \le P_{G_4} \le P_{G_6},$$

and

$$P_{G_3} \le P_{G_5} \le P_{G_6}.$$

We note that the numbers of crosspoints in N_i, $i = 1, \ldots, 6$, are the same. Thus we know that the switching network N_1 is "better" than the switching network N_2 and so forth.

IV. SERIES-PARALLEL LINEAR GRAPHS

In this section, we consider series-parallel linear graphs. Series-parallel linear graphs are sometimes preferred to spider-web linear graphs[6] because of the conditions for implementation and control. The following two theorems treat the blocking probabilities of series-parallel linear graphs.

Theorem 5: We consider the following four-stage linear graph $G_{x,y}$ (see Fig. 10).

(*i*) $m_2 = m_3 = 2$

(*ii*) $\ell(1,1,1) = \ell(1,1,2)$, $\ell(2,1,2) = \ell(2,2,1) = 0$, $\ell(3,1,1) = \ell(3,2,1)$,

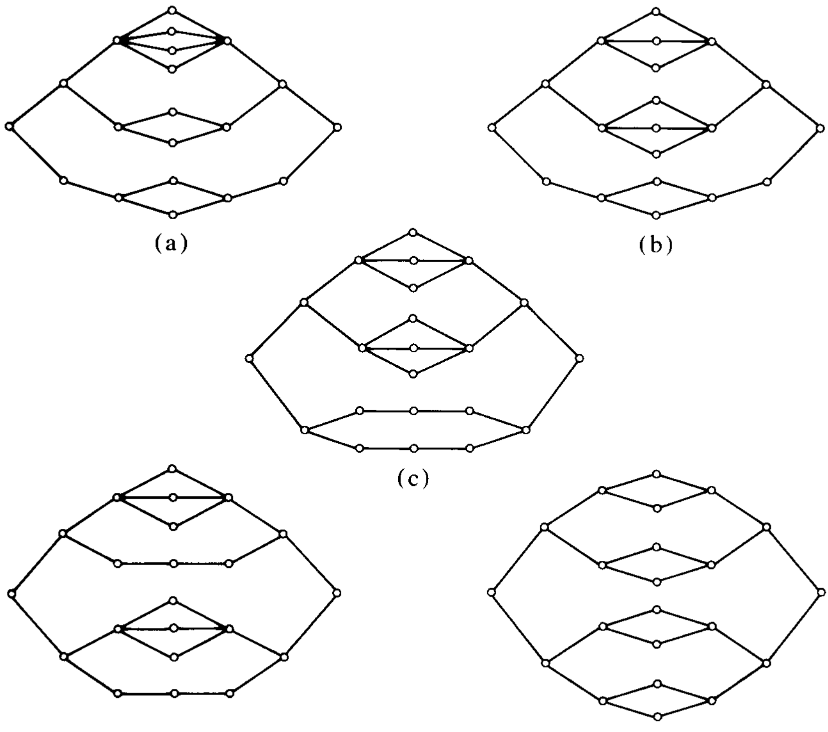
(*iii*) $\ell(2,1,1) = x$, $\ell(2,2,2) = y$.

If there are integers a and b with $x + y = a + b$, $x \le a \le b \le y$, then we have

$$P_{G_{ab}} \le P_{G_{xy}}$$

The proof of Theorem 5 is quite similar to the proof of Theorem 1—by setting $f(x) = [1 - (1 - p_1)(1 - p_2^x)(1 - p_3)]\,[1 - (1 - p_1)(1 - p_2^{c-x})(1 - p_3)]$—and is omitted.

Remark: The above theorem can be extended to multistage linear graphs by replacing each link by a linear graph under the condition that all links



(a) (b) (c) (d) (e)

Fig. 11—Examples.

between stage i and stage $i + 1$ are replaced by copies of a linear graph or by linear graphs with the same blocking probabilities.

In Fig. 11, some examples are illustrated. The linear graph in Fig. 11b has a smaller blocking probability than the linear graph in Fig. 11a by Theorems 3 and 5. The linear graph in Fig. 11c has a smaller blocking probability than the linear graph in Fig. 11b by Theorems 3 and 4.

Theorem 6: We consider the following linear graph G_{xyzw} (see Fig. 12):

(*i*) $m_i = m_j = 2$.

(*ii*) u and $s(i,1)$ are connected by a linear graph N_1. u and $s(i,2)$ are connected by a linear graph N_2. N_1 and N_2 have the same number of stages and $P_{N_1} = P_{N_2}$.

(*iii*) v and $s(j,1)$ are connected by a linear graph N_3. v and $s(j,2)$ are connected by a linear graph N_4. N_3 and N_4 have the same number of stages and $P_{N_3} = P_{N_4}$.

(*iv*) There exist $(j - i + 1)$-stage linear graphs G_1, G_2 such that $s(i,1)$

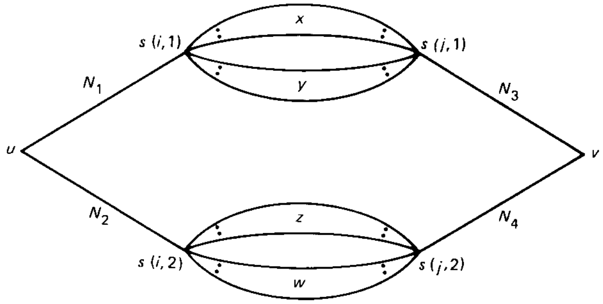


Fig. 12—Linear graph for Theorem 6.

and $s(j,1)$ are connected by x copies of G_1 and y copies of G_2 and $s(i,2)$ and $s(j,2)$ are connected by z copies of G_1 and w copies of G_2.

Suppose $x + y = z + w = c$ and $x + z = d$ for some constants c and d. We also suppose $x' + y' = z' + w' = c$, $x' + z' = d$ where $x' \leq x \leq z \leq z'$. Then we have

$$P_{G_{xyzw}} \leq P_{G_{x'y'z'w'}}.$$

Proof: Let $\alpha = (1 - P_{N_1})(1 - P_{N_3})$.

Define the following function $f(x)$:

$$f(x) = [1 - \alpha(1 - P_{G_1}^{x} P_{G_2}^{c-x})][1 - \alpha(1 - P_{G_1}^{d-x} P_{G_2}^{c-d+x})].$$

It is easy to see that $P_{G_{xyzw}} = f(x)$, $P_{G_{x'y'z'w'}} = f(x')$. Now,

$$\frac{df}{dx}(x) = \alpha(1-\alpha)(\log P_{G_1} - \log P_{G_2})(P_{G_1}^{x} P_{G_2}^{c-x} - P_{G_1}^{d-x} P_{G_2}^{c-d+x})$$

$$= \alpha(1-\alpha)(\log P_{G_1} - \log P_{G_2}) P_{G_1}^{x} P_{G_2}^{c-x} (1 - P_{G_1}^{d-2x} P_{G_2}^{2x-d}).$$

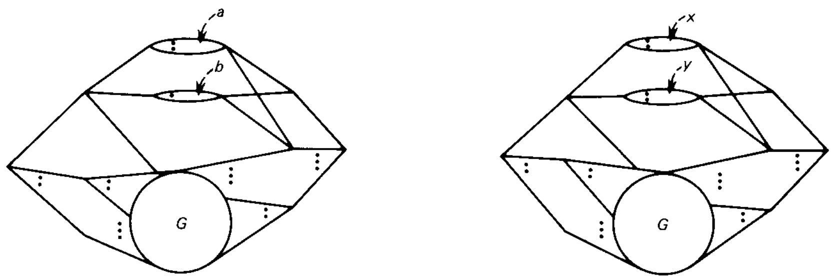
If $P_{G_1} = P_{G_2}$, we have $f(x) = f(x')$. If $P_{G_1} \neq P_{G_2}$, $f(x)$ attains its minimum at $x = d/2$. Since $f(x)$ is convex, then

$$f(x) \leq f(x') \text{ for } x' \leq x \leq \frac{d}{2}$$

Thus we have

$$P_{G_{xyzw}} \leq P_{G_{x'y'z'w'}}.$$

Theorem 5 and Theorem 6 essentially say that the more regular (i.e., evenly distributed) the linear graph, the better it is. Of course, all these results are based on the Lee model and the related independence assumption. In some existing networks, irregular linear graphs might sometimes be desirable because of the preference schemes in routing.[1]



(a) (b)

Fig. 13—Linear graphs for Theorem 7.

In Fig. 11, the linear graph in 11d has a smaller blocking probability than the linear graph in 11c by Theorem 6. By Theorem 5, the linear graph in 11e has the smallest blocking probability. We note that 11e is the most regular linear graph in Fig. 11.

Theorems 5 and 6 can be generalized to a class of spider-web linear graphs. We will state the generalized version of Theorem 5.

Theorem 7: Let $\overline{G}_{ab}$ and $\overline{G}_{xy}$ be two n-stage linear graphs satisfying the following properties (see Fig. 13).

(*i*) There exists k, $2 \leq k \leq n-2$, such that $\overline{G}_{ab} - \{s(k,1), s(k,2), s(k+1,1) s(k+2,2)\}$ is isomorphic to $\overline{G}_{xy} - \{s'(k,1), s'(k,2), s'(k+1,1), s'(k+2,2)\}$, where $\{s(i,j)\}$, $\{s'(i,j)\}$ are the sets of switches of $\overline{G}_{ab}, \overline{G}_{xy}$, respectively.

(*ii*) $\ell(k-1,i,1) = \ell(k-1,i,2) = \ell'(k-1,i,1) = \ell'(k-1,i,2)$ for $1 \leq i \leq m_{k-1}$, and $\ell(k+1,1,j) = \ell(k+1,2,j) = \ell'(k+1,1,j) = \ell'(k+1,2,j)$ for $1 \leq j \leq m_{k+1}$, where $\ell(i,j,k)$ and $\ell'(i,j,k)$ are the cardinalities of links of $\overline{G}_{ab}$, $\overline{G}_{xy}$, respectively.

(*iii*)

$$\ell(k,1,j) = \begin{cases} a & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\ell(k,2,j) = \begin{cases} b & \text{if } j = 2 \\ 0 & \text{otherwise} \end{cases}$$

Similarly,

$$\ell'(k,1,j) = \begin{cases} x & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\ell'(k,2,j) = \begin{cases} y & \text{if } j = 2 \\ 0 & \text{otherwise} \end{cases}$$

(*iv*)

$$x + y = a + b, \; x \leq a \leq b \leq y.$$

Then we have

$$P_{\overline{G}_{ab}} \leq P_{\overline{G}_{xy}}.$$

Proof: The proof is by induction on the number of stages. Suppose n = 4. Following the notation in Theorem 5, we note that $\overline{G}_{xy}$ is the parallel combination of G_{xy} and G. Thus by Theorem 5, we have

$$P_{\overline{G}_{ab}} = P_{G_{ab}} P_G \leq P_{G_{xy}} P_G = P_{\overline{G}_{xy}}.$$

For $n > 4$, we apply the same reduction scheme which is used in the proof of Theorem 4. The theorem is then proved by mathematical induction.

V. CONCLUDING REMARKS

Lee[8] first proposed the concept of a linear graph in connection with his study of the blocking probabilities of switching networks. Since then his model has been widely used. However, a systematic study of linear graphs is still far from complete. There are some results in extending Lee's method[5,9] or for studying the blocking probabilities for certain classes of series-parallel linear graphs[2]. Takagi[10,11] has defined a class of spider-web linear graphs and finds the optimal one in that class. Some of his results have been obtained earlier by Le Gall[3]. Van Bosse[12,13] extends results in Refs. 3, 10, and 11 in the sense that the occupancy distribution for links can be arbitrary. In this paper, several new methods for analyzing blocking probabilities of certain classes of switching networks are presented. We hope it will lead to more research in this direction.

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