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Multidiameters and Multiplicities

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The k-diameter of a graph Γ is the largest pairwise minimum distance of a set of k vertices in Γ, i.e., the best possible distance of a code of size k in Γ. A k-diameter for some k is called a multidiameter of the graph. We study the function $N(k, \Delta, D)$, the largest size of a graph of degree at most Δ and k-diameter D. The graphical analogues of the Gilbert bound and the Hamming bound in coding theory are derived. Constructions of large graphs with given degree and k-diameter are given. Eigenvalue upper bounds are obtained. By combining sphere packing arguments and eigenvalue bounds, new lower bounds on spectral multiplicity are derived. A bound on the error coefficient of a binary code is given.

1. INTRODUCTION

The diameter of a graph measures how far two distinct points can be; similarly, the k-diameter (definition given below) measures how far k points can be; in other words, how good can a code of size k in the graph be. Two problems on the diameter have excited a great deal of attention since the 1980s:

- the (Δ, D) graph problem: how large can a graph of bounded degree and given diameter be? [1, 6, 7]
- finding the best spectral upper bound on the diameter of a graph [3, 4].

This work is an attempt to generalize both philosophies.

First, we study a function $N(k, \Delta, D)$, the largest size of a graph of degree at the most Δ and given k-diameter D. Observe that this is a very hard problem which comprises as a special case ($k = 2$) the (Δ, D) graph problem [1, 6, 7]. We begin a tridimensional table collecting the sizes of the largest such graphs. No exact value of $N(k, \Delta, D)$ with $k > 2, \Delta > 2$ is known so far.

Next, we derive the natural analogues of the Chung *et al.* upper bounds [3, 4, 8] on the diameter. Combining the spectral bound and the Gilbert bound, we obtain some new lower bounds on the spectral multiplicity. When the graph under scrutiny is the coset graph of a binary linear code, we obtain a lower bound on the error coefficient of the dual code.

2. DEFINITIONS AND NOTATIONS

All graphs Γ considered are finite, connected, with vertex set V of cardinality v, simple, undirected, without loops or multiple edges. The graphical *distance* $d(x, y)$ between two vertices x and y is the length of a shortest path between x and y. The *girth* hereby denoted by g is the shortest size of a circuit. A *k-code* in such a graph Γ with *distance* d is a set of $k(\geq 2)$ vertices with

$$\min_{i \neq j}(d(x_i, x_j)) = d.$$

Classical codes of size k in the sense of [10] are exactly k-codes in the hypercube or more generally the q-ary hypercube or Hamming graph [2] $H(n, q)$. The *k-diameter* of Γ, say D_k, is the largest possible distance a k-code in Γ can have. Note that D_2 is the standard diameter.

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For example, we have $D_q = n$ in the q-ary Hamming graph [2] $H(n, q)$ (the direct sum of n complete graphs K_q). The sequence $k \mapsto D_k$ is nonincreasing:

$$D_2 \geq D_3 \geq \cdots D_k \geq D_{k+1} \geq \cdots.$$

A closely related and useful quantity is the *Moore function*, which is defined for integers Δ, D

$$M(\Delta, D) := 1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1},$$

or, in closed form

$$M(\Delta, D) = \begin{cases} \frac{\Delta(\Delta-1)^D-2}{\Delta-2} & \text{if } \Delta > 2, \\ 2D + 1 & \text{if } \Delta = 2. \end{cases}$$

An elementary fact of graph theory is that a graph of diameter at most D and degree at most Δ cannot have more than $M(\Delta, D)$ vertices. Its bipartite counterpart $M_b(\Delta, D)$ is $2(1+(\Delta-1) + \cdots + (\Delta - 1)^{D-1})$, or, in closed form

$$M_b(\Delta, D) = \begin{cases} \frac{2(\Delta-1)^D-2}{\Delta-2} & \text{if } \Delta > 2, \\ 2D & \text{if } \Delta = 2. \end{cases}$$

A similar but different invariant is the *covering radius* $r(C)$ of a code C which is defined by

$$r(C) := \max_{v \in V} \min_{c \in C} d(v, c).$$

Denote by T the diagonal matrix indexed by V such that $T_{x,x}$ is the degree of $x \in V$, by A the adjacency matrix of Γ and let $L = T - A$. The *Laplace operator* is then defined as

$$\mathcal{L} := T^{-1/2} L T^{-1/2}.$$

Let

$$\lambda_0 = 0 \leq \lambda_1 \leq \cdots \leq \lambda_{v-1}$$

be the eigenvalues of the Laplace operator arranged in increasing order. It is not hard to check that the whole *spectrum* fits into $[0, 2]$. In particular, if the graph is Δ-regular, then $\lambda_i = 1 - \mu_i/\Delta$, where μ_i is the ith eigenvalue (decreasing order) of the adjacency matrix.

3. Sphere Packings and Coverings

3.1. An improved Gilbert bound. The Moore function is an upper bound for the number of vertices in a graph with degree Δ and diameter D. Using the definition of the 2-diameter, the Moore bound is related to $N(2, \Delta, D)$ as follows:

$$N(2, \Delta, D) \leq M(\Delta, D). \tag{1}$$

The analogue of the Gilbert bound of coding theory [10, p. 33] is

$$N(k, \Delta, D) \leq (k - 1)M(\Delta, D),$$

which can be proved by induction on k (equation (1) is the case $k = 2$). It also follows immediately from the following result relating $N(k, \Delta, D)$ to codes with covering radius D.

Lemma 1.

$$N(k, \Delta, D) \leq F(k, \Delta, D), \tag{2}$$

where $F(k, \Delta, D)$ denotes the largest size of the vertex set V, of a graph Γ containing a $(k-1)$-code C with minimum distance at least $D + 1$, and covering radius of at most D.

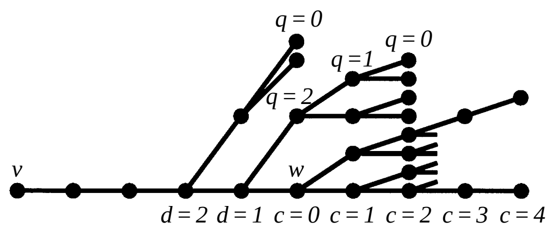

FIGURE 1. Explanation of $G(\Delta, D)$.

PROOF. Let Γ be a graph with degree at most Δ, satisfying $D_k \leq D$ on $N(k, \Delta, D)$ vertices. Consider a $(k-1)$-code C with minimum distance at least $D+1$ in Γ. Then we conclude that its covering radius is at most D, since otherwise, we would obtain a k-code with distance $D+1$, contradicting the hypothesis $D_k \leq D$. □

We are now in a position to give a small improvement on the Gilbert bound by taking into account graph connectedness and ball intersections.

THEOREM 1. *A graph with k-diameter D and maximum degree* Δ *has at most* $(k-1)M(\Delta, D) - (k-2)M(\Delta, (D-1)/2)$ *vertices if D is odd and* $(k-1)M(\Delta, D) - (k-2)M_b(\Delta, D/2)$ *if D is even.*

PROOF. If the graph is connected, we can see that if v and w are vertices at a distance $D+1$, then the number of vertices at a distance at most D from w and at least $D+1$ from v is at most,

$$I(\Delta, D) := \sum_{c=0}^{D}(\Delta-1)^c + \sum_{d=1}^{\lfloor D/2 \rfloor}(\Delta-2)\sum_{q=0}^{D-2d}(\Delta-1)^{D-d-1-q}.$$

The explanation of that formula should be clear from Figure 1. We note that $I(\Delta, D) = M(\Delta, D) - M(\Delta, (D-1)/2)$ if D is odd and $I(\Delta, D) = M(\Delta, D) - M_b(\Delta, D/2)$ if D is even. These formulae can be derived by summing the geometric series in the definition of $G(\Delta, D)$. Roughly speaking, what happens for g large enough, and say D is odd, is that the intersection of the two balls of radius D about v and w, is itself a ball of radius $\lfloor (D-1)/2 \rfloor$ centered in a point at the same distance (up to one unit) of v and w. A similar phenomenon occurs for even D.

To apply Lemma 1, we construct greedily a code of minimum distance $> D$ and covering radius at most D. The first ball covers at the most, $M(\Delta, D)$ vertices. Put the second one at distance $D+1$ from the first. It covers at the most $M(\Delta, D) - G(\Delta, D)$ vertices. After placing the $(k-1)$th ball, at most $(k-2)M(\Delta, D) - G(\Delta, D)$ vertices are covered. □

For $\Delta = 2$, the bound is $2D + 1 + (k-2)(D+1)$, instead of $(k-1)(2D+1)$, the exact value is $k(D+1) - 1$ (and the optimal graph is a cycle).

3.2. The Hamming bound. Let $e(D) := \left\lfloor \frac{D-1}{2} \right\rfloor$. Assume $D_k \leq g$. Then the analogue of the Hamming bound [10] reads

$$v \geq kM(\Delta, e(D_k)).$$

Equality corresponds to the case of *a perfect code* in a graph such that the volume of a ball of radius $e(D)$ is indeed $M(\Delta, e(D))$. This happens, for instance, if the graph is regular with girth at least $2e(D) + 1$. For instance, if $n = 2^m - 1$, the Hamming codes yield

$$N(2^{n-m}, n, 3) = 2^n = 2^{n-m}(2n + 1),$$

meeting the preceding bound with equality.

Similarly perfect Lee metric codes of length n over F_p with $p = 2n + 1$ yield

$$N(p^{n-1}, 2n, 3) = p^n = p^{n-1}(1 + 2n).$$

But the repetition codes of length $2s + 1$ yield only (for $s > 1$) the inequality

$$N(2, 2s + 1, 2s + 1) \geq 2^{2s+1}.$$

Observe that the volume of a ball of radius s in the $(2s + 1)$-hypercube is $2^{2s} < M(2s + 1, 2s + 1)$.

4. Constructions

4.1. Graphs of large girth. A relation between the girth and the k-diameter is

$$g \leq kD_k + 1.$$

This bound is of special interest in the case of incidence graphs of generalized polygons [7]. These are bipartite graphs with diameter N and girth $2N$. In that case, we obtain the estimates

$$\left\lceil \frac{2N - 1}{k} \right\rceil \leq D_k \leq N.$$

The lower bound is met with equality for the $N = 3$, $k = 3$ case of the incidence graph of a projective plane $PG(2, q)$. It can be directly verified in that case that a line l, along with a pair of points known on l, constitutes a 3-code with distance 2, and also that $D_3 \leq 2$, because a pair of points or a pair of lines are at a distance at most 2 apart. So, we obtain, for every prime power q, the estimates

$$N(3, q + 1, 2) \geq 2(q^2 + q + 1),$$

quite close to the upper bound

$$N(3, q + 1, 2) \leq 2(2 + 2q + q^2) - 2.$$

Similarly, in that case, we have $D_4 = 2$ yielding

$$N(4, q + 1, 2) \geq 2(q^2 + q + 1),$$

to be compared to the upper bound

$$N(3, q + 1, 2) \leq 3(2 + 2q + q^2) - 4.$$

A sharper lower bound will appear in Section 4.3.

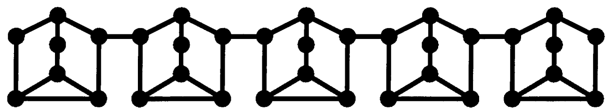

FIGURE 2. Graph showing $N(k, 3, 3) \geq 7(k-1)$.

4.2. Bipartite Graphs. Let $B(\Delta, D)$ denote the largest size of a bipartite graph of degree at most Δ and diameter D. Lower bounds for this function are tabulated for $\Delta \leq 16$ and $D \leq 10$ in [6], and an upper bound is $M_b(\Delta, D)$. Given a 3-code in such a graph, two of its points at least should lie in the same part. This leads to

$$N(3, \Delta, D-1) \geq B(\Delta, D) \qquad \text{if } D \text{ is odd.}$$

This can be generalized to multipartite graphs by denoting by $P(p, \Delta, D)$ the largest size of a p-partite graph with diameter D and degree at most Δ. For any $D = pm + a$ with $0 < a < p$ and m integer, we have

$$N(p+1, \Delta, pm) \geq P(p, \Delta, D).$$

4.3. Irregular Graphs. For $D = 2$ and $\Delta = q + 1$, we have some constructions with $n = (q^2 + q + 1)(k-1) = (\Delta^2 - \Delta + 1)(k-1)$ (take $k-1$ copies of the quotient of the incidence graph of a projective plane of order q by a polarity and add edges between the vertices of degree q to obtain a connected graph). On the other hand, the modified Gilbert bound is $(\Delta^2 - 1)(k-1) + 2$ instead of $(\Delta^2 + 1)(k-1)$.

For $q = 2$, we have the kind of graphs of Figure 2.

More generally, if we have a graph with maximum degree $\leq \Delta$ and diameter D, with n vertices among which two at least have degree $\Delta - 1$, it is easy to prove $N(k, \Delta, D) \geq (k-1)n$ by arranging a path of $k-1$ copies of G connected by edges.

This gives $N(k, \Delta, 1) \geq (k-1)\Delta$, to be compared by the upper bound coming from $k - 1 \geq \lceil \frac{n}{\Delta} \rceil$, namely $N(k, \Delta, 1) \leq 1 + (k-1)\Delta$.

For $D = 3, 5$, the generalized quadrangles and hexagons also give good results: a (s, t)-quadrangle Q has $(s+1)(st+1)$ vertices of degree $t+1$ and $(t+1)(st+1)$ vertices of degree $s+1$; one of the components of the Kronecker product of Q by itself is regular of degree $(t+1)(s+1)$, has $2(s+1)(t+1)(st+1)^2$ vertices and admits an obvious polarity, the quotient is a diameter 3 graph with $(s+1)(t+1)(st+1)^2$ vertices of maximum degree $\Delta = (t+1)(s+1)$, with $(s+1)(t+1)(st+1)$ vertices having degree $st+t+s$. Hence $N(k, (t+1)(s+1), 3) \geq (k-1)(s+1)(t+1)(st+1)^2$, provided that there exists a (s, t)-quadrangle.

For each $m \geq 0$, some quadrangles with $s = t = 2^{2m+1}$ already have a polarity and give $N(k, s+1, 3) \geq (k-1)(s+1)(s^2+1)$.

For example, $15(k-1) \leq N(k, 3, 3) \leq 18(k-1) + 4$.

Similarly, the (s, t)-hexagons yield graphs with $(s+1)(t+1)(s^2t^2 + st + 1)^2$ vertices, maximum degree $\Delta = (t+1)(s+1)$, with $(s+1)(t+1)(s^2t^2 + st + 1)$ vertices having degree $st + t + s$. Hence, $N(k, (t+1)(s+1), 5) \geq (k-1)(s+1)(t+1)(s^2t^2 + st + 1)^2$, provided that there exists a (s, t)-generalized hexagon, and hexagons of order $s = t = 3^{2m+1}$ admitting a polarity give $N(k, s+1, 5) \geq (k-1)(s+1)(s^2 + s + 1)$.

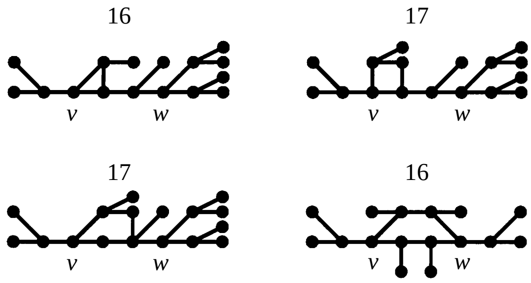

FIGURE 3. Explanation of $N(3, 3, 2) \leq 16$.

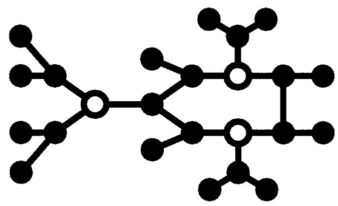

FIGURE 4. Explanation of $N(4, 3, 2) \leq 24$.

4.4. Small values For $D = 2$, $k = 3$ and $\Delta = 3$, we have $M = 10$ and $F = 8$; the improved Gilbert bound is 18.

If there is a vertex v of degree ≤ 2, there are at the most seven vertices at a distance ≤ 2 from v. If w is at a distance 3 from v, there are at the most eight vertices at a distance ≤ 2 from w and at a distance at least 3 from v. Thus, a graph with maximum degree 3 and 3-diameter 2 having a vertex of degree 2 has at the most 15 vertices. Thus, there is no graph with 17 vertices, maximum degree 3 and 3-diameter 2.

Let us consider now 3-regular connected graphs. Since $18 < M(3, 3) = 22$, every vertex of a 3-regular connected graph with 3-diameter 2 lies in a cycle of length < 7, thus the bound 18 cannot be attained. See Figure 3.

Thus we can state $14 \leq N(3, 3, 2) \leq 16$.

Similarly, from the improved Gilbert bound for $N(4, 3, 2)$, that is 26, we can see that the girth is at the most 7, and this implies $N(4, 3, 2) \leq 24$ because we can either find a vertex of degree ≤ 2 leading to a bound of $7 + 8 + 8 = 23$ or use the results for a cycle of length ≤ 6, or start with a cycle of length 7 as shown in Figure 4. The proof is easily generalized to $N(k, 3, 2) \leq 8(k - 1)$.

TABLE 1.
Small values.

k	Δ	D	$N(k, \Delta, D)$	G	IG	graph
3	3	2	14–16	20	18	$PG(2, 2)$
3	3	4	56–68	72	68	[6]
3	3	3	30–40	66	40	4.3
3	4	2	26–32	34	32	$PG(2, 3)$
4	3	2	21–24	30	26	Section 4.3
4	3	3	45–58	66	58	Section 4.3

The fourth column gives the known bounds, the fifth and sixth ones present the Gilbert and improved Gilbert bounds.

5. SPECTRAL BOUNDS

5.1. Main bounds. Recall that the inverse hyperbolic cosine is given by

$$\cosh^{-1}(x) = \log\Big(x + \sqrt{x^2 - 1}\Big),$$

for $x \geq 1$.

THEOREM 2. *Suppose a connected graph G is not a complete graph. For $X, Y \subset V(G)$ and X not equal to the complement $\bar{Y}$ of Y, we have*

$$d(X, Y) \leq \left\lceil \frac{\cosh^{-1}\sqrt{\frac{\mathrm{vol}\,\bar{X}\,\mathrm{vol}\,\bar{Y}}{\mathrm{vol}\,X\,\mathrm{vol}\,Y}}}{\cosh^{-1}\frac{\lambda_{v-1}+\lambda_1}{\lambda_{v-1}-\lambda_1}} \right\rceil. \tag{3}$$

PROOF. For $X \subset V(G)$, we define

$$\psi_X(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

If we can show that for some integer t and some polynomial $p_t(z)$ of degree t,

$$\langle T^{1/2}\psi_Y, p_t(\mathcal{L})(T^{1/2}\psi_X)\rangle > 0,$$

then, there is a path of length at the most t joining a vertex in X to a vertex in Y. Therefore, we have $d(X, Y) \leq t$.

Let a_i denote the Fourier coefficients of $T^{1/2}\psi_X$, i.e.,

$$T^{1/2}\psi_X = \sum_{i=0}^{n-1} a_i\phi_i,$$

where the ϕ_i's are the orthonormal eigenfunctions of $\mathcal{L}$. In particular, we have

$$a_0 = \frac{\langle T^{1/2}\psi_X, T^{1/2}\mathbf{1}\rangle}{\langle T^{1/2}\mathbf{1}, T^{1/2}\mathbf{1}\rangle} = \frac{\mathrm{vol}\,X}{\mathrm{vol}\,G},$$

where vol X is the sum of degrees of the vertices in X and **1** denotes all the 1's functions.

Similarly, we write

$$T^{1/2}\psi_Y = \sum_{i=0}^{v-1} b_i\phi_i,$$

Suppose we choose $p_1(z) = \frac{2z}{\lambda_{v-1}+\lambda_1} - 1$ and $p_t(z) = (p_1(z))^t$. Since G is not a complete graph, $\lambda_1 \neq \lambda_{v-1}$, and

$$|p_t(\lambda_i)| \leq (1-\lambda)^t$$

for all $i = 1\ldots, v-1$, where $\lambda = 2\lambda_1/(\lambda_{v-1}+\lambda_1)$. Therefore, we have

$$\begin{aligned}
\langle T^{1/2}\psi_Y, p_t(\mathcal{L})T^{1/2}\psi_X\rangle &= a_0b_0 + \sum_{i>0} p_t(\lambda_i)a_ib_i \\
&\geq a_0b_0 - (1-\lambda)^t\sqrt{\sum_{i>0} a_i^2 \sum_{i>0} b_i^2} \\
&= \frac{\mathrm{vol}\,X\,\mathrm{vol}\,Y}{\mathrm{vol}\,G} - (1-\lambda)^t\,\frac{\sqrt{\mathrm{vol}\,X\,\mathrm{vol}\,\bar{X}\,\mathrm{vol}\,Y\,\mathrm{vol}\,\bar{Y}}}{\mathrm{vol}\,G},
\end{aligned}$$

by using the fact that

$$\sum_{i>0} a_i^2 = \|T^{1/2}\psi_X\|^2 - \frac{(\mathrm{vol}\, X)^2}{\mathrm{vol}\, G} = \frac{\mathrm{vol}\, X\ \mathrm{vol}\, \bar{X}}{\mathrm{vol}\, G}.$$

We note that in the above inequality, the equality holds if and only if $a_i = cb_i$ for $i > 0$, for some constant c. This can only hold when $X = Y$ or $X = \bar{Y}$. Since the theorem obviously holds for $X = Y$ and we have the hypothesis that $X \neq \bar{Y}$, we may assume that the inequality is strict. If we choose

$$t \geq \frac{\log \sqrt{\frac{\mathrm{vol}\, \bar{X}\ \mathrm{vol}\, \bar{Y}}{\mathrm{vol}\, X\ \mathrm{vol}\, Y}}}{\log \frac{1}{1-\lambda}},$$

we have

$$\langle T^{1/2}\psi_Y, p_t(\mathcal{L})T^{1/2}\psi_X\rangle > 0,$$

This proves $D(G) \leq \left\lceil \frac{\log \sqrt{\frac{\mathrm{vol}\, \bar{X}\ \mathrm{vol}\, \bar{Y}}{\mathrm{vol}\, X\ \mathrm{vol}\, Y}}}{\log \frac{1}{1-\lambda}} \right\rceil$.

It can be improved with another choice for p_t, namely the normalized Chebychev polynomial such that

$$\cosh\left(t \cosh^{-1}\left(\frac{1}{1-\lambda}\right)\right) p_t(z) := \cosh\left(t \cosh^{-1}\left(\frac{1}{1-\lambda} - \frac{2z}{\lambda_{v-1} - \lambda_1}\right)\right).$$

Then, for $\lambda_1 \leq z \leq \lambda_{v-1}$, we have $|p_t(z)| \leq \frac{1}{\cosh\left(t \cosh^{-1}\left(\frac{1}{1-\lambda}\right)\right)}$,

$$\begin{aligned}\langle T^{1/2}\psi_Y, p_t(\mathcal{L})T^{1/2}\psi_X\rangle &= a_0 b_0 + \sum_{i>0} p_t(\lambda_i) a_i b_i \\ &\geq a_0 b_0 - \frac{1}{\cosh\left(t \cosh^{-1}\left(\frac{1}{1-\lambda}\right)\right)} \sqrt{\sum_{i>0} a_i^2 \sum_{i>0} b_i^2} \\ &= \frac{\mathrm{vol}\, X\ \mathrm{vol}\, Y}{\mathrm{vol}\, G} - \frac{1}{\cosh\left(t \cosh^{-1}\left(\frac{1}{1-\lambda}\right)\right)} \frac{\sqrt{\mathrm{vol}\, X\ \mathrm{vol}\, \bar{X}\ \mathrm{vol}\, Y\ \mathrm{vol}\, \bar{Y}}}{\mathrm{vol}\, G}.\end{aligned}$$

Note that

$$\sum_{i>0} a_i^2 = \|T^{1/2}\psi_X\|^2 - \frac{(\mathrm{vol}\, X)^2}{\mathrm{vol}\, G} = \frac{\mathrm{vol}\, X\ \mathrm{vol}\, \bar{X}}{\mathrm{vol}\, G}.$$

If we choose

$$t \geq \frac{\cosh^{-1} \sqrt{\frac{\mathrm{vol}\, \bar{X}\ \mathrm{vol}\, \bar{Y}}{\mathrm{vol}\, X\ \mathrm{vol}\, Y}}}{\cosh^{-1} \frac{1}{1-\lambda}},$$

we have

$$\langle T^{1/2}\psi_Y, p_t(\mathcal{L})T^{1/2}\psi_X\rangle > 0.$$

This proves Theorem 2 □

As an immediate consequence of Theorem 2, we have:

COROLLARY 1. *Suppose G is a regular graph which is not complete. Then*

$$D(G) \le \left\lceil \frac{\cosh^{-1}(v-1)}{\cosh^{-1}\frac{\lambda_{v-1}+\lambda_1}{\lambda_{v-1}-\lambda_1}} \right\rceil.$$

To generalize Theorem 2 to distances among k subsets of the vertices, we need the following geometric lemma [5].

LEMMA 2. *Let $x_1, x_2, \ldots, x_{d+2}$ denote $d+2$ arbitrary vectors in d-dimensional Euclidean space. Then there are two of them, say, v_i, v_j, $(i \ne j)$ such that $\langle v_i, v_j\rangle \ge 0$.*

THEOREM 3. *Suppose G is not a complete graph. For $X_i \subset V(G), i = 0, 1, \ldots, k$, we have*

$$\min_{i\ne j} d(X_i, X_j) \le \max_{i\ne j} \left\lceil \frac{\cosh^{-1}\sqrt{\frac{\mathrm{vol}\,\bar{X}_i\,\mathrm{vol}\,\bar{X}_j}{\mathrm{vol}\,X_i\,\mathrm{vol}\,X_j}}}{\cosh^{-1}\frac{1}{1-\lambda_k}} \right\rceil,$$

if $1-\lambda_k \ge \lambda_{v-1}-1$.

PROOF. Let X and Y denote two distinct subsets among the X_i's. We consider

$$\langle T^{1/2}\psi_Y, (I-\mathcal{L})^t T^{1/2}\psi_X\rangle \ge a_0b_0 + \sum_{i=1}^{k-1}(1-\lambda_i)^t a_i b_i - \sum_{i\ge k}(1-\lambda_k)^t |a_i b_i|.$$

For each X_i, $i = 0, 1, \ldots, k$, we consider the vector consisting of the Fourier coefficients of the eigenfunctions $\varphi_1, \ldots, \varphi_{k-1}$ in the eigenfunction expansion of X_i. Suppose we define a scalar product for two such vectors $(a_1, \ldots, a_{k-1})$ and $(b_1, \ldots, b_{k-1})$ by

$$\sum_{i=1}^{k-1}(1-\lambda_i)^t a_i b_i.$$

From Lemma 2, we know that we can choose two of the subsets, say, X and Y with their associated vectors satisfying

$$\sum_{i=1}^{k-1}(1-\lambda_i)^t a_i b_i \ge 0.$$

Therefore, we have

$$\langle T^{1/2}\psi_Y, (I-\mathcal{L})^t T^{1/2}\psi_X\rangle > \frac{\mathrm{vol}\,X\,\mathrm{vol}\,Y}{\mathrm{vol}\,G} - (1-\lambda_k)^t\frac{\sqrt{\mathrm{vol}\,X\,\mathrm{vol}\,\bar{X}\,\mathrm{vol}\,Y\,\mathrm{vol}\,\bar{Y}}}{\mathrm{vol}\,G}$$

and we proved that

$$\min_{i\ne j} d(X_i, X_j) \le \max_{i\ne j} \left\lceil \frac{\log\sqrt{\frac{\mathrm{vol}\,\bar{X}_i\,\mathrm{vol}\,\bar{X}_j}{\mathrm{vol}\,X_i\,\mathrm{vol}\,X_j}}}{\log\frac{1}{1-\lambda_k}} \right\rceil.$$

By using the Chebychev polynomial $p_t(x)$ instead of $(1-x)^t$, we can improve the above bound by replacing log with $\cosh^{-1}$ and Theorem 3 is proved. □

We note that the condition $1 - \lambda_k \geq \lambda_{v-1} - 1$ can be eliminated by modifying the λ's as follows:

THEOREM 4. *For $X_i \subset V(G)$, $i = 0, 1, \ldots, k$, we have*

$$\min_{i \neq j} d(X_i, X_j) \leq \max_{i \neq j} \left\lceil \frac{\cosh^{-1} \sqrt{\frac{\operatorname{vol} \bar{X}_i \operatorname{vol} \bar{X}_j}{\operatorname{vol} X_i \operatorname{vol} X_j}}}{\cosh^{-1} \frac{\lambda_{v-1}+\lambda_k}{\lambda_{v-1}-\lambda_k}} \right\rceil$$

if $\lambda_k \neq \lambda_{v-1}$.

Observing that the denominator is a decreasing function of $\lambda_{v-1} \leq 2$, we obtain the following useful corollary.

COROLLARY 2. *For $X_i \subset V(G)$, $i = 0, 1, \ldots, k$, we have*

$$\min_{i \neq j} d(X_i, X_j) \leq \max_{i \neq j} \left\lceil \frac{\cosh^{-1} \sqrt{\frac{\operatorname{vol} \bar{X}_i \operatorname{vol} \bar{X}_j}{\operatorname{vol} X_i \operatorname{vol} X_j}}}{\cosh^{-1} \frac{2+\lambda_k}{2-\lambda_k}} \right\rceil$$

if $\lambda_k \neq \lambda_{v-1}$.

Eventually, taking all X_i's of size unity we obtain:

COROLLARY 3. *If G is a graph of size v distinct from the complete graph its k-diameter is bounded above as*

$$D_{k+1}(G) \leq \max_{i \neq j} \left\lceil \frac{\cosh^{-1} v}{\cosh^{-1} \frac{2+\lambda_k}{2-\lambda_k}} \right\rceil$$

if $\lambda_k \neq \lambda_{v-1}$.

5.2. Spectral Multiplicity. We now derive an application of the preceding bounds to estimate the spectral multiplicity of graphs.

THEOREM 5. *If A_1 denotes the multiplicity of λ_1 for Δ regular graph on v vertices, then*

$$A_1 + 1 \geq v/M\left(\Delta, \left\lceil \frac{\cosh^{-1}(v)}{\cosh^{-1}\left(\frac{2+\lambda_2}{2-\lambda_2}\right)} \right\rceil\right).$$

PROOF. Consider an $(A_1 + 1)$-code of minimum distance D_{A_1+1} in the considered graph. Then, by Corollary 3, its minimum distance is at most $\lceil \cosh^{-1}(v)/\cosh^{-1}(\frac{2+\lambda_2}{2-\lambda_2})\rceil$. We apply the analogue of the Gilbert bound mentioned before Lemma 1, keeping in mind that an upper bound on a ball of radius D in the graph we consider is $M(\Delta, D)$. □

This can be generalized further to obtain bounds on the distribution of, say, the first m eigenvalues.

THEOREM 6. *If A_i denotes the multiplicity of the ith distinct eigenvalue for a Δ regular graph on v vertices, and $k = \sum_{j=1}^{m} A_j$, then*

$$\sum_{j=1}^{m} A_j + 1 \geq v/M\left(\Delta, \left\lceil \frac{\cosh^{-1}(v)}{\cosh^{-1}\left(\frac{2+\lambda_{k+1}}{2-\lambda_{k+1}}\right)} \right\rceil\right).$$

More generally, if we have a graph G with balls of radius D bounded by a function $M_G(D)$, we obtain the following result.

THEOREM 7. *With the above hypothesis, if A_i denotes the multiplicity of the ith distinct eigenvalue for a Δ regular graph on v vertices, and $k = \sum_{j=1}^{m} A_j$, then*

$$\sum_{j=1}^{m} A_j + 1 \geq v/M_G \left(\left\lceil \frac{\cosh^{-1}(v)}{\cosh^{-1}\left(\frac{2+\lambda_{k+1}}{2-\lambda_{k+1}}\right)} \right\rceil \right).$$

For instance for Abelian Cayley graphs of degree $\Delta = k_1 + 2k_2$ (k_1 = number of generators of order 2), we obtain from [9]

$$M_G(D) = (2D + k)^k / k!,$$

where $k = k_1 + k_2$. Another graph that already received some attention in [8] is the *coset graph* of a linear code. Let C denote a k-dimensional binary linear code of length n, and minimum distance d. Then, a graph $G(C)$ can be built on the 2^k cosets of the dual code with degree n and spectrum related in a simple way to the weights w_i of C. Specifically, in the notations of Section 1, we have $\lambda_i = \frac{2w_i}{n}$, and $\lambda_1 = 2d/n$ with multiplicity A_d the so-called error coefficient in coding theory, which is the leading term in error probability calculations. It is straightforward to show that for this graph a suitable bound on the size of balls is

$$M_{G(C)}(D) = \sum_{j=0}^{D} \binom{n}{j},$$

or, using the entropy function $H(x) := -x \log_2(x) - (1-x)\log_2(1-x)$, and not necessarily performing asymptotics

$$M_{G(C)}(D) = 2^{nH(D/n)}.$$

Theorem 5 applied to that situation yields, noting that $G(C)$ is the complete graph if C is the dual of a perfect Hamming code.

THEOREM 8. *Let C be an $[n, k, d]$ binary linear code, which is not the dual of a perfect Hamming code and weights $w_1 = 0, w_1 = d, w_2, \ldots, w_{n-1}$. Let*

$$U(n, k, x) := \left\lceil \frac{\cosh^{-1}(2^k)}{\cosh^{-1}\left(\frac{n+x}{n-x}\right)} \right\rceil.$$

Then the error coefficient of C is bounded below as

$$A_d \geq 2^{k - nH(U(n,k,w_2)/n)}.$$

We leave it as an open problem to see if the exponent of the exponential on the RHS can be made > 0.

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