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ON PARTITIONS OF GRAPHS INTO TREES

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We consider the minimum number $\tau(G)$ of subsets into which the edge set $E(G)$ of a graph G can be partitioned so that each subset forms a tree. It is shown that for any connected G with n vertices, we always have $\tau(G) \leqslant \lceil \frac{1}{2}n \rceil$.

1. Introduction

We consider finite undirected graphs[1] without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. G is said to be a *forest* if G contains no cycles. A connected forest is called a *tree*. The *arboricity* $\gamma(G)$ of G is defined to be the minimum number of subsets into which $E(G)$ can be partitioned so that the graph formed by each of the subsets is a forest. Similarly, the *vertex-arboricity* $\gamma'(G)$ of G is defined to be the minimum number of subsets into which $V(G)$ can be partitioned so that the graph induced by each of the subsets is a forest.

In [4], Nash-Williams gives the following expression for $\gamma(G)$ (assuming $|E(G)| \geqslant 1$):

$$\gamma(G) = \max_H \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil, \tag{1}$$

where H ranges over all nontrivial induced subgraphs of G (and, as usual, $|X|$ denotes the cardinality of the set X and $\lceil x \rceil$ denotes the least integer greater than or equal to x).

On the other hand, Chartrand and Kronk [2] give the following upper bound on $\gamma'(G)$:

$$\gamma'(G) \leqslant 1 + \left\lfloor \tfrac{1}{2} \max_H \delta(H) \right\rfloor, \tag{2}$$

where, as before, H ranges over all induced subgraphs of G, $\delta(H)$ denotes the maximum degree of any vertex of H and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

In this paper, we consider the minimum number $\tau(G)$ of subsets into which the edge set $E(G)$ of G can be partitioned so that each subset forms a *tree*. This

[1] We follow the terminology of Behzad and Chartrand [1].

problem was suggested by M. Foregger and T. Foregger who showed that for a connected graph G, the minimum number $\tau'(G)$ of subsets into which $V(G)$ can be partitioned so that each subset induces a tree satisfies (see [3])

$$\tau'(G) \leq \lceil \tfrac{1}{2} |V(G)| \rceil. \tag{3}$$

We will show that $\tau(G)$ satisfies the same inequality, i.e., for a connected graph G,

$$\tau(G) \leq \lceil \tfrac{1}{2} |V(G)| \rceil. \tag{4}$$

It is easy to see that

$$\tau(G) \geq \gamma(G) \tag{5}$$

and for the complete graph K_n on n vertices

$$\tau(K_n) = \gamma(K_n) = \lceil \tfrac{1}{2} n \rceil, \tag{6}$$

which shows that equality in (3) and (4) can be achieved.

2. Preliminaries

In this section, we assume G is connected. A *tree partition* of a graph G is a set $\mathcal{T} = \{T_1, T_2, \ldots, T_r\}$ of trees with the following properties:

(i) Each T_i is a (not necessarily induced) subgraph of G;
(ii) For $i \neq j$, T_i and T_j are edge disjoint, i.e., $E(T_i) \cap E(T_j) = \emptyset$.
(iii) G is the union of the T_i, $1 \leq i \leq r$. i.e., $V(G) = \bigcup_i V(T_i)$, $E(G) = \bigcup_i E(T_i)$.

We write $G = \sum_{i=1}^{r} T_i$ and we say G is *partitioned* by $\mathcal{T} = \{T_1, \ldots, T_r\}$. Thus $\tau(G)$ is just the minimum number of trees any tree partition of G can have.

First, let us prove the following auxiliary result.

Lemma 2.1. *Let G be a connected graph and suppose $|V(G)| > 2$. Then there exist two vertices x, y of G such that the following holds:*

(a) *The graph $G - \{x, y\}$ with vertex set $V(G) - \{x\} - \{y\}$ and edge set $\{e \in E(G) : x \notin e$ and $y \notin e\}$ is connected.*

(b) *Either x is adjacent to y or there is a vertex z which is adjacent to both x and y.*

Proof. Since G is connected, there exists a spanning tree T of G. For any two vertices v_1, v_2 of G, there is a unique path $P(v_1, v_2)$ in T joining v_1 and v_2. Let $d(v_1, v_2)$ denote the length of $P(v_1, v_2)$, i.e., the number of edges in $P(v_1, v_2)$. Let v^* be a fixed vertex of G.

Let x be a vertex with $d(v^*, x) \geq d(v^*, v)$ for any vertex v of G. Now, x is of degree 1 in T since otherwise there is a vertex v' adjacent in T to x which is not in $P(v^*, x)$ and therefore $d(v', v^*) > d(v^*, x)$, which is impossible. Let v_1 denote the unique vertex adjacent to x in T. There are several cases to be considered:

Case 1. v_1 is of degree 1 in T. Then T consists of two vertices and $|V(G)| = 2$, which contradicts our assumption that $|V(G)| > 2$.

Case 2. v_1 is of degree 2 in T. It is easy to see that $T - \{x, v_1\}$ is still a tree. Thus, $G - \{x, v_1\}$ is connected since it has a spanning tree, and x is adjacent to v_1, so that (a) and (b) are satisfied in this case.

Case 3. v_1 is of degree greater than 2 in T. Let y be a vertex adjacent to v_1 in T which is not in $P(v^*, x)$. Thus, y must be of degree 1 in T since $d(v^*, y) = d(v^*, x)$. It is clear that $T - \{x, y\}$ is a tree. Therefore $G - \{x, y\}$ is connected and v_1 is a vertex adjacent to both x and y.

Combining the preceding cases, the lemma is proved.

3. The main theorem

Instead of just showing $\tau(G) \leq \lceil \frac{1}{2} |V(G)| \rceil$ for G connected, we will prove the following stronger theorem.

Theorem 3.1. *Let G be a connected graph and let n denote $|V(G)|$. Then there is a tree partition $\mathcal{T} = \{T_1, \ldots, T_{\lceil n/2 \rceil}\}$ of G and a function $\lambda : V(G) \to \{1, 2, \ldots, \lceil \frac{1}{2} n \rceil\}$ such that:*

(i) $\lambda(v) = i$ *implies* $v \in V(T_i)$;
(ii) *For any* k, $1 \leq k \leq \lceil \frac{1}{2} n \rceil$, $|\{v : \lambda(v) = k\}| \leq 2$.

In other words, for the tree partition $\mathcal{T}$, we can assign vertices to trees such that each tree has at most two vertices assigned to it and contained in it.

Proof. First, let us verify Theorem 3.1 for $n = 1$ and 2, since we will prove the theorem by induction. When $n = 1$ or 2, $\lceil \frac{1}{2} n \rceil = 1$ and G is itself a tree T_1. Define $\lambda(v) = 1$ for all $v \in V(G)$. Then λ satisfies (i) and (ii) and the theorem holds in this case.

Suppose the thoerem holds for all connected graphs G with $|V(G)| \leq n$. It suffices to show that the theorem holds for any connected graph G with $|V(G)| = n + 2$.

By the lemma we can choose $x, y \in V(G)$ such that $G - \{x, y\}$ is connected where either x is adjacent to y or there is a $z \in V(G)$ which is adjacent to both x and y. We consider the corresponding two cases.

Case 1. x is adjacent to y and the graph $G' = G - \{x, y\}$ is connected. By the induction assumption, there is a tree partition $\mathcal{T}' = \{T'_1, T'_2, \ldots, T'_{\lceil n/2 \rceil}\}$ of G' and

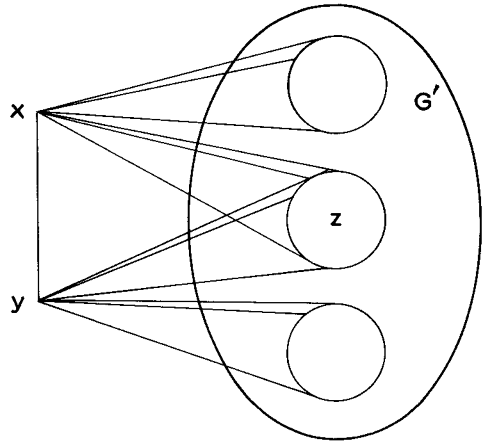

Fig. 1.

there is a function $\lambda': V(G') \to \{1, 2, \ldots, \lceil \frac{1}{2}n \rceil\}$, such that $\lambda'(v) = i$ implies $v \in V(T_i')$, and for any k, $1 \leq k \leq \lceil \frac{1}{2}n \rceil, |\{v : \lambda'(v) = k\}| \leq 2$.

Let Z be a subset of $V(G')$ (see Fig. 1) defined by

$$Z \equiv \{v \in V(G) : v \text{ is adjacent to both } x \text{ and } y \text{ in } G\}.$$

Now we choose a tree partition $\mathcal{T}$ as follows.

(a) If i is not $\lambda'(v)$ for any vertex v in Z, we let $T_i = T_i'$.

(b) If there is exactly one vertex h in Z with $\lambda'(h) = i$, we choose T_i to have vertex set $V(T_i) = V(T_i') \cup \{x, y\}$ and edge set $E(T_i) = E(T_i') \cup \{(h, x), (h, y)\}$. T_i is connected and $|E(T_i)| = |E(T_i')| + 2 = |V(T_i')| + 1 = |V(T_i)| - 1$. Thus, T_i is a tree.

(c) If there are two vertices h, k in Z with $\lambda'(h) = \lambda'(k) = i$, we choose T_i to have vertex set $V(T_i) = V(T_i') \cup \{x, y\}$ and edge set $E(T_i) = E(T_i') \cup \{(h, x), (k, y)\}$. We note as before that T_i is a tree since $|E(T_i)| = |V(T_i)| - 1$ and T_i is connected.

From (a), (b) and (c), we have now defined T_i for $1 \leq i \leq \lceil \frac{1}{2}n \rceil$. Let $T^* = T_{\lceil n/2 \rceil + 1}$ be the graph formed by the edges $E(T^*) = E(G) - \bigcup_{i=1}^{\lceil n/2 \rceil} E(T_i)$. Since (x, y) is in $E(T^*)$ and any vertex in T^* is adjacent to exactly one of x and y, T^* is connected and in fact is a tree. Moreover, all T_i, $1 \leq i \leq \lceil \frac{1}{2}n \rceil + 1$, are mutually edge disjoint. Thus we have constructed a tree-partition of G consisting of $\lceil \frac{1}{2}n \rceil + 1 = \lceil \frac{1}{2} |V(G)| \rceil$ trees. Define $\lambda : V(G) \to \{1, 2, \ldots, \lceil \frac{1}{2}n \rceil + 1\}$ by

$$\lambda(v) = \begin{cases} \lceil \frac{1}{2}n \rceil + 1 & \text{if } v = x \text{ or } v = y, \\ \lambda'(v) & \text{otherwise.} \end{cases}$$

It is easy to see that (i) and (ii) are satisfied by this definition of λ.

Case 2. x is not adjacent to y but there exists z such that z is adjacent to both x and y and $G' = G - \{x, y\}$ is connected. In this case, the construction of a tree partition of G is slightly different from that of Case 1. Let $\mathcal{T}' = \{T_1', T_2', \ldots, T_{\lceil n/2 \rceil}'\}$ be a tree partition of G' guaranteed by the induction hypothesis and let λ' be the corresponding function on $V(G')$.

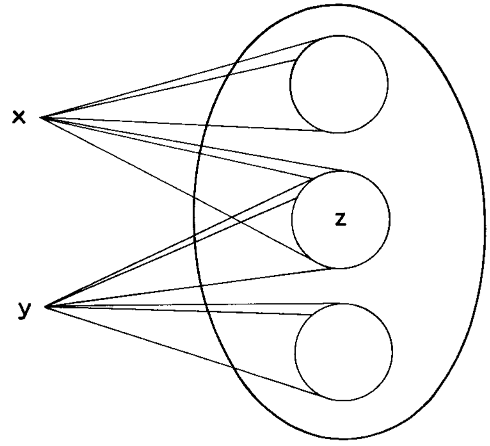

Fig. 2.

We define Z to be the subset of $V(G)$ consisting of all vertices of G which are adjacent to both x and y (see Fig. 2). Note that Z is non-empty since $z \in Z$. Denote $\lambda'(z)$ by i^*. We define T_i, $1 \leq i \leq \lceil \frac{1}{2}n \rceil$, $i \neq i^*$, exactly as in Case 1. T_{i^*} is defined as follows:

(a′) If there exists a vertex v in Z with $\lambda'(v) = i^*$ and $v \neq z$, we choose T_{i^*} to have vertex set $V(T_{i^*}) = V(T'_{i^*}) \cup \{x, y\}$ and edge set $E(T_{i^*}) = E(T'_{i^*}) \cup \{(v, x), (v, y)\}$.

(b) Otherwise we choose $T_{i^*} = T'_{i^*}$.

Let $T^* = T_{\lceil n/2 \rceil + 1}$ be the graph formed by the edges $E(T^*) = E(G) - \bigcup_{i=1}^{\lceil n/2 \rceil} E(T_i)$. Since (x, z) and (z, y) are in $E(T^*)$ and any vertex in $V(T^*) - \{x, y, z\}$ is adjacent to exactly one of x and y, T^* is a tree. Thus $\mathcal{T} = \{T_1, \ldots, T_{\lceil n/2 \rceil + 1}\}$ is a tree partition of G. Define $\lambda : V(G) \to \{1, 2, \ldots, \lceil \frac{1}{2}n \rceil + 1\}$ by

$$\lambda(v) = \begin{cases} \lceil \frac{1}{2}n \rceil + 1 & \text{if } v = x \text{ or } v = y, \\ \lambda'(v) & \text{otherwise.} \end{cases}$$

Again, it is easily checked that (i) and (ii) are satisfied by the definition of λ.

Thus, in each case, it is possible to choose the appropriate tree partition and λ for G. This completes the induction step and the theorem is proved.

Theorem 3.2. *For a connected graph G, with n vertices, $n > 1$, and e edges, we have*

$$\left\lceil \frac{e}{n-1} \right\rceil \leq \tau(G) \leq \lceil \tfrac{1}{2}n \rceil .$$

Proof. By Theorem 3.1, we have $\tau(G) \leq \lceil \frac{1}{2}n \rceil$. Let $\mathcal{T} = \{T_1, \ldots, T_r\}$ be a tree partition of G where $r = \tau(G)$. Since every tree has at most $n-1$ edges, we have

$$e = \sum_{i=1}^{r} |E(T_i)| \leq (n-1)\tau(G).$$

Therefore,

$$\left\lceil \frac{e}{n-1} \right\rceil \leqslant \tau(G),$$

and Theorem 3.2 is proved.

The same methods can be used to establish the following generalization of Theorem 3.1 (cf. [3]).

Theorem 3.3. *Let G be a graph having n vertices, e edges, and k connected components, k < n. Then*

$$k-1+\left\lceil \frac{e}{n-k} \right\rceil \leqslant \tau(G) \leqslant \lceil \tfrac{1}{2}(n+k-1) \rceil . \tag{7}$$

Proof. Let G be a graph having components with $n_1, n_2, \ldots, n_k$ vertices and $e_1, e_2, \ldots, e_k$ edges, respectively. The minimum number of trees which cover G is bounded above by

$$\tau(G) \leqslant \lceil \tfrac{1}{2}n_1 \rceil + \lceil \tfrac{1}{2}n_2 \rceil + \cdots + \lceil \tfrac{1}{2}k_2 \rceil \leqslant \tfrac{1}{2}(n+k)$$

Since $\tau(G)$ is an integer, we have $\tau(G) \leqslant \lceil \frac{1}{2}(n+k-1) \rceil$.

On the other hand, by assuming $n_i > 1$ for $1 \leqslant i \leqslant h$ and $n_i = 1$ for $h < i \leqslant k$, we have

$$\tau(G) \geqslant \left\lceil \frac{e_1}{n_1-1} \right\rceil + \left\lceil \frac{e_2}{n_2-1} \right\rceil + \cdots + \left\lceil \frac{e_h}{n_h-1} \right\rceil + k - h$$

Without loss of generality, we may assume $n_1 \leqslant n_2 \leqslant \cdots \leqslant n$. We note that $n_h \leqslant n-k+1$. It is easy to verify that

$$\left\lceil \frac{e_1}{n_1-1} \right\rceil + \left\lceil \frac{e_2}{n_2-1} \right\rceil \geqslant 1 + \left\lceil \frac{e_1+e_2-n_1+1}{n_2-1} \right\rceil .$$

Thus, we have

$$\begin{aligned}
\tau(G) &\geqslant k-1+\left\lceil \frac{e-(n_1+\cdots+n_{h-1})+h-1}{n_h-1} \right\rceil \\
&\geqslant k+\left\lceil \frac{e-n+h}{n_h-1} \right\rceil \\
&\geqslant k+\left\lceil \frac{e-n+h}{n-k} \right\rceil \\
&= k-1+\left\lceil \frac{e}{n-k} \right\rceil
\end{aligned}$$

and Theorem 3.3 is proved.

4. Conclusions

It would be desirable to have an exact formula for $\tau(G)$, in the spirit of (1). The relation between $\tau(G)$ and $\tau'(G)$ is not completely understood. Let G_1 be the graph shown in Fig. 3(a). We have

$$\tau(G_1)=3>\tau'(G_1)=2.$$

On the other hand, for the graph G_2 shown in Fig. 3(b) (provided by R.L. Graham), we have

$$\tau'(G_2)=4>\tau(G_2)=2.,$$

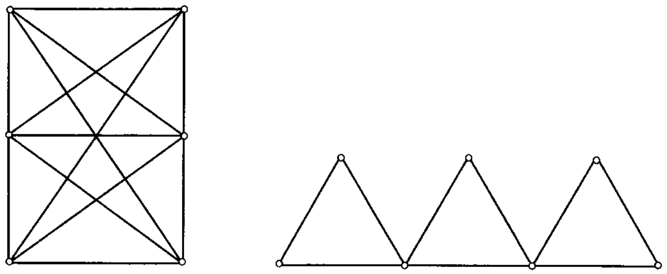

Fig. 3

It would be interesting to know how large and how small the ratio $\tau(G)/\tau'(G)$ can be for a connected graph G (e.g., in terms of $|V(G)|$). Obvious generalizations of the graphs in Fig. 3 show that there are arbitrarily large graphs G and G' having n vertices and satisfying

$$\frac{\tau(G)}{\tau'(G)}>\tfrac{1}{8}(n+4),$$

and

$$\frac{\tau'(G')}{\tau(G')}>\tfrac{1}{4}n.$$

There are many possible variations of this problem. For example, we might ask, for a given graph G, what is the minimum number of subsets into which $E(G)$ can be partitioned so that each subset forms a graph with certain specified properties, e.g., each graph is bipartite, has at most one cycle, has only odd cycles, has chromatic number $\leqslant k$, etc.

Of course, we could also consider the corresponding problem of determining the minimum number of subsets into which the vertex set can be partitioned so that each subset *induces* a graph with the desired properties.

Acknowledgment

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