Annals of Discrete Mathematics 2 (1978) 173–200.

STEINER TREES FOR LADDERS

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Introduction

Suppose we are given a finite set X of points in the plane and we are required to form a network $N(X)$ connecting up all the points of X so that the total length of $N(X)$ is as small as possible. As might be expected, the difficulty of this task depends not only on the particular structure X may have but also on just which candidates are to be allowed for $N(X)$. For example, if $N(X)$ must be formed by placing straight line segments between appropriate pairs of points of X (with the length of $N(X)$ being the sum of the (Euclidean) lengths of these segments) then $N(X)$ is called a *minimum spanning tree* for X and efficient procedures are known for generating such networks (see [12]). On the other hand, suppose we are first allowed to add additional points to X, forming some set Y containing X, and we then choose $N(X)$ to be a minimum spanning tree for Y. (Extra points *can* help; for example, suppose X is the set of vertices of an equilateral triangle.) Such a network is called a *minimum Steiner tree* for X. For this case, however, not only are no efficient algorithms known for constructing general minimum Steiner trees but, in fact, there is strong evidence that no such algorithms can even exist in principle. There are several reasons why the construction of a minimum Steiner tree $S^*(X)$ for X can be difficult. (The fact that it is even a finite problem was not known until 1961 [10].) It may happen that there are many additional points which must be added to form Y from X (these points are called *Steiner points*) and the potential topologies for connecting all these points together are both complicated and numerous. On the other hand, it may happen that while there are relatively few potential Steiner points and only very simple topologies for them, there are a tremendous number of choices among just which ones to choose. It was this second situation on which the NP-completeness proof for the minimum Steiner tree problem in [6] was based. It is the first situation that will occupy our attention in this paper.

Minimum Steiner trees have been studied extensively for some time and a substantial number of results concerning their structure are known (e.g., see [4, 5, 7, 10]). In particular, restricting ourselves to minimum Steiner trees which have no Steiner points of degree 2 (nothing essential is lost by this restriction), it is known that any minimum Steiner tree $S^*(X)$ for X can have at most $|X|-2$ Steiner points (where, as usual, $|X|$ denotes the cardinality of X). Those trees $S^*(X)$ which have the maximum number $|X|-2$ of Steiner points are called *full* Steiner trees. It is also known that $S^*(X)$ may always be decomposed into sets $S^*(X_1), \ldots, S^*(X_t)$, where $X_1, \ldots, X_t$ are subsets of X with $|X_i \cap X_j| \leq 1$ for all $i \neq j$, $S^*(X_k)$ is a *full* minimum Steiner tree for X_k and the edges of the $S^*(X_k)$ form a partition of the edges of $S^*(X)$ (see Fig. 1).

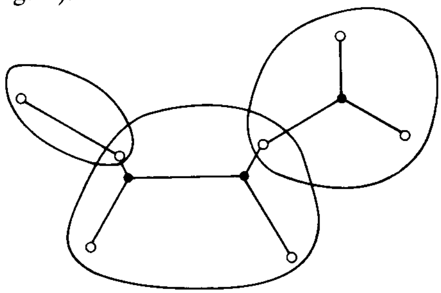


Fig. 1.

Currently, the most successful algorithms[1] for generating minimum Steiner trees for general sets X involve (cleverly) choosing small trial sets for the X_k, constructing full minimum Steiner trees on them and then piecing everything together (see [3]). Thus, it becomes important to understand the structure of sets X, which can support a full minimum Steiner tree. For example, it would be wonderful[2] if no set with more than 100 points could have a full minimum Steiner tree.

Perhaps the simplest[3] infinite family of sets whose minimum Steiner trees one might study are the *ladders*, so named by Boyce, who first focussed attention on them in [3]. A *ladder* L_n consists of $2n$ points arranged in a rectangular 2 by n array with adjacent pairs of points forming a square (see Fig. 2).

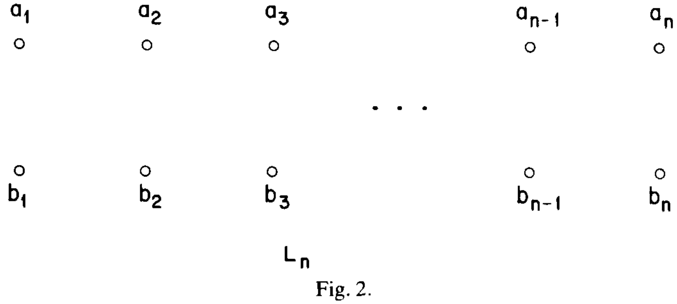


Fig. 2.

In this paper, we determine the minimum Steiner trees $S^*(L_n)$ for L_n. In particular, it turns out, as suspected by Boyce, that for n odd, $S^*(L_n)$ is a full Steiner tree (for n even, $S^*(L_n)$ degenerates into a union of $S^*(L_1)$'s and $S^*(L_2)$'s).

[1] These work quite well when $|X|=10$; problems with $|X|=20$, however, appear to be hopeless by these techniques.

[2] In fact, more wonderful than one might first think, in view of the previously mentioned NP-completeness of the problem.

[3] Actually, a set of collinear points is even simpler but minimum Steiner trees for such sets are highly uninteresting.

This furnishes the first example of arbitrarily large point sets having *full* minimum Steiner trees.

It was found that structure of the class of all full Steiner trees on L_n, i.e., full trees in which each Steiner point is still the intersection of 3 incident edges, each meeting the other two at 120°, but whose total length may not be minimal, is surprisingly rich. The analysis of this structure involves a rather delicate interplay between geometry and diophantine approximation. We summarize some of the results at the end of the paper. The detailed proofs will be given in a later paper.

We should make a few remarks at this point regarding the style of the paper. Rather than include full proofs for all assertions made (which would result in a paper of formidable length), we have elected just to sketch most of the proofs, giving hints where helpful, but in sufficient detail so that the interested reader will be able to construct complete proofs if desired. Our object will be not so much to convince but rather, in the words of Halmos [8], "to induce a benevolent feeling of credulity." For any undefined terminology, the reader may consult [5] (for Steiner trees), [9] (for graph theory) and [1] (for complexity of algorithms).

Preliminaries

We begin by fixing a standard set of points for L_n. By definition L_n will consist of the $2n$ points

$$\{a_1, \ldots, a_n, b_1, \ldots, b_n\} \quad \text{where } a_k = (2k-2, 1)$$

and

$$b_k = (2k-2, -1) \quad \text{for } 1 \leq k \leq n.$$

The set $A = \{a_1, \ldots, a_n\}$ is called the *top row* of L_n; the set $B = \{b_1, \ldots, b_n\}$ is called the *bottom row* of L_n. A set $\{a_k, b_k\}$ is called a *column* of L_n. Let S^* be a minimum Steiner tree for L_n with vertex set $S \cup A \cup B$. The points $A \cup B = L_n$ are called the *regular* points of S^*; the points S are called the *Steiner* points of S^*. Edges of S^* are taken to be closed line segments between various pairs of points of S^*. We assume w.l.o.g. that every Steiner point is incident to at least 3 edges of S^*.

Fact 1 (see [5]). All Steiner points of S^* are incident to exactly 3 edges of S^*, each meeting the other two at 120°. S^* has at most $n-2$ Steiner points.

By the *Steiner hull* of X we mean the complement of the union of all (infinite) closed 120° sectors which do not intersect X.

Fact 2 (see [5]). All Steiner points of S^* lie in the Steiner hull of L_n.

Note that the Steiner hull of X is a subset of the convex hull of X.

Fact 3 (see [5, 7]). No edge of S^* can have length exceeding 2.

Proof. If there were such an edge then we could remove it, forming two connected components C_1 and C_2, which could then be reconnected by adjoining *some* edge of a minimum spanning tree for L_n. Since there is a spanning tree for L_n with all edges having length equal to 2 then the new Steiner tree for L_n has smaller length than that of S^*, which is a contradiction. □

Let R denote the infinite closed region shown in Fig. 3. We call R a *pointed strip*; $t(R)$ is called the *tip* of R. R can have any position and orientation in the plane.

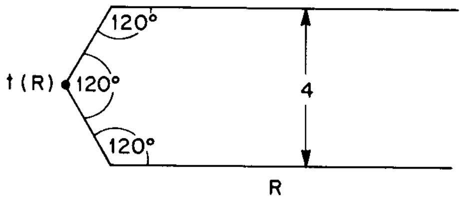


Fig. 3.

Fact 4 (see [7]). If $R \cap L_n = \emptyset$, then $t(R)$ cannot be a Steiner point of S^*.

Idea of proof. If we assume that $t(R)$ is a Steiner point of S^*, then using Fact 3, we can (carefully) choose a path in S^* which never leaves R (essentially, always try to go toward the middle of the strip). Hence, if $R \cap L_n = \emptyset$, then the path cannot terminate and so S^* must have infinitely many Steiner points, which contradicts Fact 1. □

Note that Fact 2 is an immediate consequence of Fact 4.

Fact 5 (see [5]). The angle formed by any two edges of S^* with a common endpoint must be at least 120°.

Proof. If the edges $[x, y]$ and $[x, z]$ make an angle of less than 120°, then adding a Steiner point in the triangle determined by x, y and z results in a shorter total length, which is impossible. □

We remark here that of course no two edges of S^* can intersect except at a common endpoint (see [10]).

Fact 6 (see [4]). There exist (unique) subsets $X_1, \ldots, X_t \subseteq L_n$ and full minimum Steiner trees $S^*(X_k)$ on X_k such that:

(i) $|X_i \cap X_j| \leq 1$ for $i \neq j$,

(ii) $$S^* = \bigcup_{k=1}^{t} S^*(X_k).$$

We call the $S^*(X_k)$ the *full tree components* of S^*.

Fact 7 (see [7]). Let $x, y \in L_n$. Then x and y belong to the same full tree component of S^* iff x and y are the only regular points on the path in S^* between x and y (where the path is defined to be the (unique) minimal connected set containing x and y, which can be formed from the union of edges of S^*.)

Proof. In a full Steiner tree, a point has degree 1 iff it is regular. □

Fact 8. Let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ be two points in the plane. Then the point $p = (x, y)$ for which p_1, p_2 and p form a counterclockwise equilateral triangle is given by

$$x = \tfrac{1}{2}(x_1 + x_2 + \sqrt{3}(y_1 - y_2)),$$

$$y = \tfrac{1}{2}(y_1 + y_2 - \sqrt{3}(x_1 - x_2)).$$

(see Fig. 4). Furthermore, if C is the centroid of the triangle and z is any point on the arc of a circle through x and y centered at C, then:

(i) length $[p, z]$ = length $[p_1, z]$ + length $[p_2, z]$,

(ii) $\angle p_1 z p = \angle p_2 z p = 60°$.

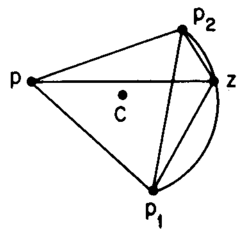


Fig. 4.

Proof. Elementary geometry (see [10]). □

Minimum Steiner trees for L_n

We are now in a position to begin a more detailed analysis of the structure of S^*.

Fact 9. Suppose a_{k-1} and a_{k+1} are in the same full tree component $S^*(X_i)$ of S^*. Then a_k is also in S^*.

Idea of proof. Suppose $a_k \notin S^*(X_i)$. By Fact 7, there is a path $P = (a_{k-1}, s_1, s_2, \ldots, s_r, a_{k+1})$ in S^* from a_{k-1} to a_{k+1} containing no regular points except a_{k-1} and a_{k+1}. By Fact 2, P lies in the Steiner hull of L_n. Hence, P must intersect the open line segment (a_k, b_k), say at the point x (see Fig. 5) (In fact, there may be more than one such intersection.)

We next claim that at least one of the points a_{k-1}, a_{k+1} does not belong to a full

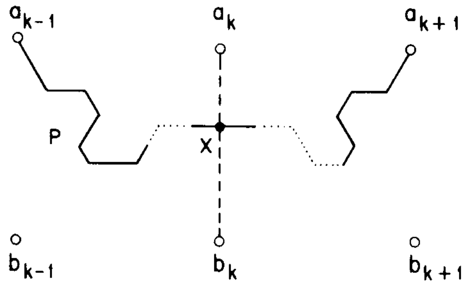


Fig. 5.

tree component which also contains a_k. For suppose they both do, i.e., suppose there are full tree components $S^*(X_u)$ and $S^*(X_v)$ with a_{k-1}, $a_k \in S^*(X_u)$ and $a_k, a_{k+1} \in S^*(X_v)$. If $u = v$, then

$$\{a_{k-1}, a_{k+1}\} \subseteq X_u \cap X_i$$

and so $u = i$, which is impossible since we have assumed $a_k \notin S^*(X_i)$. If $u \neq v$, then by Fact 7 there are paths P_u in $S^*(X_u)$ from a_{k-1} to a_k and P_v in $S^*(X_v)$ from a_k to a_{k+1} and thus, a path $P' = P_u \cup P_v$ from a_{k-1} to a_{k+1} in S^*, which is different from P. However, in a tree this is impossible and the claim follows. We assume w.l.o.g. that a_{k-1} and a_k do not belong to a common full tree component. Again, by Fact 7, there is a path P_1 in S^* from a_{k-1} to a_k which contains some regular point y different from a_{k-1} and a_k. Since no edge of P_1 can intersect any edge of P (except at a_{k-1}) then by Fact 2, the only possibility is that $y = a_{k+1}$ (actually, a weakened form of the Jordan Curve Theorem [11] is used here). If it were the case that a_{k+1} and a_k also do not belong to a common full tree component, then the same argument applies and we get a contradiction, since there would exist a path from a_{k-1} to a_k containing a_{k+1} and a path from a_{k+1} to a_k containing a_{k-1}. Thus, we must have that a_{k+1} and a_k belong to a common full tree component $S^*(X_j)$. Since $S^*(X_j) \cap P = \{a_{k+1}\}$ then $X_j = \{a_k, a_{k+1}\}$ and $S^*(X_j)$ is just the line segment $[a_k, a_{k+1}]$. However, the length of $[a_k, x]$ is less than 2 while the length of $[a_k, a_{k+1}]$ is equal to 2 so that replacing the edge $[a_k, a_{k+1}]$ in S^* by the edge $[a_k, x]$ we obtain a tree with shorter length than S^*. This is impossible and Fact 9 follows. □

It is clear that the same result also holds for points in the *bottom* row of L_n. More generally, the following holds.

Fact 10. If $i < j < k$ and a_i and a_k are in a common full tree component $S^*(X_m)$, then a_j is also in $S^*(X_m)$.

Idea of proof. Assume $a_j \notin S^*(X_m)$. Let P be the path in S^* from a_i to a_k. As in the argument for Fact 9, the path P' from a_i to a_j can only intersect P in the point a_i. If P' has a Steiner point s then by Fact 4, some regular point b_t must be connected to s by a path not intersecting P (since $s \notin S^*(X_m)$) and this is clearly impossible. If P' has no Steiner point then by Fact 3, $j = i + 1$ and $[a_i, a_j]$ is an edge of S^*. But as in the proof of Fact 9, we can replace this edge by a shorter edge from a_j to P, which is a contradiction. □

Fact 11. Suppose a_k, a_{k+1} belong to a common full tree component $S^*(X_i)$. Then either $X_i = \{a_k, a_{k+1}\}$ or $X_i \cap \{b_k, b_{k+1}\} \neq \emptyset$.

Idea of proof. Suppose $X_i \neq \{a_k, a_{k+1}\}$. Consider the closed shaded region Q (part of the Steiner hull of L_n) shown in Fig. 6. By Fact 7, the path P from a_k to a_{k+1} has at least one Steiner point. Since all Steiner points must lie in the Steiner hull of L_n

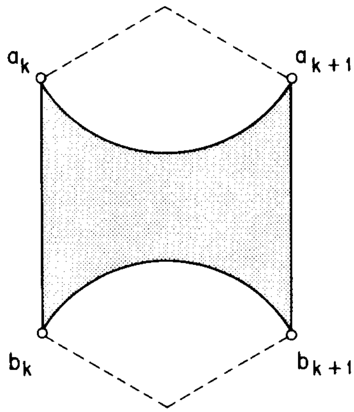


Fig. 6.

and no edge of P has length exceeding 2 (Facts 2 and 3), then some Steiner point s of P must lie in Q. But it is not difficult to see that *some* pointed strip R can be placed with the tip $t(R) = s$ so that $B \cap R \subseteq \{b_k, b_{k+1}\}$. Hence, the desired result follows from Fact 4. □

We now know that if a full tree component $S^*(X_i)$ has

$$a_j, a_{j+1}, \ldots, a_k \in X_i \quad \text{and} \quad b_p, b_{p+1}, \ldots, b_q \in X_i,$$

then $|j - p| \leq 1$, $|k - q| \leq 1$.

Fact 12. If $n > 1$, $[a_k, b_k]$ cannot be an edge of S^*.

Idea of proof. Suppose $[a_k, b_k]$ is an edge of S^*. Consider the paths P from a_k to a_{k+1} and P' from b_k to b_{k+1} (if $k = n$, use a_{k-1} and b_{k-1}). We must have either $b_k \notin P$ or $a_k \notin P'$. In the first case the angle between $[a_k, b_k]$ and the edge of P leaving a_k is $\leq 90°$, which contradicts Fact 5. The second case is similar. □

Fact 13. The only two possible minimum Steiner trees for L_2 are as shown in Fig. 7.

Proof. This is well known (see [5]). □

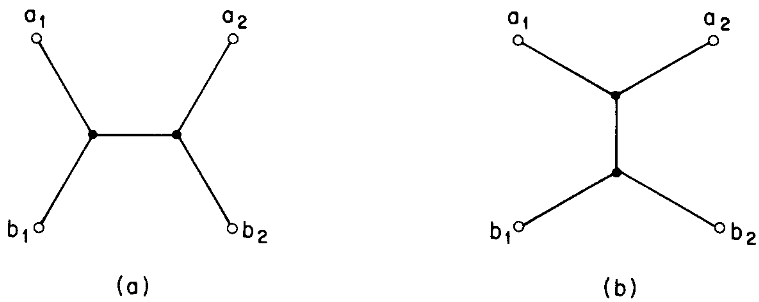


Fig. 7.

We come now to the crux of the matter. Let L_m^* denote a set of points formed from L_m by possibly deleting a_m.

Fact 14. Suppose $m \geqslant 3$ and T^* is a *full* minimum Steiner tree for L_m^*. Then a_1 and b_1 must be joined to a common Steiner point.

Idea of proof. Suppose a_1 and b_1 are not joined to a common Steiner point. By Fact 12, $[a_1, b_1]$ is not an edge of T^*. Thus, the path P from a_1 to b_1 has the Steiner points (in order) $s_1, \ldots, s_r$ where $r \geqslant 2$.

(i) Suppose $r \geqslant 3$. By angle considerations, we see that as we move along P from a_1 to b_1 we cannot always turn in the same direction at each s_i since P would then leave the Steiner hull of L_m^* (see Fig. 8). Hence, for some i, P must turn to the left (counterclockwise) as it leaves s_i (see Fig. 9). By Fact 4, there must be a path P' from s_i to some regular point v of T^*, where (by a suitable orientation of a pointed strip R) we may assume $v \neq a_1, b_1$. Furthermore, $P \cap P' = \{s\}$ since T^* is a tree. But this is impossible (by the Jordan Curve Theorem) since v must be on the *outside* of the simple closed curve C formed by P and $[a_1, b_1]$ while the first edge of P' from s_i is on the *inside* of C.

Thus, we may assume

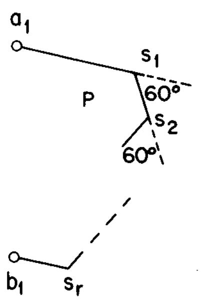


Fig. 8.

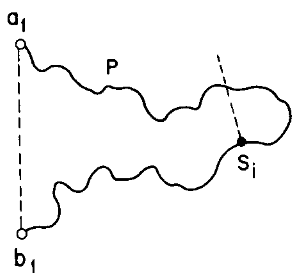


Fig. 9.

(ii) P has exactly *two* Steiner points s_1 and s_2. Clearly, P must turn to the right (clockwise) at both s_1 and s_2. Consider the two "3[rd]" lines L_1 and L_2 leaving s_1 and s_2, respectively (see Fig. 10).

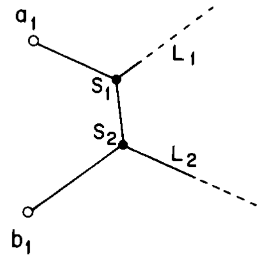


Fig. 10.

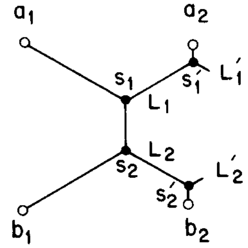


Fig. 11.

There are several possibilities:

(a) Both L_1 and L_2 go directly to regular points. This is impossible because T^* is a *full* Steiner tree on L_m^*.

(b) Both L_1 and L_2 go to further Steiner points s_1' and s_2' (see Fig. 11). By placing suitably oriented pointed strips with tips at s_1' and s_2' we can conclude by Fact 4 that there are nonintersecting paths from s_1' to a bottom row point and s_2' to a top row point. This of course is impossible.

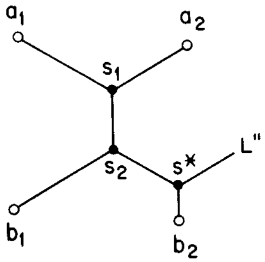


Fig. 12.

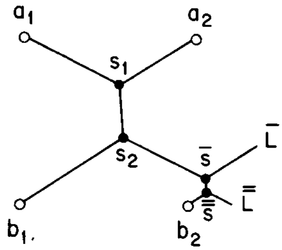


Fig. 13.

(c) L_1 goes directly to a regular point and L_2 goes to a Steiner point (the case with L_1 and L_2 interchanged is similar). Let $\bar{P}$ denote the path from s_2 to b_2. By hypothesis, $\bar{P}$ contains at least one Steiner point.

(1) Suppose $\bar{P}$ contains exactly *one* Steiner point s^* (see Fig. 12). Geometrical considerations force the slope of the line through s_1 and s_2 to be *negative*. Thus, the x-coordinate of s^* is less than 2. Consider the "3rd" line segment L'' leaving s^* (parallel to $[b_1, s_2]$). If L'' terminates at a point $\bar{s}$ with x-coordinate less than 4 then $\bar{s}$ is a Steiner point and the tip of a suitable pointed strip can be placed at $\bar{s}$ so that the conclusion of Fact 4 cannot hold. On the other hand, if the x-coordinate of $\bar{s}$ is at least 4, then the length of $L'' = [s^*, \bar{s}]$ is > 2 which contradicts Fact 3 (which applies to L_m^* as well as L_m).

(2) Suppose $\bar{P}$ contains at least 3 Steiner points. In this case, an argument similar to that used in (i) applies and we reach a contradiction.

(3) $\bar{P}$ contains exactly 2 Steiner points $\bar{s}$ and $\bar{\bar{s}}$. Let $\bar{L} = [\bar{s}, \bar{p}]$ and $\bar{\bar{L}} = [\bar{\bar{s}}, \bar{p}]$ be the corresponding 3rd line segments (see Fig. 13). It is immediate that $\bar{P}$ must turn to the right at $\bar{s}$ and $\bar{\bar{s}}$ as shown in Fig. 13. As before, the slope of the line through s_1 and s_2 must be negative. Since $[s_1, s_2]$ is parallel to $[\bar{s}, \bar{\bar{s}}]$ then in order for $\bar{L}$ to be able to terminate, its extension must pass through or below a_3. Therefore, by

comparing what happens in the parallel situation as we go along the path from b_1 to s_2 to s_1 to a_2, we must have length $[s_1, s_2] \geq$ length $[\bar{s}, \bar{\bar{s}}]$. There are now 4 cases to consider:

(A) $\bar{p}$ and $\bar{\bar{p}}$ are both regular points. Thus, we must have $\bar{p} = a_3$ and $\bar{\bar{p}} = b_3$ (see Fig. 14(a)) and $L_m^* = L_3$. In this case, a simple calculation shows that the length of T^* is

$$2\sqrt{15 + 6\sqrt{3}} = 10.078\ldots .$$

However, the length of the Steiner tree for L_3 shown in Fig.14(b) is only

$$\sqrt{44 + 24\sqrt{3}} = 9.251\ldots,$$

so that T^* is not a minimum Steiner tree.

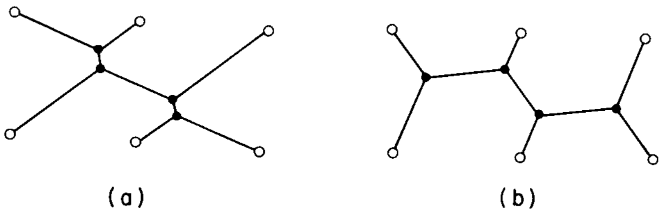


Fig. 14.

(B) $\bar{p}$ and $\bar{\bar{p}}$ are both Steiner points. In this case we reach the same difficulty we had in (ii) (b), where the extensions of the 3rd lines from the 2 Steiner points ran into each other.

(C) $\bar{p}$ is a regular point and $\bar{\bar{p}}$ is a Steiner point. Thus, $\bar{p} = a_3$ (see Fig. 15). Note that since the slope of $[b_1, s_2]$ is less than 1, then the slope of $[a_1, s_1]$ is less than $-\tan 15°$.

(α) Suppose the extension of $\bar{\bar{L}}$ passes *below* b_4. Thus, $[\bar{\bar{p}}, b_3]$ must be an edge of T^* and so, the x-coordinate of $\bar{\bar{p}}$ is less than 4. But this implies that the length of $[\bar{\bar{p}}, z]$, the 3rd line leaving $\bar{\bar{p}}$ parallel to $[b_1, s_2]$, must exceed 2 (since the x-coordinate of z must exceed 6) and this is impossible.

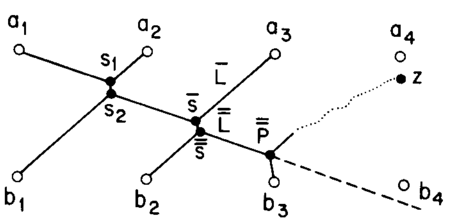


Fig. 15.

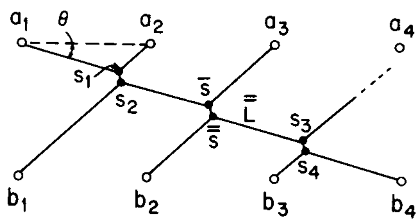


Fig. 16.

(β) Suppose the extension of $\bar{\bar{L}}$ passes through or *above* b_4. Of course, it must pass below b_5, since $\tan 15° > 1/4$. Based on Facts 4 and 9, $\bar{\bar{L}}$ must go to a Steiner point s_3. Also, there is another Steiner point s_4 so that $[s_3, s_4]$, $[b_3, s_4]$ and $[b_4, s_4]$ are edges in T^* (see Fig. 16).

We shall show length $[b_1, s_2] > 2$ so that this configuration can not be part of a minimum Steiner tree for L_m^*.

Claim. Length $[b_1, s_2] > 2$.

Proof. Let θ denote the angle $\angle a_2 a_1 s_1$. It is easy to see that

$$\tan\theta < \tfrac{1}{3}. \tag{1}$$

Let v be determined so that $[s_1, v]$ is parallel to $[a_1, a_2]$ and v is on the line $[s_2, b_1]$ (see Fig. 17). Let u denote the point at the intersection of $[s_1, v]$ and the extension of $[s_2, \bar{s}]$.

It is easily verified that

$$\text{length}\,[s_1, s_2] = \text{length}\,[\bar{s}, \bar{\bar{s}}] \geqslant \text{length}\,[s_3, s_4].$$

Let z denote the point at the intersection of $[a_4, b_4]$ and the extension of $[a_1, s_1]$ (see Fig. 18). It is clear that

$$\tan\theta = \tfrac{1}{6}(2 - \text{length}\,[z, b_4]).$$

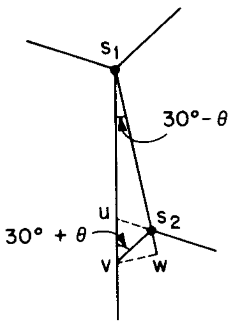


Fig. 17.

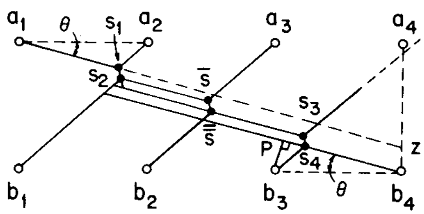


Fig. 18.

On the other hand

$$\text{length}\,[z, b_4] \leqslant 3\,\text{length}\,[s_1, u].$$

Thus

$$\tan\theta \geqslant \tfrac{1}{6}(2 - 3\,\text{length}\,[s_1, u]). \tag{2}$$

The angle $\angle s_2 b_1 b_2$ is equal to $60° - \theta$. From Fig. 19 we see that

$$x = \text{length}\,[s_1, v] = 2 - 2\tan(60° - \theta). \tag{3}$$

Also, we have

$$\text{angle}\,\angle v s_1 s_2 = 30° - \theta,$$

$$\text{angle}\,\angle s_1 v s_2 = 30° + \theta.$$

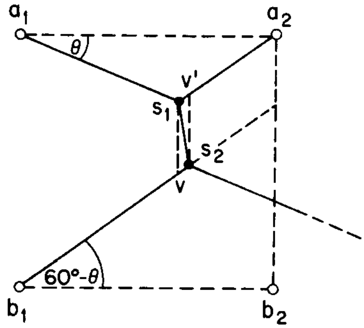


Fig. 19.

Let w denote the point on the extension of $[s_1, s_2]$ so that $\angle s_1 w v$ is 90° (see Fig. 17). Then,

$$\text{length}\,[v, s_2] = \frac{2}{\sqrt{3}}\,\text{length}\,[v, w]$$

$$= \frac{2}{\sqrt{3}}\,x \sin(30° - \theta),$$

$$\text{length}\,[s_2, w] = \frac{1}{\sqrt{3}}\,x \sin(30° - \theta),$$

$$\text{length}\,[s_1, w] = x \cdot \cos(30° - \theta),$$

$$\text{length}\,[s_1, s_2] = x\,(\cos(30° - \theta) - \frac{1}{\sqrt{3}} \sin(30° - \theta))$$

$$= x \cdot \frac{2}{\sqrt{3}} \sin(30° + \theta).$$

Now,

$$\text{length}\,[s_1, u] + \text{length}\,[u, v] = x.$$

On the other hand,

$$\frac{\text{length}\,[s_1, u]}{\text{length}\,[u, v]} = \frac{\text{length}\,[s_1, s_2]}{\text{length}\,[v, s_2]} = \frac{\sin(30° + \theta)}{\sin(30° - \theta)},$$

$$\text{length}\,[s_1, u]\left(1 + \frac{\sin(30° - \theta)}{\sin(30° + \theta)}\right) = x,$$

$$\text{length}\,[s_1, u] = x \sin(30° + \theta)/\cos\theta.$$

From (3), we get

$$\text{length}\,[s_1, u] = 2(1 - \tan(60° - \theta)) \cdot \sin(30° + \theta)/\cos\theta$$

$$= (1 - \sqrt{3}) + (\sqrt{3} + 1)\tan\theta.$$

Therefore, by (2),

$$6\tan\theta \geqslant 2-3((1-\sqrt{3}+(\sqrt{3}+1)\tan\theta),$$
$$\tan\theta > 0.2955\ldots, \tag{4}$$
$$\theta > 16.46.$$

In Fig. 18,

$$\begin{aligned}\text{length}\,[b_3, s_4] &= \frac{2}{\sqrt{3}}\text{length}\,[p, b_3]\\ &= \frac{2}{\sqrt{3}}\cdot 2\cdot\sin\theta,\end{aligned}$$

$$\begin{aligned}\text{length}\,[b_1, s_2] &\geqslant 3\,\text{length}\,[b_3, s_4] + \text{length}\,[\bar{s}, \bar{\bar{s}}]\\ &\geqslant 4\sqrt{3}\sin\theta + \frac{2}{\sqrt{3}}\sin(30^\circ+\theta)\cdot(2-2\tan(60^\circ-\theta))\\ &\geqslant 4\sqrt{3}\sin(16.46^\circ) + \frac{2}{\sqrt{3}}\sin(46.46^\circ)\cdot(2-2\tan(60^\circ-16.46^\circ))\\ &\geqslant 2.046\ldots.\end{aligned}$$

This proves the claim and consequently, case (β) cannot occur. This concludes case (C).

(D) $\bar{p}$ is a Steiner point and $\bar{\bar{p}}$ is a regular point. Thus, $\bar{\bar{p}} = b_3$ (see Fig. 20).

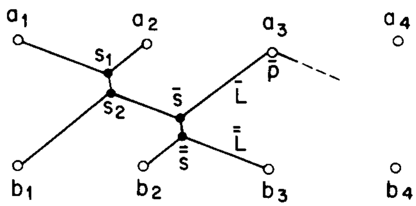


Fig. 20.

Let $[\bar{p}, p^*]$ be the edge leaving $\bar{p}$ parallel to $[s_1, s_2]$. If p^* is a Steiner point, then the slope of $[b_2, \bar{s}]$ would have to be less than $\frac{1}{2}$ (i.e., the extension of $[b_2, \bar{s}]$ cannot pass above a_4). However, it is clear that the slope of $[a_1, s_1]$ is greater than $-\frac{1}{2}$ because of the path from a_1 to b_3. Since the angles $\angle a_1s_1s_2$ and $\angle s_1s_2b_1$ are 120° then we have a contradiction. Thus, p^* is a regular point and, in fact, $p^* = a_3$.

We now claim that if we are able to show that

$$\text{length}\,[a_3, \bar{p}] < 2\sqrt{15+6\sqrt{3}} - \sqrt{44+24\sqrt{3}} = 0.8278\ldots, \tag{5}$$

then we are finished. For, the length of the tree spanned by $\{a_1, a_2, a_3, b_1, b_2, b_3\}$

(and Steiner points $\{s_1, s_2, \bar{s}, \bar{\bar{s}}, \bar{p}\}$) in Fig. 20 is at least as long as that of the tree in Fig. 14(a) (which is the minimum length for a Steiner tree for L_3 with that topology). Hence, in Fig. 20, if the edges $[a_1, s_1]$, $[b_1, s_2]$, $[s_1, s_2]$, $[s_1, a_2]$, $[s_2, \bar{s}]$, $[\bar{s}, \bar{\bar{s}}]$, $[b_2, \bar{\bar{s}}]$, $[\bar{\bar{s}}, b_3]$, $[\bar{s}, \bar{p}]$ are replaced by the tree shown in Fig. 14(b) (leaving $[a_3, \bar{p}]$ in), then this new tree for L_m^* has a length which is less than that of T^* by at least

$$2\sqrt{15+6\sqrt{3}} - \sqrt{44+24\sqrt{3}} - \text{length}\,[a_3, \bar{p}].$$

Hence, if (5) holds we reach a contradiction which would finally complete the proof of Fact 14.

We have seen (see Fig. 21) that $\tan\theta < \frac{1}{2}$, i.e., $\tan\alpha = \tan(60° - \theta) > 5\sqrt{3} - 8$. Thus,

$$\text{length}\,[t, b_3] \geq 2(5\sqrt{3} - 8).$$

Since

$$\text{length}\,[a_3, \bar{p}] \leq \text{length}\,[a_3, \bar{t}] < 2 - \text{length}\,[t, b_3] \leq 2(9 - 5\sqrt{3}) = 0.6795\ldots,$$

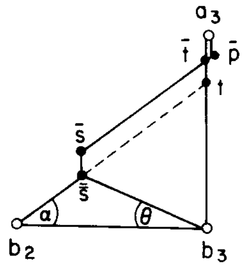


Fig. 21

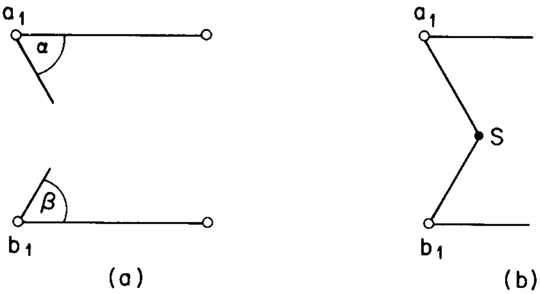


Fig. 22

then (5) easily holds. This completes the proof outline for Fact 14. □

An immediate corollary of this result is the following.

Fact 15. If T^* is a full minimum Steiner tree for L_m^*, $m \geq 3$, then one of the angles α, β is $\geq 60°$ (see Fig. 22(a)).

Proof. By Fact 14, a_1 and b_1 have a common Steiner point s. Thus, $\alpha + \beta = 120°$ and Fact 15 follows. □

Let us call a full tree component $S^*(X_i)$ *trivial* if $|X_i| = 2$. By Fact 12, such an X_i must be $\{a_k, a_{k+1}\}$ or $\{b_k, b_{k+1}\}$ for some k.

Fact 16. Suppose a minimum Steiner tree S^* for L_n, $n \geq 2$, has a full tree component $S^*(X_i)$, where

$$X_i = \{a_r, a_{r+1}, \ldots, a_s\} \cup \{b_r, b_{r+1}, \ldots, b_s\} \quad \text{for some } r < s.$$

(We call this a *rectangular* component with $s-r+1$ columns.) Then either $s = r+1$ or $X_i = L_n$.

Idea of proof. If $n = 2$, then the result is immediate. Assume $n > 2$ and suppose w.l.o.g. that $s < n$. Since S^* is a tree on L_n, then either a_s and a_{s+1} or b_s and b_{s+1} belong to a common full tree component. Assume a_s and a_{s+1} both belong to $S^*(X_j)$ for some $j \neq i$ (see Fig. 23).

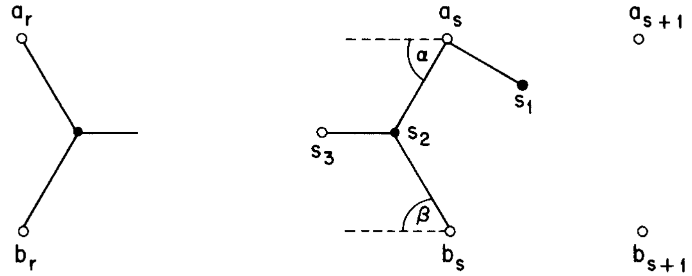


Fig. 23.

Let $[a_s, s_1]$ be the first edge in the path from a_s to a_{s+1} ($s_1 \neq s_2$ since $X_i \cap X_j = \{a_s\}$). By Fact 15, one of the angles α, β is $\geqslant 60°$. If $\beta \geqslant 60°$, then *reflect* $S^*(X_i)$ about the x-axis so that the "new" full tree component for $\{a_r, \ldots, a_s\} \cup \{b_r, \ldots, b_s\}$ now has $\alpha \geqslant 60°$. Thus we may assume $\alpha \geqslant 60°$. By Fact 5 the angle between $[s_2, a_s]$ and $[s_1, a_s]$ must be exactly equal to 120°. Therefore, $[s_2, s_3]$ is parallel to the x-axis. However, it now follows by an argument similar to that of Fact 14(ii)(b) that $[s_3, a_{s-1}]$ and $[s_3, b_{s-1}]$ are edges of S^*, i.e., $r = s-1$ which is the desired result. □

We denote by $F(2)$ a rectangular component with 2 columns (see Fig. 7).

Fact 17. The full tree components $S^*(X_i)$ of L_n are either rectangular or trivial. Furthermore, if two full tree components intersect then one of them is rectangular and the other one is trivial.

Idea of proof. Suppose both $[a_k, a_{k+1}]$ and $[b_k, b_{k+1}]$ are edges of S^*. Then the only way for these two components to be connected is with a common Steiner point s for either a_k and b_k or a_{k+1} and b_{k+1} (by Fact 15). Suppose a_k and b_k have a common Steiner point (the other case is similar). Then, replacing $[a_k, a_{k+1}]$ by $[a_{k+1}, b_{k+1}]$, we obtain a minimum Steiner tree for L_n with a pair of edges meeting at 90°, which contradicts Fact 5.

Suppose S^* has a nontrivial, nonrectangular full tree component $S^*(X_i)$. Thus, by Fact 11, for some k, we have (w.l.o.g.) $a_{k-1} \notin X_i$ and $a_k, b_{k-1}, b_k \in X_i$ (see Fig. 24). Now, $[a_{k-1}, a_k]$ cannot be an edge of S^* since if it were, it could be replaced by the equal length edge $[a_{k-1}, b_{k-1}]$, forming a minimum Steiner tree for L_n with an angle of less than 120°, contradicting Fact 5. If a_{k-1} and a_k were in a common full tree component then by applying either Fact 3 or Fact 4 to s, the first Steiner point

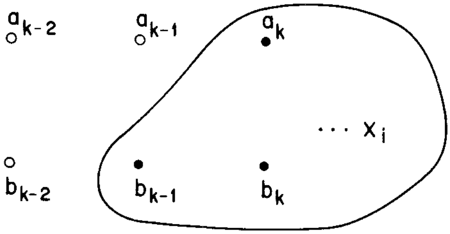


Fig. 24.

in the path from a_k to a_{k-1}, we reach a contradiction. Therefore, a_{k-1} and a_k do not belong to a common full tree component. Also, if no b_j belongs to a common full tree component with a_{k-1}, then we would have the edge $[a_{k-2}, a_{k-1}]$ in S^*, which is similarly impossible. If $a_{k-1}, b_k \in S^*(X_j)$ for some j, then we would also have $b_{k-1} \in S^*(X_j)$, which contradicts Fact 6. By symmetry, we also reach a contradiction if $a_{k-1}, b_{k-1} \notin S^*(X_j)$ for some j. Hence, we may assume that $a_{k-1}, b_{k-1} \in S^*(X_j)$ for some j. Therefore, by Fact 14, they share a common Steiner point s, i.e., so that $[s, a_{k-1}]$ and $[s, b_{k-1}]$ are edges of $S^*(X_j)$ (see Fig. 25). But by Fact 15, one of the angles α, β is $\geqslant 60°$. As before, we may assume it is β (by reflecting the portion of T^* on $\{a_1, \ldots, a_{k-1}\} \cup \{b_1, \ldots, b_{k-1}\}$ if necessary). This implies that the angle between $[s, b_{k-1}]$ and $[s', b_{k-1}]$ is $< 120°$ which contradicts Fact 5. Hence we cannot have a nonrectangular, nontrivial full tree component of S^*. Of course, two rectangular components cannot intersect (since if they did, their intersection would have at least two points, which is impossible.) □

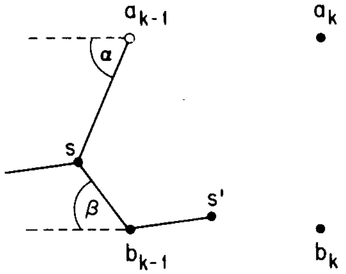


Fig. 25.

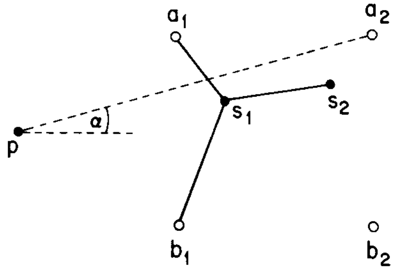


Fig. 26.

We have now reduced the study of minimum Steiner trees on ladders L_n to the study of *full* minimum Steiner trees on (possibly smaller) *subladders* L_m of L_n.

Let F^* denote a *full* minimum Steiner tree for a ladder L_m, $m \geqslant 3$ (see Fig. 26).

Fact 18. The slope σ of $[s_1, s_2]$ satisfies

$$|\sigma| < 2 - \sqrt{3}.$$

Idea of proof. Assume (w.l.o.g.) that $\sigma \geqslant 0$. Let $p = (-\sqrt{3}, 0)$ be the point shown in Fig. 26 forming an equilateral triangle with a_1 and b_1. If s_1 lies above the line through p and a_2 then there is no way (by a suitable application of Fact 4) to complete F^*. □

We will assume hereafter (w.l.o.g.) that $\sigma \geqslant 0$. It follows from Fact 18 that $\alpha < 15°$.

Let T^* denote the subtree of F^* induced by the Steiner points of F^*.

Fact 19. T^* contains no point of degree exceeding 2.

Idea of proof. Let m_k denote the number of points of T^* which have degree k, $k \geqslant 1$, and assume $m_3 + m_4 + \cdots > 0$. If $|T^*|$ denotes the number of points of T^* (i.e., the number of Steiner points of F^*), then

$$\sum_{v \in T^*} \deg v = 2|T^*| - 2.$$

Thus,

$$\sum_{v \in T^*} (\deg v - 2) = -2$$

$$= m_1 + \sum_{k \geqslant 2} (k-2) m_k.$$

Therefore,

$$m_1 = 2 + \sum_{k \geqslant 2} (k-2) m_k$$

$$\geqslant 2 + m_3 + m_4 + \cdots > 2$$

by hypothesis. Careful consideration of the facts established up to this point now shows that there must exist a pair of adjacent points in some row which are not endpoints and which are connected to a common Steiner point s_1.

(i) Suppose the points are b_k, b_{k+1} for some k, $1 < k < m-1$ (see Fig. 27).

Let us call the directions of line segments $[b_k, s_1]$, $[b_{k+1}, s_1]$ and $[s_1, s_2]$, *directions* I, II *and* III, respectively. Since the slope of $[b_k, s_1]$ is $< 2 - \sqrt{3}$ by Fact 18, then the slope of $[b_{k+1}, s_1]$ is < -1. Since we have assumed the slope of $[b_k, s_1]$ is $\geqslant 0$, then

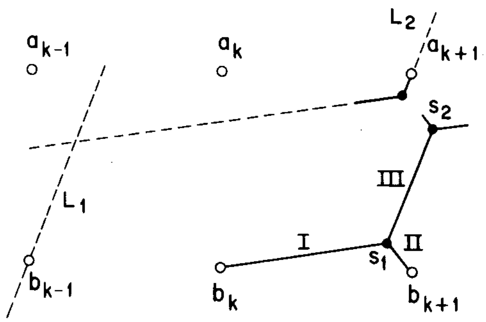


Fig. 27.

the slope of $[s_1, s_2]$ is ≥ 0. Thus, if we start at b_{k+1} and proceed along the path P determined by alternately choosing the directions II and III, until we terminate at a point a_t in the top row of L_m then we must have $t \geq k+1$. In fact, it is not hard to see that $t = k+1$. Let L_1 and L_2 denote the lines through b_{k-1} and a_{k+1}, respectively, having direction III. It is now not hard to see that some edge $[s, s']$ of F^* must have s to the left of (or on) L_1 and s' to the right of (or on) L_2. However, this forces length $[s, s'] > 2$, which contradicts Fact 3.

(ii) The case in which the two points are in the top row is handled by rotating the preceding arguments by 180°. □

Fact 19 implies that T^* is a *path*. Let s_0 denote the Steiner point of F^* common to a_1 and b_1 and let s_{2m-3} denote the Steiner point of F^* common to a_m and b_m. Every other Steiner point s is connected to a *unique* regular point $p(s) \in L_m$. Let us label the consecutive Steiner points proceeding along T^* from s_0 to s_{2m-3} by $s_0, s_1, s_2, \ldots, s_{2m-3}$ (see Fig. 28).

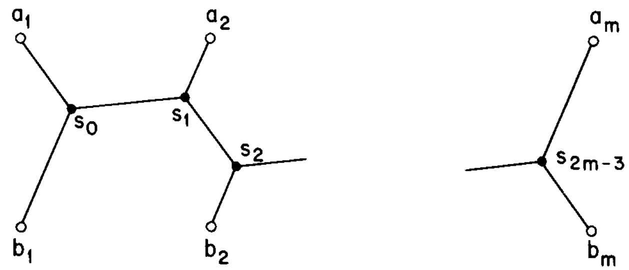


Fig. 28.

Fact 20. For any k, $1 \leq k < 2m-3$, the 3 points $p(s_k)$, $p(s_{k+1})$, $p(s_{k+2})$ cannot belong to 3 different columns of L_m.

Idea of proof.

(i) Suppose $p(s_k)$, $p(s_{k+1})$, $p(s_{k+2})$ all belong to the bottom row of L_m. Then for some i, we must have

$$p(s_k) = b_i, \qquad p(s_{k+1}) = b_{i+1}, \qquad p(s_{k+2}) = b_{i+2}$$

(see Fig. 29). The directions of the various edges must be as shown (since T^* must turn in the same direction at s_k, s_{k+1} and s_{k+2}). As in the proof of Fact 19, if we start from b_{i+1} and follow the path determined by alternately choosing directions III and II we must terminate at a_{i+1}. Of course, $[s_k, a_{i+1}]$ cannot be an edge of F^*. But now, as in the proof of Fact 19 (since $i > 1$), some edge must span the region bounded by lines through b_{i-1} and a_{i+1} having direction III. This forces its length to be > 2, which is impossible.

(ii) The other various possibilities are similar and will be left to the reader. □

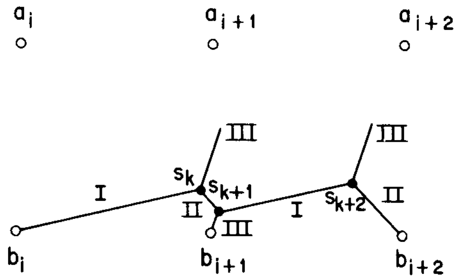


Fig. 29.

It follows from Fact 20 that the points in successive columns are connected to T^* in order as we go from s_0 to s_{2m-3}. If, for some j,

$$p(s_j) = a_k, \qquad p(s_{j+1}) = b_k,$$

we say to the $(k-1)^{st}$ column is a *top-first* column. Otherwise we say it is a *bottom-first* column (see Fig. 30). Since, we have made the normalizing assumption that the slope of $[s_0, s_1]$ is $\geqslant 0$ then the 2^{nd} column (i.e., containing a_1 and b_1) is a top-first column.

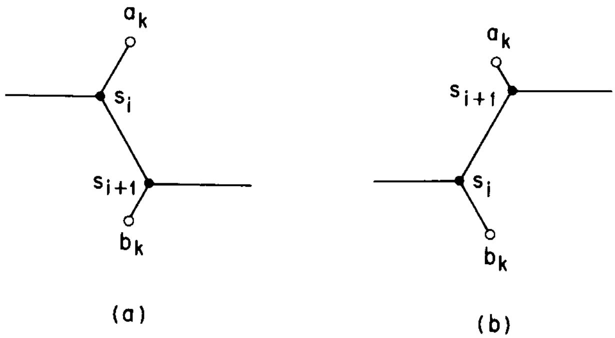


Fig. 30.

Fact 21. Suppose F^* has b bottom-first columns and $m-b-2$ top-first columns. Then

$$\text{length } F^* = ((m(2+\sqrt{3})-2)^2 + (2b+2-m)^2)^{1/2}. \tag{6}$$

Idea of proof. Let X be any set in the plane and suppose S is a minimum Steiner tree in which x and x' are regular points having a common Steiner point (see Fig. 31(a)). Let X' be formed from X by removing x and x' and adjoining t, the "equilateral triangle" point determined by x and x' (cf. Fact 8). Form the Steiner tree S' by deleting $[s, x]$, $[s, x']$ and $[s, s']$ from S and adding $[t, s']$ (see Fig. 31(b)). By Fact 8, S' is a minimum Steiner tree for X'.

Starting with L_m, we can successively replace pairs of points by the appropriate equilateral points, eventually forming a set with *two* points (it is rather easy to form

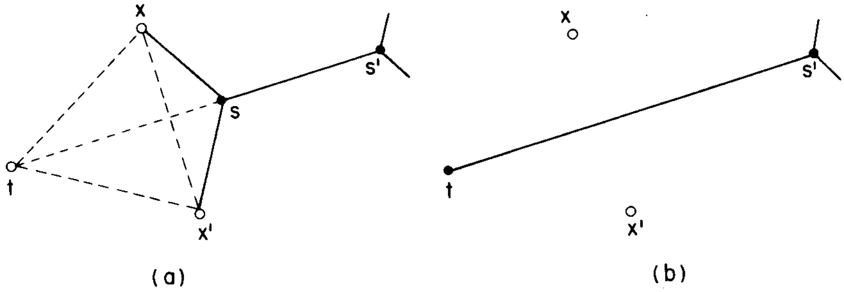


Fig. 31.

minimum Steiner trees for such sets). A useful observation in such a reduction is the following. If the $(k-1)^{st}$ column is a top-first column (see Fig. 32(a)) and t, a_k and b_k are replaced by $T'=(X', Y')$, then

$$x = x - \sqrt{3},$$

$$y' = y - 1.$$

Similarly, if the $(k-1)^{st}$ column is a bottom-first column (Fig. 32(b)), then

$$x' = x - \sqrt{3},$$

$$y = y + 1.$$

(These expressions follow at once from Fact 8.) Hence, replacing a_1 and b_1 by $t_0 = (-\sqrt{3}, 0)$ and a_m and b_m by $t' = (2m-2+\sqrt{3}, 0)$, we see that

$$t_{2m-4} = (x_{2m-4}, y_{2m-4})$$

with

$$x_{2m-4} = x_0 - (m-2)\sqrt{3} = -(m-1)\sqrt{3},$$

$$y_{2m-4} = y_0 - b + (m-2-b) = m-2-2b.$$

Thus, the length of F^* is just the length of $[t_{m-4}, t']$ which is

$$((m(2+\sqrt{3})-2)^2 + (2b+2-m)^2)^{1/2}$$

and Fact 21 is proved. □

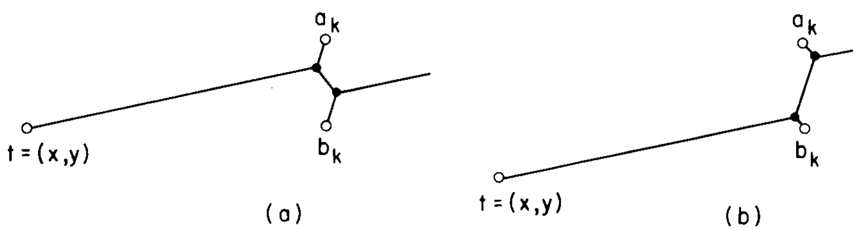


Fig. 32.

Note that the slope σ of a line with direction I (e.g., $[s_0, s_1]$) is given by

$$\sigma = \frac{m-2-2b}{m(2+\sqrt{3})-2}. \tag{7}$$

It is easily seen that once the slope σ is determined then the *order* of the top-first and bottom-first columns is completely determined. Of course, to minimize length F^*, one should choose $2b+2-m$ as close to zero as possible. The normalization $\sigma \geq 0$ implies $b \leq [m/2]-1$. If m is *odd* then by choosing $b=[m/2]-1=(m-3)/2$ we achieve the minimum length of F^*, which is

$$l_m = ((m(2+\sqrt{3})-2)^2+1)^{1/2}. \tag{8}$$

However, if m is *even*, then if $b=(m-2)/2$ is chosen, we obtain $\sigma = 0$ and we know in this case (by an argument in Fact 16) that we must have $m=2$. Thus, for m even and >2, the best choice for b is $(m-4)/2$ and the length of F^* in this case is

$$((m(2+\sqrt{3})-2)^2+4)^{1/2}.$$

However, for m even, a direct comparison shows that this exceeds $m(2+\sqrt{3})-2$, the length of the Steiner tree on L_m formed by alternating $F(2)$'s and edges (see Fig. 33). Furthermore, an easy calculation shows that

$$l_{m+2} < l_m + 2 + \text{length } F(2)$$

$$= l_m + 4 + 2\sqrt{3}.$$

This implies that if X_i is a full tree component isomorphic to L_m, m odd, and $S^*(X_i)$

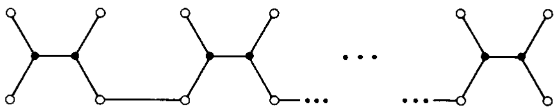


Fig. 33.

is connected to an adjacent $F(2)$ in S^* by an edge, then we should replace that portion of S^* by a full tree on L_{m+2} (which will be shorter).

These observations allow us to conclude the main result of the paper.

Theorem. *The minimum Steiner trees $S^*(L_n)$ on L_n are given as follows:*

(i) *For n odd, $S^*(L_n)$ is a full Steiner tree, unique up to reflection, having $(n-1)/2$ top-first columns alternating with $(n-3)/2$ bottom-first columns (see Fig. 34). The slope of $[s_0, s_1]$ is $(n(2+\sqrt{3})-2)^{-1}$. The length of $S^*(L_n)$ is $((n(2+\sqrt{3})-2)^2+1)^{1/2}$.*

(ii) *For n even, $S^*(L_n)$ has $n/2$ full tree components connected by edges (see Fig. 35(a)). The length of $S^*(L_n)$ is $n(2+\sqrt{3})-2$.*

Note that for n even there are in fact 2^{n-1} different minimum Steiner trees on L_n,

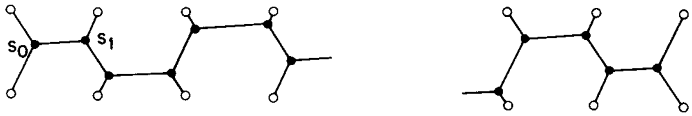


Fig. 34.

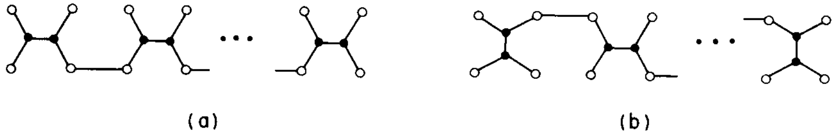


Fig. 35.

corresponding to the different choices for the orientation of the $F(2)$'s and the rows of the connecting edges.

Concluding remarks

The preceding analysis leads naturally to the consideration of the structure of the class of *all* full (not necessarily minimum) Steiner trees on L_n. As mentioned in the introduction, this turns out to be surprisingly complicated. We give a brief summary of some of the relevant results. The details will be given in a future paper.

To begin with, we restrict ourselves to full Steiner trees S^* for L_n in which all Steiner points are incident to exactly 3 equiangular edges (i.e., each meeting the other two at 120°).

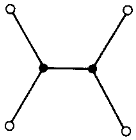
(i) Suppose the Steiner points of S^* induce a path with each pair a_1, b_1 and a_n, b_n having common Steiner points and with each a_k, b_k, $1 < k < n$, connected to a unique Steiner point. Such a Steiner tree we call a *Type I* Steiner tree for L_n. As before, it can be shown that the points of L_n are connected to Steiner points in successive columns, so that the columns can again be classified as top-first or bottom-first. Thus, the tree is specified by the sequence $C = (c_2, c_3, \ldots, c_{n-1})$ where

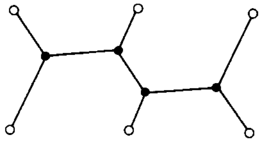
$$c_k = \begin{cases} 1 & \text{if the } k^{\text{th}} \text{ column is top-first,} \\ -1 & \text{if the } k^{\text{th}} \text{ column is bottom-first.} \end{cases}$$

It can be shown that if we define $\delta_k = \sum_{i=2}^{k} c_i$, then C corresponds to a realizable tree (where we have assumed $c_2 = 1$) iff

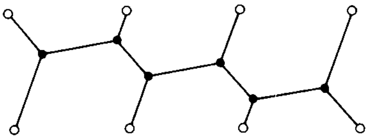
$$\frac{\delta_k}{k} > \frac{\delta_{n-1}}{n+2\sqrt{3}-3} > \frac{\delta_k - 1}{k+2\sqrt{3}-3} \quad \text{for } c_k = 1,$$

$$\frac{\delta_k}{k} < \frac{\delta_{n-1}}{n+2\sqrt{3}-3} < \frac{\delta_k + 1}{k+2\sqrt{3}-3} \quad \text{for } c_k = -1,$$

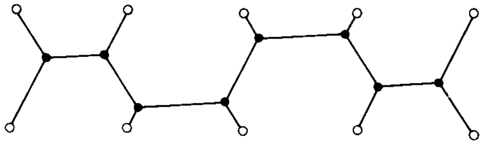




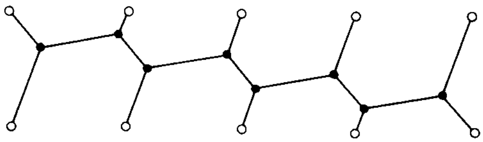
$n = 2, b = 0, L = 5.464\ldots$ $n = 3, b = 0, L = 9.250\ldots$



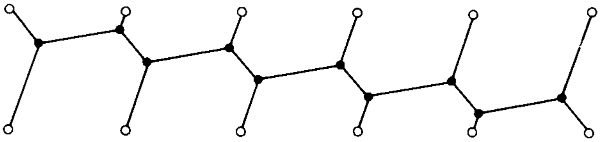
$n = 4, b = 0, L = 13.082\ldots$



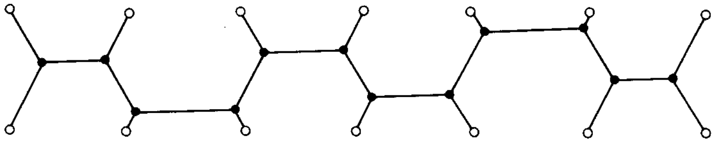
$n = 5, b = 1, L = 16.690\ldots$



$n = 5, b = 0, L = 16.928\ldots$

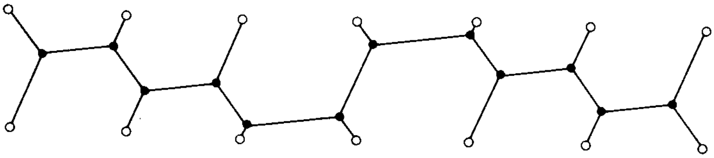


$n = 6, b = 0, L = 20.781\ldots$

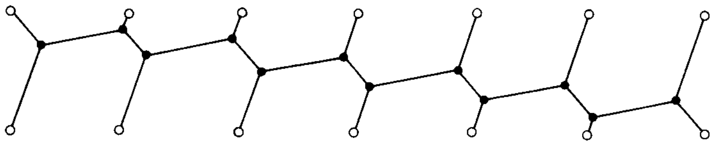


$n = 7, b = 2, L = 24.145\ldots$

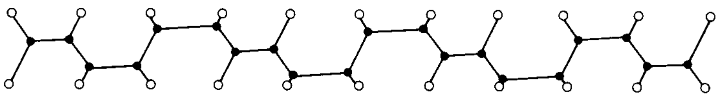
Fig. 36.



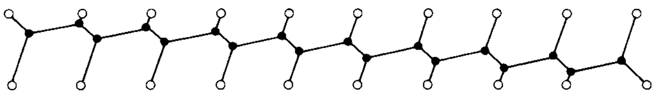
$n = 7, b = 2, L = 24.310\ldots$



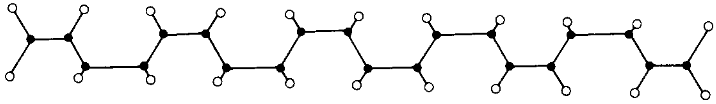
$n = 7, b = 2, L = 24.637\ldots$



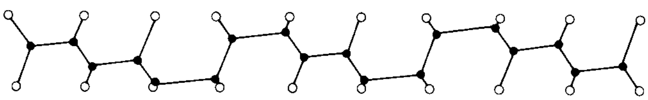
$n = 8, b = 2, L = 27.928\ldots$



$n = 8, b = 0, L = 28.495\ldots$

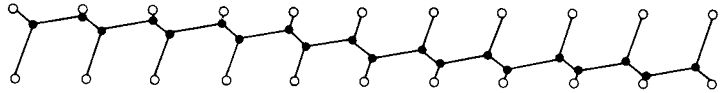


$n = 9, b = 3, L = 31.604\ldots$

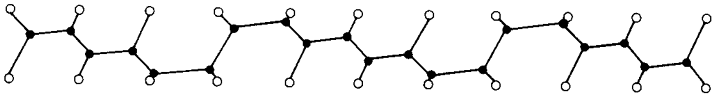


$n = 9, b = 1, L = 31.982\ldots$

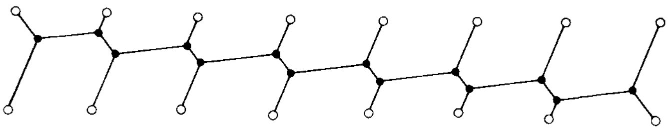
Fig. 36 (contd.)



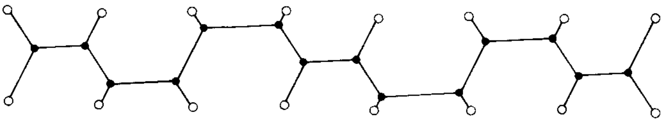
$n = 9, b = 0, L = 32.355\ldots$



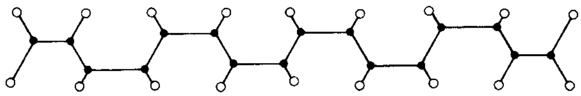
$n = 10, b = 2, L = 35.827\ldots$



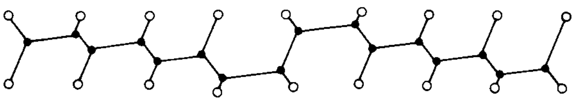
$n = 10, b = 0, L = 36.215\ldots$



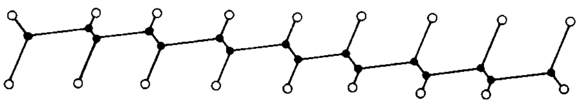
$n = 11, b = 4, L = 39.065\ldots$



$n = 11, b = 3, L = 39.168\ldots$



$n = 11, b = 2, L = 39.371\ldots$



$n = 11, b = 0, L = 40.076\ldots$

Fig. 36 (contd.).

for $2 \leqslant k \leqslant n-1$. A very intricate analysis of this problem yields the following result.

Theorem. *Let $F(n)$ denote the number of Type* I *full Steiner trees for L_n. Then*

$$F(n) = \begin{cases} 1 & \text{if } n = 2, 3, 4, \\ d^*(n-2) + d^*(3n-2) + 1 & \text{if } n > 4 \text{ is even}, \\ d^*(n-2) + d^*(3n-2) + d^*(n-1) & \text{if } n > 4 \text{ is odd}, \end{cases}$$

where $d^*(x)$ denotes the number of odd divisors of x which are greater than 1.

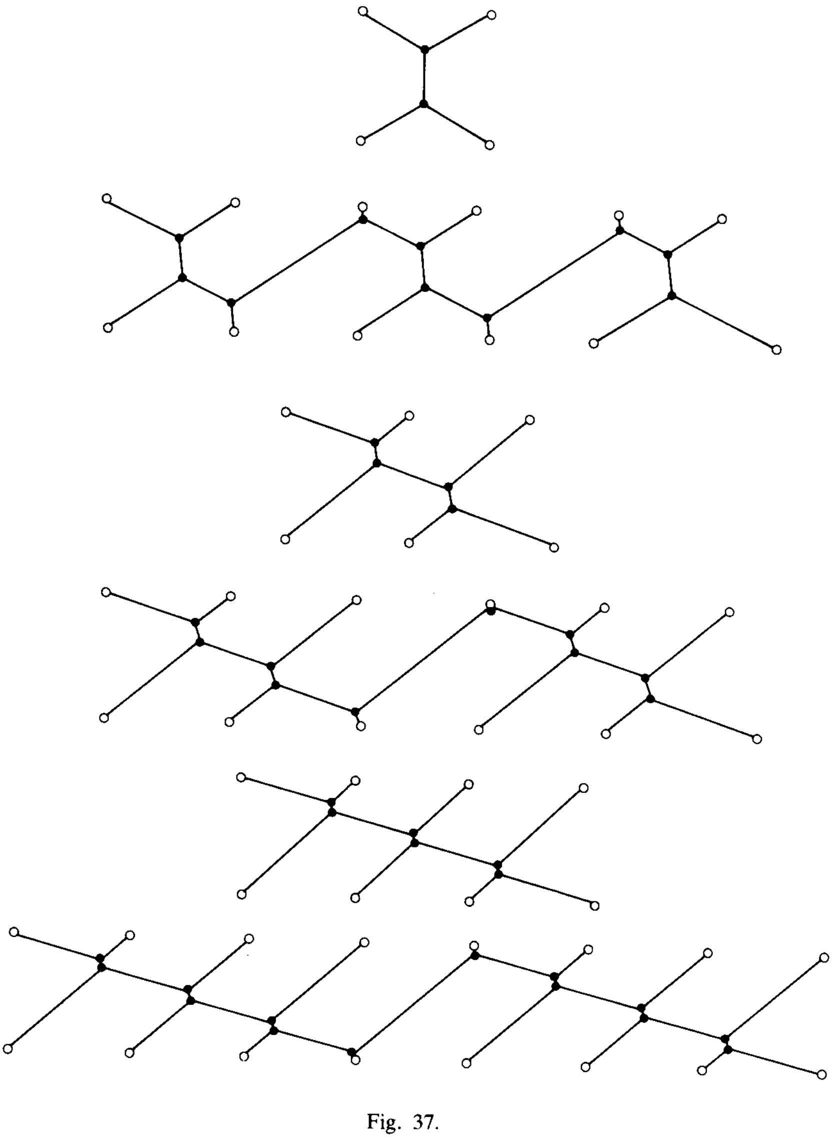


Fig. 37.

In Table 1 we tabulate a few small values of $F(n)$. We show some of the corresponding trees in Fig. 36.

Table 1

n	$F(n)$	n	$F(n)$
2	1	7	3
3	1	8	2
4	1	9	3
5	2	10	2
6	1	11	4

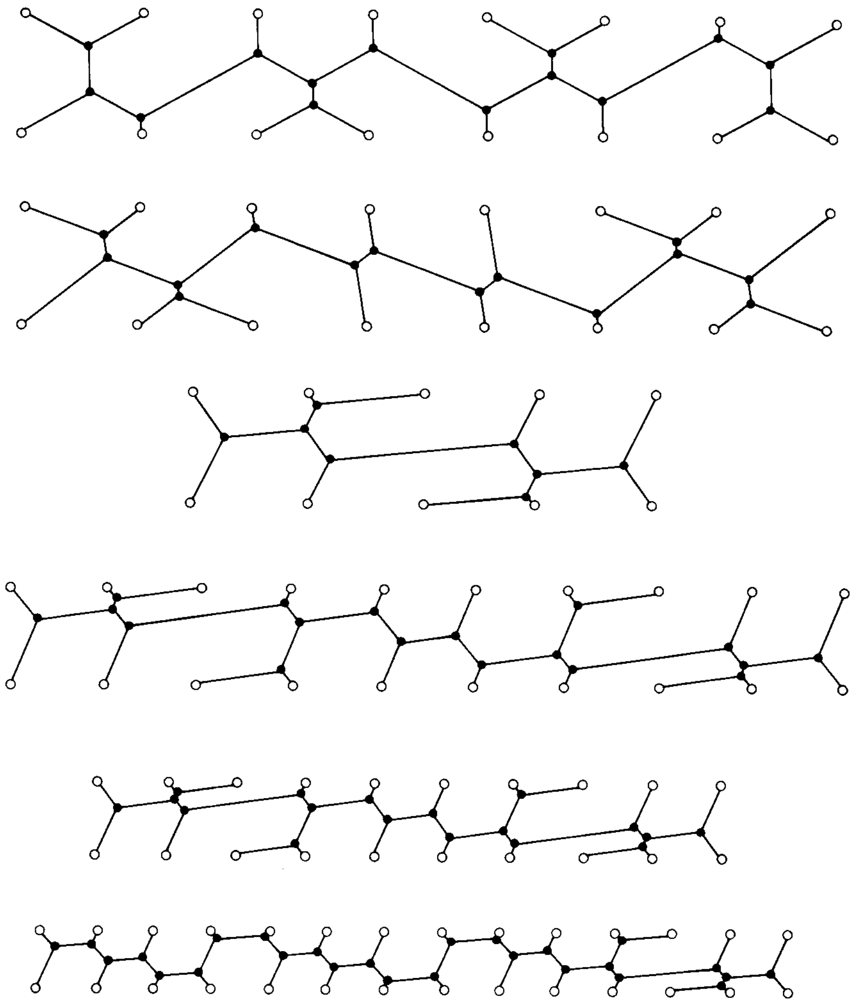


Fig. 38.

(ii) There is another class of full Steiner trees on L_n for which the Steiner points still induce a path but the pairs a_1, b_1 and a_n, b_n no longer have common Steiner points. We call these *Type II* trees. Their analysis turns out to similar to that of Type I (although somewhat simpler). We show representatives of several families of Type II trees in Fig. 37.

(iii) It happens that there are full Steiner trees on L_n whose Steiner points induce trees which are *not* paths. These are called *Type III* trees (what else?). At present, their structure is incompletely understood. We show some of these trees in Fig. 38. Notice the last tree which is not symmetric about the center. It seems likely (but has not yet been proved) that for some $c > 1$, there are more than c^n Type III trees on L_n for n sufficiently large.

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