Subgraphs of a Hypercube Containing No Small Even Cycles

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ABSTRACT

We investigate several Ramsey–Turán type problems for subgraphs of a hypercube. We obtain upper and lower bounds for the maximum number of edges in a subgraph of a hypercube containing no four-cycles or more generally, no $2k$-cycles C_{2k}. These extremal results imply, for example, the following Ramsey theorems for hypercubes: A hypercube can always be edge-partitioned into four subgraphs, each of which contains no six-cycle. However, for any integer t, if the n-cube is edge-partitioned into t subgraphs, then one of the subgraphs must contain an eight-cycle, provided only that n is sufficiently large (depending only on t).

1. INTRODUCTION

Let Q_n denote the n-cube with node set $N = N(Q_n)$ consisting of all binary n-tuples and edge set $E = E(Q_n)$ consisting of all pairs of n-tuples that differ at exactly one coordinate. So, Q_n has $|N(Q_n)| = 2^n$ nodes and $|E(Q_n)| = e(Q_n) = n \cdot 2^{n-1}$ edges.

Paul Erdös raised the following question about 15 years ago [6]:

How many edges can a subgraph of Q_n have that contains no 4-cycles?

This problem has been studied by many researchers [1, 2, 5, 9, 11, 13] and we note that it is in fact related to various fault-tolerant properties of hypercubes when used as parallel computation architectures [13, 15]. In this paper, a number of results related to the above question are obtained.

Theorem 1. A subgraph of Q_n containing no C_4 can have at most $(\alpha + o(1))n2^{n-1}$ edges where $\alpha \approx .623$ satisfies $9\alpha^3 + 5\alpha^2 - 5\alpha - 1 = 0$.

Let $f(n)$ denote the maximum number of edges in a subgraph of Q_n containing no C_4. For small n, it is known that $f(1) = 1, f(2) = 3, f(3) = 9,$

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$f(4) = 24$ and $f(5) = 56$ (see [5]). The best construction known so far is due to Guan [11], yielding $f(n) \geq (n + 3)2^{n-2}$ + l.o.t. where the lower order term (l.o.t.) has a lower bound of $-(n - 3\lfloor (n - 1)/3 \rfloor)2^{2\lfloor (n-1)/3 \rfloor}$. The conjecture of Erdös—"$f(n) = (\frac{1}{2} + o(1))e(Q_n)$?"—remains unresolved.

It seems natural to consider the question of determining the maximum numbers $f_{2k}(n)$ of edges in a subgraph of Q_n containing no C_{2k}. Clearly, $f_4 = f$ and only even cycles are of interest since Q_n is bipartite. Erdös asked that if it is true that every subgraph of Q_n containing $\varepsilon e(Q_n)$ edges must contain C_6 for every $\varepsilon > 0$ provided n is sufficiently large. This question is answered in the negative. Theorems 2 and 3 give upper and lower bounds for $f_6(n)$.

Theorem 2. A subgraph of Q_n containing no C_6 can have at most $(\sqrt{2} - 1 + o(1))n2^{n-1}$ edges.

Theorem 3. The edge set $E(Q_n)$ of Q_n can be partitioned into four subgraphs, each of which contains no C_6.

As an immediate consequence of Theorem 3, we have $f_6(n) \geq \frac{1}{4}e(Q_n)$. It turns out that the above question by Erdös can be answered affirmatively for cycles C_{4k}, $k \geq 2$.

Theorem 4. Every subgraph of Q_n containing $cn^{-1/4}e(Q_n)$ edges must contain C_8 and, in general all C_{4t}, for $2 \leq t \leq k$, where the constant c depends only on k.

Theorem 4 can be strengthened as follows:

Theorem 5. Every subgraph of Q_n containing $cn^{(1/2)+(1/2k)}e(Q_n)$ edges must contain C_{4k} for $k \geq 2$.

Theorem 5 leads to the following Ramsey-type result:

Theorem 6. For any integer t and an integer $k \geq 2$ if Q_n is edge-partitioned into t subgraphs, then one of the subgraphs must contain C_{4k} provided that n is sufficiently large (depending only on t and k).

We observe that, for $m < n$, Q_n can be covered by Q_m's so that each edge of Q_n is contained in the same number of Q_m's. Consequently, we have

$$\frac{f_{2k}(n)}{e(Q_n)} \leq \frac{f_{2k}(m)}{e(Q_m)} \quad \text{for } m \leq n\,.$$

Therefore $\sigma_{2k} = \lim_{n\to\infty}(f_{2k}(n))/(e(Q_n))$ exists. While σ_4 and σ_6 are undetermined, we show that $\sigma_{4k} = 0$, for any $k \geq 2$. Many questions in the spirit of Erdös–Stone [7, 8] can be asked for subgraphs of an n-cube. Numerous related questions remain open, some of which we mention here.

(i) Is it true that $f_{2k}(n) \geq f_{2k+2}(n)$? Does the strict inequality hold?
(ii) From Theorem 2, we have $\sqrt{2} - 1 \geq \sigma_6 \geq 1/4$. Is it true that $\sigma_6 = \frac{1}{4}$?
(iii) From Theorem 5, we have $\sigma_8 = 0$. So, the next unsettled case is σ_{10}. Is it true that $\sigma_{10} = 0$? Does Theorem 6 hold if we consider 10-cycles instead?
(iv) Of course, the most interesting question is to determine σ_4. We know that $.623 \geq \sigma_4 \geq 1/2$. Is it true that $\sigma_4 = 1/2$?
(v) A graph h is said to be t-Ramsey if any edge-coloring of Q_n in t colors must contain H, provided that n is sufficiently large. We have shown that C_{4k}, for $k \geq 2$ is t-Ramsey for any t and C_6 is 2-Ramsey but not 4-Ramsey. Is C_6 3-Ramsey? It would be of interest to characterize t-Ramsey graphs for each t.

The paper is organized as follows: Theorem 2 and Theorem 3 are proved in Section 2, which deals with subgraphs of Q_n containing no 6-cycles. Theorems 4, 5, and 6 are given in Section 3, which considers C_{2k}-free subgraphs of Q_n for general $k \geq 4$. In Section 4, we establish upper bounds for the number of edges in a C_4-free subgraph of Q_n by proving a series of facts that lead to Theorem 1.

2. SUBGRAPHS OF Q_n CONTAINING NO 6-CYCLES

Suppose G is a subgraph of Q_n. Let d_v denote the degree of vertex v in G, and let α denote the edge density of G. In other words $\Sigma_v d_v = \alpha n 2^n$. We also denote $\bar{d}_v = n - d_v$. Suppose H is a subgraph of Q_n. We let $G \cap H$ denote the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$.

We consider subgraphs of G in a subcube Q_2 of Q_n. There are $\binom{n}{2}2^{n-2}$ such Q_2's in Q_n. Let $\chi_0, \chi_1, \chi_2, \chi_2', \chi_3$, and χ_4 denote the fraction of the total number of Q_2's with $G \cap Q_2$ isomorphic to the graphs in Figure 1 (a), (b), (c), (d), (e), and (f), respectively. For example, there are $\chi_0\binom{n}{2}2^{n-2}$ Q_2's in Q_n with $G \cap Q_2$ isomorphic to the graph shown in Fig. 1(a) containing no edges.

We have

$$\chi_0 + \chi_1 + \chi_2' + \chi_2 + \chi_3 + \chi_4 = 1\,. \tag{1}$$

Let us assume G is a subgraph of Q_n containing no C_6. We will first prove Theorem 2 by showing G has at most $(\sqrt{2} - 1 + o(1))n2^{n-1}$ edges.

Proof of Theorem 2.

Fact 1. $\chi_4 = o(1)$ and $\chi_3 = o(1)$.

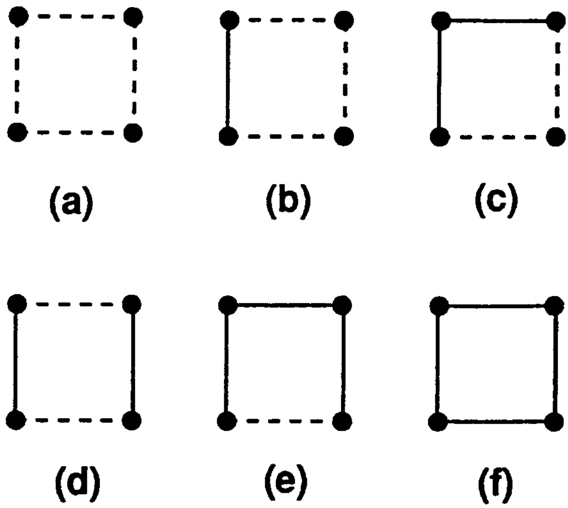

FIGURE 1

Proof. For each $\{u, v\} \in E(Q_n)$ there is at most one 2-subcube Q_2 in Q_n so that $E(Q_2) - \{\{u, v\}\}$ contains three edges of G, since G is C_6-free. Therefore we have

$$n2^{n-1} = e(Q_n) \geq (4\chi_4 + \chi_3)\binom{n}{2}2^{n-2}.$$

This implies $\chi_4 \leq 1/n$ and $\chi_3 \leq 4/n$.

By counting the number of edges in $Q_2 \cap G$ ranging over all 2-subcubes Q_2 in Q_n, we have

$$(4\chi_4 + 3\chi_3 + 2\chi_2 + 2\chi_2' + \chi_1)\binom{n}{2}2^{n-2} = \alpha 2^{n-1}n(n - 1),$$

i.e.,

$$4\chi_4 + 3\chi_3 + 2\chi_2 + 2\chi_2' + \chi_1 = 4\alpha + o(1).$$

Thus, by Fact 1 and (1) we have

$$2 + o(1) \geq 2\chi_2 + 2\chi_2' + \chi_1 = 4\alpha + o(1).$$

This implies $\alpha \leq \frac{1}{2} + o(1)$.

To improve the above bound, we proceed as follows:

Fact 2. $$\chi_2 = \frac{\sum_v \binom{d_v}{2}}{\binom{n}{2}2^{n-2}} + o(1) \geq 4\alpha^2 + o(1).$$

Proof. We consider the number of paths of two edges in G in subgraphs Q_2 of Q_n. We have

$$(4\chi_4 + 2\chi_3 + \chi_2)\binom{n}{2}2^{n-2} = \sum_v \binom{d_v}{2}.$$

Fact 2 follows immediately by using Fact 1 and the Cauchy–Schwarz inequality.

For each node v we define a graph G_v with node set $M(v) = \{u: \{u, v\} \in E(Q_n), \{u, v\} \notin E(G)\}$. For two nodes u and w in $M(v)$, we say u is adjacent to w in G_v if the four-cycle containing u, v, w contains two edges. Since G contains no C_6, G_v contains no triangle for all v. Turán's theorem [15, 16] implies that the number of edges $e(G_v)$ in G_v satisfies the following:

$$e(G_v) \le \frac{|M(v)|^2}{4} \le \frac{(n - d_v)^2}{4}.$$

Therefore, by Fact 2 we have

$$\begin{aligned}\tfrac{1}{4}\sum_v (n - d_v)^2 \ge \sum_v e(G_v) &= \chi_2\binom{n}{2}2^{n-2} + \text{l.o.t.}\\ &\ge \sum_v \binom{d_v}{2} + \text{l.o.t.}\end{aligned}$$

Consequently,

$$\tfrac{1}{4}n^2 2^n - \tfrac{1}{2}n\sum_v d_v \ge \tfrac{1}{4}\sum d_v^2 + \text{l.o.t.}$$

Using Cauchy–Schwarz again, we have

$$\tfrac{1}{4}n^2 2^n - \alpha n^2 2^{n-1} \ge \tfrac{1}{4}2^n \alpha^2 n^2 + \text{l.o.t.}$$

Therefore

$$1 - 2\alpha - \alpha^2 + o(1) \ge 0$$

and

$$\alpha \le \sqrt{2} - 1 + o(1).$$

This completes the proof of Theorem 2.

To establish a lower bound of $(\frac{1}{4} + o(1))e(Q_n)$ for $f_6(n)$, we construct four graphs A_n, B_n, C_n, and D_n, satisfying the following conditions:

(i) A_n, B_n, C_n, and D_n have the same node set as $N(Q_n)$.
(ii) $A_1 = Q_1$, $B_1 = C_1 = D_1 =$ the trivial graph with no edge.
(iii) A_2, B_2, C_2, and D_2 each consists of one distinct edge of Q_2.
(iv) A_{n+2}, B_{n+2}, C_{n+2}, and D_{n+2} are constructed recursively as shown in Figure 2.

For example, A_{n+2} can be viewed as the union of two copies of A_n, denoted by $A_n(0,0)$, $A_n(1,1)$ (on nodes with prefix 00 and 11, respectively) and two copies of B_n, denoted by $B_n(0,1)$, $B_n(1,0)$. Between $A_n(0,0)$ and $B_n(0,1)$, there is an odd matching (where an edge is said to be odd if the total number of coordinates with value 1 in both of the end points is odd, otherwise it is said to be even). Also, between $B_n(1,0)$ and $A_n(1,1)$, there is an odd matching. It is easy to verify that A_n, B_n, C_n, and D_n satisfy the following properties:

(a) $E(A_n)$, $E(B_n)$, $E(C_n)$, and $E(D_n)$ are disjoint and the union of all of them is $E(Q_n)$.

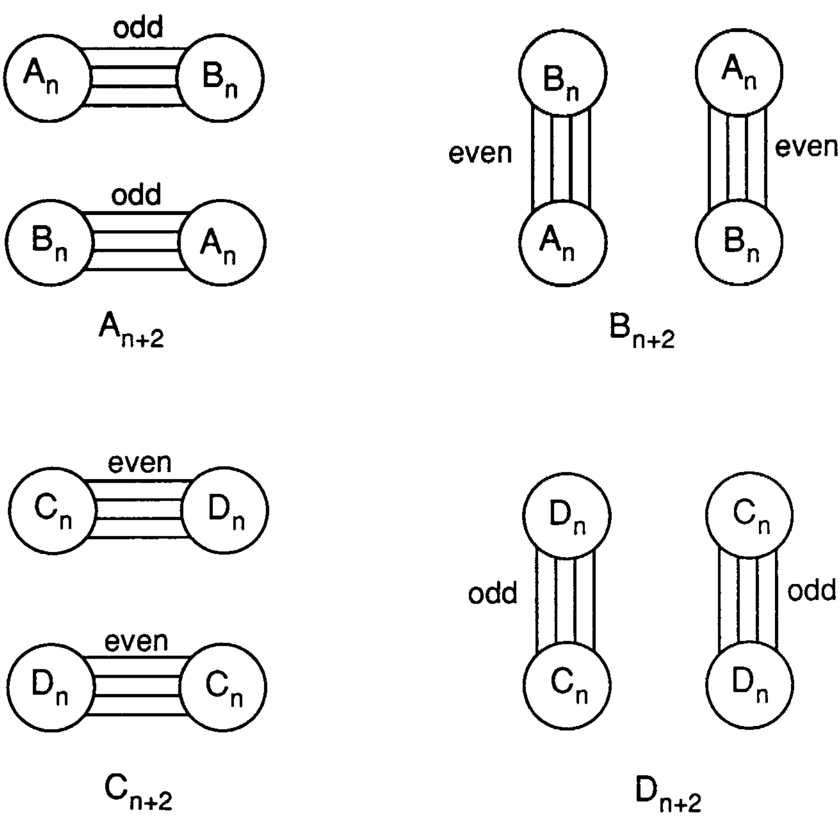

FIGURE 2

(b) $E(A_n) \cup E(B_n)$ and $E(C_n) \cup E(D_n)$ are C_4-free.
(c) A_n, B_n, C_n, and D_n are C_6-free.

The proofs of (a), (b), and (c) are by induction on n. It is easy to see that (a) holds trivially, and (b) follows from (a), and (c) follows from (b). This completes the proof of Theorem 3.

3. SUBGRAPHS OF Q_n CONTAINING NO C_{2k}, $k \geq 4$

For $k \geq 4$, suppose G is a subgraph of Q_n containing no C_{2k}. We will establish upper bounds of $f_{4k}(n)$ by proving Theorem 4.

Proof of Theorem 4. We will first show that if $e(G) \geq \varepsilon e(Q_n)$ for any $\varepsilon > 0$, then G contains C_{4k} for fixed $k \geq 2$, provided n is sufficiently large.

We consider a graph H_v, for each node v, defined as follows: (We note that H_v is similar to but different from G_v as defined in Section 2). The node set of H_v consists of all u so that $\{u, v\} \in E(Q_n)$, and u and w are adjacent if the Q_2 containing u, w, v has the property that $E(Q_2) - \{\{u, v\}, \{w, v\}\}$ contains two edges in $E(G)$. Since G is C_{2k}-free, H_v cannot contain k-cycles. Therefore, for $k \geq 4$, and $k \equiv 0 (\text{mod } 2)$ H_v can have at most $n^{1+(1/k)}$ edges (see [3]). Therefore we have

$$\sum_v \binom{d_v}{2} \leq (4\chi_4 + 3\chi_3 + \chi_2)\binom{n}{2}2^{n-2} = \sum_v e(H_v)$$
$$\leq n^{1+(1/k)} \cdot 2^n.$$

Therefore, $\chi_4 = o(1)$, $\chi_3 = o(1)$ and $\chi_2 = o(1)$. By using similar arguments as in the proof of Fact 2, we have

$$4\alpha^2 + \text{o}(1) \leq 8n^{-1+1/k}.$$

Hence,

$$\alpha \leq (\sqrt{2} + o(1))n^{(-1/2)+(1/2k)},$$

and Theorems 4–6 are proved.

4. SUBGRAPHS OF Q_n CONTAINING NO 4-CYCLES

In this section, we assume that G is a subgraph of Q_n containing no C_4. We want to establish upper bounds for the number of edges $e(G)$ of G. We follow the notation in previous sections and we note that $\chi_4 = 0$. First, we will prove some helpful facts.

Lemma 1.

$$2\chi_3 + \chi_2 = \frac{8}{n(n-1)2^n} \sum_v \binom{d_v}{2}.$$

Proof. The number of $K_{1,2}$ (i.e., a path with 2 edges) in G is equal to $\Sigma_v \binom{d_v}{2}$. Since each $K_{1,2}$ is contained in exactly one subcube Q_2 of Q_n, the number of $K_{1,2}$ is exactly $(2\chi_3 + \chi_2)\binom{n}{2}2^{n-2}$. Therefore Lemma 1 holds.

Lemma 2. Let $\bar{d}_v = n - d_v$.

$$\chi_2 + 2\chi_1 + 4\chi_0 = \frac{8}{n(n-1)2^n} \sum_v \binom{\bar{d}_v}{2}.$$

Proof. We count the number of $K_{1,2}$ in $\bar{G}$, the complement of G in Q_n, in two ways as in Lemma 1.

Lemma 3.

$$\chi_0 - \chi_2' = \frac{4}{n(n-1)2^n} \sum_v \left(\binom{d_v}{2} + \binom{\bar{d}_v}{2} \right) - 1.$$

This implies $\chi_0 - \chi_2' \geq (2\alpha - 1)^2 + O(1/n)$.

Proof. This follows from (1) and the addition of two equalities in Lemmas 1 and 2.

Lemma 4. Any subcube Q_3 in Q_n can contain at most two nodes of degree 3 in $G \cap Q_3$.

Proof. Suppose v is of degree 3 in $G \cap Q_3$. We consider two possibilities.

Case 1. If no neighbor of v in $G \cap Q_3$ is of degree 3 in $G \cap Q_3$, no vertex of distance 2 from v in Q_3 can have degree 3 since G does not contain a 4-cycle. Therefore there are at most two vertices of degree 3 in $G \cap Q_3$.

Case 2. v has a neighbor of degree 3 in $G \cap Q_3$. It is easy to check that no other vertex can have degree 3 in $G \cap Q_3$. Lemma 4 is proved.

Let a_i denote the fraction so that $a_i\binom{n}{3}2^{n-3}$ subcubes Q_3 of Q_n contain i nodes of degree 3 in $G \cap Q_3$ where $i = 0, 1$, and 2. By definition, we have $a_2 + a_1 + a_0 = 1$ and the a_i's satisfy the following:

Lemma 5.

$$2a_2 + a_1 = \frac{48}{n(n-1)(n-2)2^n} \sum_v \binom{d_v}{3}.$$

Proof. We consider the number of degree 3 nodes in $G \cap Q_3$ ranging over all 3-subcube Q_3 in Q_n. On one hand, this number is $(2a_2 + a_1)\binom{n}{3}2^{n-3}$. On the other hand, each $K_{1,3}$ (i.e., a star with 3 edges) is contained in a unique 3-subcube and therefore the above number is equal to the number of occurrences of $K_{1,3}$ in G, which is exactly $\Sigma_v\binom{d_v}{3}$. Lemma 5 is proved.

From Lemma 5, we can deduce a simple (but weak) upper bound for α as follows:

Lemma 6. Suppose a subgraph G of Q_n contains no 4-cycles and has $\alpha n 2^{n-1}$ edges. Then α satisfies $(n-1)(n-2) \geq 4\alpha^3 n^2 - 12\alpha^2 n + 8\alpha$. This implies $\alpha \leq (1 + o(1))(\frac{1}{4})^{1/3} \approx .630$.

Proof. From Lemma 5 we have

$$2 - a_1 - 2a_0 \geq \frac{48}{n(n-1)(n-2)2^n} \sum_v \binom{d_v}{3}. \tag{2}$$

Since $\Sigma_v\binom{d_v}{3} \geq 2^n\binom{\alpha n}{3}$, we have

$$n(n-1)(n-2) \geq 4\alpha n(\alpha n - 1)(\alpha n - 2).$$

If we focus on the first order terms, we get

$$1 + o(1) \geq 4\alpha^3.$$

That is,

$$\alpha < (1 + o(1))(\tfrac{1}{4})^{1/3} \approx .630.$$

In order to improve this bound, further work is needed. For each vertex v, we consider $M(v)$ and G_v as defined in Section 2. That is, $M(v) = \{u\colon u$ is adjacent to v in Q_n but u is not adjacent to v in $G\}$. For each pair u and w in $M(v)$, we say $\{u, w\}$ is blue if the unique 4-cycle containing u, v, w contains two edges of G other than $\{u, v\}$ and $\{v, w\}$.

For each v we consider triples $\{t, u, w\}$ in $M(v)$. We say $\{t, u, w\}$ is of type (v, i) if exactly i of the three pairs $\{t, u\}$, $\{t, w\}$, $\{u, w\}$ are blue.

In a 3-subcube Q_3 in Q_n, we say a node v is admissible if there are three nodes t, u, w in Q_3 and $\{t, u, w\}$ is of type $(v, 3)$ or type $(v, 0)$.

A result of Goodman [10] states:

Lemma 7. Let X be a graph on t nodes with $p\binom{t}{2}$ edges. Then, the number of monochromatic triangles (i.e., triples with all pairs being edges or all being nonedges) is at least $(1 - 3p + 3p^2)\binom{t}{3}$.

The proofs for the following three lemmas are by case-to-case analysis, which is quite straightforward and will be omitted.

Lemma 8. In a 3-subcube Q_3 of Q_n, if $G \cap Q_3$ contains two nodes of degree 3, then it contains no admissible nodes in $G \cap Q_3$.

Lemma 9. If $G \cap Q_3$ contains one node of degree 3, then it contains at most one admissible node.

Lemma 10. If $G \cap Q_3$ contains no node of degree 3, then it contains at most eight admissible nodes.

Summarizing Lemmas 7–10, we have the following:

Lemma 11. $(a_1 + 2a_0)\binom{n}{3}2^{n-3} \geq (1/16)\Sigma_v\binom{\bar{d}_v}{3}$.

Proof. We consider $\frac{1}{4}$ times the total number of admissible nodes in Q_3's of Q_n. From Lemmas 8 to 10, this number is no more than $(a_1 + 2a_0) \cdot \binom{n}{3}2^{n-3}$, while Lemma 7 provides the lower bound since $1 - 3p + 3p^2 \geq \frac{1}{4}$ for $0 \leq p \leq 1$. Lemma 11 is proved.

We can improve Lemma 6 as follows:

Lemma 12. Suppose a subgraph G of Q_n contains no 4-cycles and has $\alpha n 2^{n-1}$ edges. Then α satisfies $(5\alpha^3 + \alpha^2 - \alpha - 1)n^2 + (5 - 2\alpha - 9\alpha^2)n + 10\alpha - 2 \leq 0$. This implies $\alpha \leq .628$.

Proof. From (2) and Lemma 11 we have

$$\begin{aligned} 2 &\geq \frac{48}{n(n-1)(n-2)2^n} \sum_v \binom{d_v}{3} + a_1 + 2a_0 \\ &\geq \frac{1}{n(n-1)(n-2)2^n}\left(48 \sum_v \binom{d_v}{3} + 3 \sum_v \binom{\bar{d}_v}{3}\right) \\ &\geq \frac{1}{n(n-1)(n-2)}\left(48\binom{\alpha n}{3} + 3\binom{(1-\alpha)n}{3}\right). \end{aligned}$$

This implies

$$(5\alpha^3 + \alpha^2 - \alpha - 1)n^2 + (-9\alpha^2 - 2\alpha + 5)n + 10\alpha - 2 \leq 0,$$

and $\alpha \leq .628$ provided n is sufficiently large.

To improve upon Lemma 11, more careful analysis is needed. We will define weighting functions to help keep track of various counts in subgraphs of G. Although the weighting functions look a little complicated, they serve as convenient and useful tools.

For each node v and a 3-subcube Q_3 and Q_n, we define

$$f(v, Q_3) = \begin{cases} 1, & \text{if } v \text{ is of degree 3 in } G \cap Q_3; \\ \frac{1}{4}, & \text{if } v \text{ is admissible;} \\ 0, & \text{otherwise.} \end{cases}$$

A second weighting function g is defined on a pair u, v of nodes in a 3-subcube Q_3 in Q_4. We say $g(u, v, Q_3) = 0$ if $\{u, v\} \in E(G)$. Suppose $\{u, v\} \notin E(G)$. We define $g(u, v, Q_3) = \frac{1}{4}$ if each Q_2 containing u, v in Q_3 has exactly two edges of G where one of the edges contains u and both edges share a node (see Figure 3) and 0 otherwise.

The weighting functions defined above satisfy the following properties.

Lemma 13. For a subcube Q_3 in Q_n we define

$$f(Q_3) = \sum_v f(v, Q_3)$$

and

$$g(Q_3) = \sum_{\{u,v\}} g(\{u, v\}, Q_3) .$$

Then for every Q_3 in Q_n, we have

$$f(Q_3) + g(Q_3) \leq 2 . \tag{3}$$

We note that Lemmas 4, 8, and 9 yield $f(Q_3) \leq 2$. Still, there is some gap between $f(Q_3)$ and 2 in many cases depending on the occurrences of edges

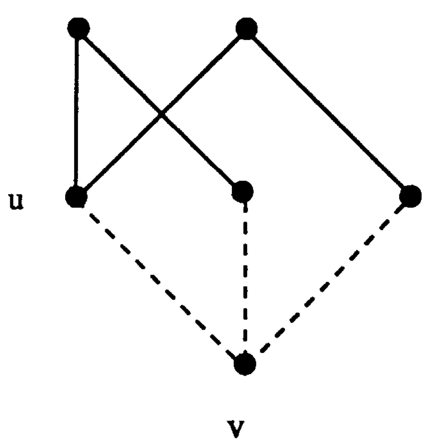

FIGURE 3

in $G \cap Q_3$. Roughly speaking, the way that g is defined is intended to capture such gaps.

Proof of Lemma 13. There are several possibilities.

Case 1. If there are two nodes v_1 and v_2 of degree 3 in $G \cap Q_3$, v_1 and v_2 are of distance 1 or 3. In either case it is easy to check that for $v \neq v_1$ and v_2, we have $f(v, Q_3) = 0$ and $g(u, v, Q_3) = 0$ for all v.

Case 2. If there is exactly one node v_i of degree 3 in $G \cap Q_3$, there is at most one node v with $f(v, Q_3) = \frac{1}{4}$. If there is one node v with $f(v, Q_3) = \frac{1}{4}$, it is done since there is at most three pairs u, v with $g(u, v, Q_3) \leq \frac{1}{4}$. We may assume there is no node v with $f(v, Q_3) = \frac{1}{4}$. It can be checked that there are at most two pairs u, v with $g(u, v, Q_3) \leq \frac{1}{4}$.

Case 3. Suppose there is no node of degree 3. If g is nonzero for some choice of u and v, then there are at most six pairs u, v with $g(u, v, Q_3) = \frac{1}{4}$ and at most two nodes v with $f(v, Q_3) = \frac{1}{4}$. If g only has zero value, then there are at most 8 nodes v with $f(v, Q_3) = \frac{1}{4}$. Lemma 13 is proved.

Lemma 14. $\Sigma_{Q_3} f(Q_3) \geq \frac{1}{4} \Sigma_v \binom{\bar{d}_v}{3}(1 - 3\rho_v + 3\rho_v^2) + \Sigma_v \binom{d_v}{3}$ where Q_3 ranges over all 3-subcubes of Q_n and there are $\rho_v \binom{\bar{d}_v}{2}$ blue pairs in $M(v)$ for all v in Q_n.

Proof. $f(Q_3)$ is equal to the sum of degree-three nodes and $\frac{1}{4}$ times that total number of admissible nodes. Therefore Lemma 14 follows from Lemma 7.

Lemma 15. $\Sigma_{Q_3} g(Q_3) \geq \frac{1}{4} \Sigma_v \bar{d}_v \binom{\rho_v \bar{d}_v}{2}$.

Proof. For each v, $\Sigma_{u,Q_3} g(u, v, Q_3)$ is exactly the sum of $\frac{3}{4}$ times the number of blue triangles in $M(v)$ and $\frac{1}{4}$ times the number of triangles with exactly two blue pairs in $M(v)$. Let D_u denote the number of blue pairs containing u in $M(v)$. We have

$$\sum_{u,Q_3} g(u, v, Q_3) \geq \frac{1}{4} \sum_{u \in M(v)} \binom{d_u}{2} \geq \frac{1}{4} \bar{d}_v \binom{\rho_v \bar{d}_v}{2}$$

Combining Lemmas 13–15 we have

$$2 \cdot \binom{n}{3} 2^{n-3} \geq \sum_v \binom{d_v}{3} + \frac{1}{4} \sum_v \left(\binom{\bar{d}_v}{3}(1 - 3\rho_v + 3\rho_v^2) + \bar{d}_v \binom{\rho_v \bar{d}_v}{2} \right).$$

We note that

$$\binom{\bar{d}_v}{3}(1 - 3\rho_v + 3\rho_v^2) + \bar{d}_v\binom{\rho_v\bar{d}_v}{2} \geq \binom{\bar{d}_v}{3}(1 - 3\rho_v + 6\rho_v^2) - \rho_v\bar{d}_v^2/6$$
$$\geq \frac{5}{8}\binom{\bar{d}_v}{3} - \rho_v\binom{\bar{d}_v}{2}\Big/3$$

since the function $1 - 3\chi + 6\chi^2$ has the minimum value 5/8 at $\chi = 1/4$. Also, $\Sigma\rho_v\binom{\bar{d}_v}{2} = \chi_2\binom{n}{2}2^{n-2}$. Using the Cauchy–Schwarz inequality, we have

$$\alpha\binom{n}{3}2^{n-3} \geq 2^n\binom{\alpha n}{3} + \frac{5}{32}\binom{(1-\alpha)n}{3} - \binom{n}{2}2^{n-2}/12.$$

Therefore, α satisfies

$$\frac{1}{4} \geq \alpha^3 + \frac{5}{32}(1 - \alpha)^3 + o(1).$$

This completes the proof of Theorem 1.

Added Remark. Recently, A. E. Brouwer, I. J. Dejter, and Carsten Thomassen [4] proved Theorem 3 independently.

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