Do Stronger Players Win More Knockout Tournaments?

F. R. K. CHUNG and F. K. HWANG*

We study knockout tournaments using a set of players with which a (pairwise) preference scheme is associated. We show that if the preference scheme is of the Bradley–Terry type, then for any knockout tournament, if player i has a larger merit value than player j, he also has a greater probability of winning the tournament. When the preference scheme is not of the Bradley–Terry type, counterexamples are given for several of the main results.

KEY WORDS: Bradley–Terry model; Knockout tournaments; Paired comparisons; Rooted binary trees.

1. INTRODUCTION

A knockout tournament (David 1963, Hartigan 1966, Hwang 1977, Moon 1968) among n players can be defined as a partially ordered set of games, each involving two players, which satisfies the following properties:

(i) Each game has a winner and a loser; a loser of a game is not involved in any further game.
(ii) The tournament ends when all players but one have lost a game; the player left is the winner of the tournament.
(iii) The subset of games involving any one player are linearly ordered.

It is easy to verify that a knockout tournament among n players will end after exactly $n - 1$ games.

It is convenient to describe this problem in the terminology of graph theory. For our purpose, we represent a knockout tournament among n players by a rooted binary tree with n terminal nodes (a terminal node is a node of indegree one and outdegree zero) each labeled with a distinct player. Thus the set of terminal nodes represents the starting positions of the players in the tournament. The tournament starts with any pair of labeled nodes which have the same father-node (a node whose two outgoing links connect the pair). This pair of nodes and the links which go to them are then eliminated from the binary tree, while the winner of the game now labels the father-node. Thus the tournament among n players is reduced to a tournament among $n - 1$ players, and we proceed inductively until the only node with indegree zero (called root) of the rooted binary tree is labeled by the winner of the tournament.

The rooted binary tree, ignoring the assignment of players to nodes, is called a knockout tournament plan. Thus a knockout tournament plan for n players can generate $n!$ knockout tournaments, each with a distinct assignment.

* F.R.K. Chung and F.K. Hwang are both members of the technical staff in the Department of Discrete Mathematics, Bell Laboratories, Murray Hill, NJ 07974. The authors wish to thank A.M. Odlyzko, the Editor, and the referees for useful comments.

Let π_{ij} denote the constant probability that player C_i beats player C_j in a game, so that $\pi_{ij} + \pi_{ji} = 1$. Then $\pi = \{\pi_{ij}\}$ is known as a preference scheme. David (1963) considers the following types of preference schemes:

(i) If $\pi_{ij} \geq \frac{1}{2}$, $\pi_{jk} \geq \frac{1}{2} \Rightarrow \pi_{ik} \geq \frac{1}{2}$, π is said to satisfy stochastic transitivity.
(ii) If $\pi_{ij} \geq \frac{1}{2}$, $\pi_{jk} \geq \frac{1}{2} \Rightarrow \pi_{ik} \geq \max(\pi_{ij}, \pi_{jk})$, π is said to satisfy strong stochastic transitivity.
(iii) Suppose that player C_i has true merit M_i when judged on some characteristic, and π_{ij} can be expressed for all i, j as $H(M_i - M_j)$, where $H(x)$ increases monotonically from $H(-\infty) = 0$ to $H(\infty) = 1$, and $H(-x) = 1 - H(x)$. Then π is said to be a linear preference scheme.

It is clear that strong stochastic transitivity implies stochastic transitivity, and a linear preference scheme always satisfies strong stochastic transitivity. When a linear preference scheme satisfies

$$H(M_i - M_j) = \frac{1}{4}\int_{-(M_i-M_j)}^{\infty} \operatorname{sech}^2 \tfrac{1}{2}y\,dy \ ,$$

we obtain the important subcase (known as the Bradley–Terry model (1952)),

$$\pi_{ij} = \pi_i/(\pi_i + \pi_j) \ , \quad \text{where} \quad \pi_i = e^{M_i} \ ,$$

which has been extensively studied in the literature (see Bradley 1976, David 1963, Davidson and Farquhar 1976 for some references). We will call such a π a B–T preference scheme.

Let T_n denote a knockout tournament plan among n players and A an assignment of players to the starting positions. Let $P_i(T_n, A; \pi)$ be the probability that player C_i wins the tournament (T_n, A) under the preference scheme π. Presumably, $P_i(T_n, A; \pi)$ depends on the starting position of C_i in the tournament. So if we only want to compare the players, we can neutralize the effect of the starting positions by averaging over all $n!$ assignments. Thus we define

$$P_i(T_n, \pi) = \frac{1}{n!}\sum_A P_i(T_n, A; \pi) \ .$$

If π is such that the players can be naturally ordered, such as in the three cases David (1963) has considered, then a stronger player by that ordering should have a

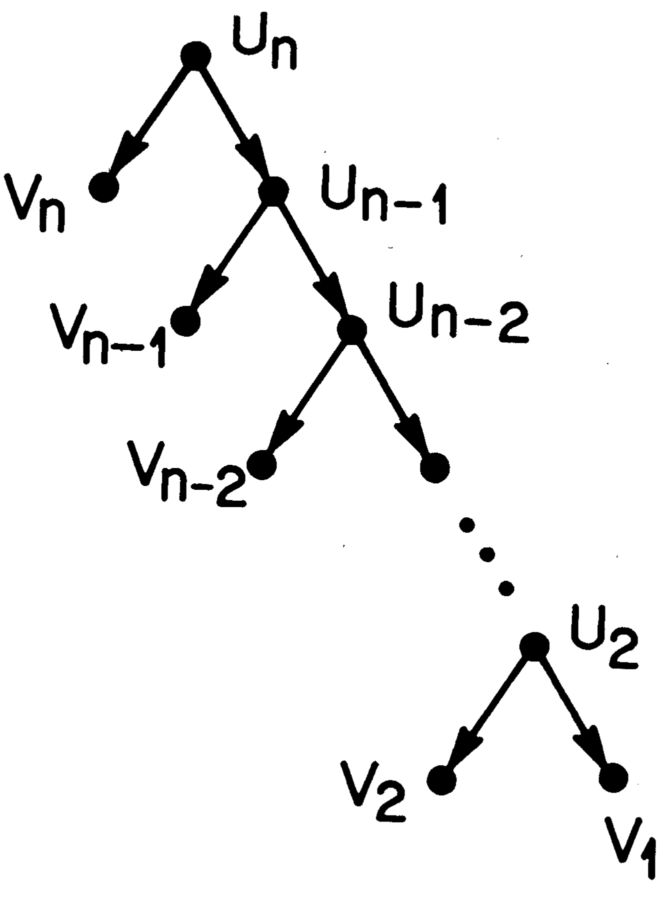
© Journal of the American Statistical Association
September 1978, Volume 73, Number 363
Theory and Methods Section

larger $P_i(T_n, \pi)$ for any T_n. However, it is easy to see that the property of stochastic transitivity cannot guarantee this result—counterexamples are readily found; e.g., $\pi_{12} = \pi_{13} = .5 + \epsilon$, and $\pi_{23} = 1 - \epsilon$ for small ϵ in a three-player tournament. We believe that the property of strong stochastic transitivity for π is sufficient to preserve the ordering in $\{P_i(T_n, \pi)\}$. But this is a surprisingly difficult conjecture to prove. The main result of this note is a proof that if π is a B–T preference scheme, then C_i stronger than C_j in π implies $P_i(T_n, \pi) \geq P_j(T_n, \pi)$ for any T_n.

2. THE BRADLEY–TERRY PREFERENCE SCHEME

If T_n is of the form shown in Figure A; i.e., the correspondence of T_n to a sequence of games is unique, then T_n is called a ladder tree and is denoted by L_n. We name the highest terminal node v_n, the second highest v_{n-1}, and so on; we name the internal nodes $u_n, \ldots, u_2$, where u_i is the father-node of v_i. Consider the assignment A to L_n: $C_i \to v_i$ for $i = 1, \ldots, n$.

A. A Ladder Tree L_n



Lemma 1: Let π be a B–T preference scheme and π' another B–T preference scheme obtained from π by changing π_1 to π_1'. Then we have

$$\frac{\pi_n + \pi_1'}{\pi_n + \pi_1} \geq \frac{P_n(L_n, A; \pi)}{P_n(L_n, A; \pi')} \geq \frac{\pi_n - \pi_1'}{\pi_n + \pi_1} \quad \text{if} \quad \pi_1' \geq \pi_1 .$$

The proof is given in the Appendix.

Only the left inequality is needed in the rest of this article (chiefly in Lemma 2). However, the right inequality is necessary in our induction proof of the left inequality.

Lemma 2: Consider the ladder tree L_n with assignment $A: C_i \to v_i$, $i = 1, \ldots, n$, and a B–T preference scheme π. Let A_{ij} be the assignment obtained from A by interchanging C_i and C_j. Then $\pi_n \geq \pi_1$ implies

(i) $P_n(L_n, A_{1n}; \pi) \geq P_1(L_n, A; \pi)$;
(ii) $P_i(L_n, A_{1n}; \pi) \geq P_i(L_n, A; \pi)$ for $i = 2, \ldots, n-1$;
(iii) $P_1(L_n, A_{1n}; \pi) \leq P_n(L_n, A; \pi)$.

Proof: Lemma 2 is trivially true for $n = 2$. Assume $n \geq 3$. Let A^i be the assignment A restricted to the terminal nodes of L_i.

(i) $P_n(L_n, A_{1n}; \pi)$
$$= \prod_{j=1}^{n-1} \frac{\pi_n}{\pi_n + \pi_j} \geq \prod_{j=2}^{n} \frac{\pi_1}{\pi_1 + \pi_j} = P_1(L_n, A; \pi) .$$

(ii) $P_i(L_n, A_{1n}; \pi)$
$$= P_i(L_i, A_{1n}{}^i; \pi) \frac{\pi_i}{\pi_i + \pi_1} \prod_{j=i+1}^{n-1} \frac{\pi_i}{\pi_i + \pi_j}$$
$$\geq P_i(L_i, A^i; \pi) \frac{\pi_i}{\pi_i + \pi_n} \prod_{j=i+1}^{n-1} \frac{\pi_i}{\pi_i + \pi_j}$$
$$= P_i(L_n, A; \pi) \text{ by Lemma 1 .}$$

(iii) $$P_1(L_n, A_{1n}; \pi) = 1 - \sum_{i=2}^{n} P_i(L_n, A_{1n}; \pi)$$
$$\leq 1 - \sum_{i=1}^{n-1} P_i(L_n, A; \pi)$$
$$= P_n(L_n, A; \pi) \text{ by using (i) and (ii) .}$$

Theorem: Let π be a B–T preference scheme and T_n a knockout tournament plan for n players. Then

$$\pi_i \geq \pi_j \Rightarrow P_i(T_n, A; \pi) \geq P_j(T_n, A_{ij}; \pi)$$

for any A.

Proof: The theorem is trivially true for $n = 2$. We prove the general case by induction.

Case 1: There exists a first-round game involving neither C_i nor C_j.

Let C_h and C_k be the two players involved in a first-round game, where $\{h, k\} \cap \{i, j\} = \emptyset$. Let T_{n-1} be the tournament plan obtained from T_n by deleting the two terminal nodes labeled C_h and C_k under A. Let $A^{(h)}$ $(A^{(k)})$ be the assignment on T_{n-1} obtained from A by assigning C_h (C_k) to the new terminal node. Then

$$P_i(T_n, A; \pi) = \frac{\pi_h}{\pi_h + \pi_k} P_i(T_{n-1}, A^{(h)}; \pi) + \frac{\pi_k}{\pi_h + \pi_k} P_i(T_{n-1}, A^{(k)}; \pi)$$
$$\geq \frac{\pi_h}{\pi_h + \pi_k} P_j(T_{n-1}, A_{ij}{}^{(h)}; \pi) + \frac{\pi_k}{\pi_h + \pi_k} P_j(T_{n-1}, A_{ij}{}^{(k)}; \pi)$$
$$= P_j(T_n, A_{ij}; \pi) \text{ by induction .}$$

Case 2: Every first-round game involves C_i or C_j. There are two possible subcases. In the first, there is exactly one first-round game involving either C_i or C_j (or both). In the second, there are exactly two first-round games that involve C_i and C_j separately. These two subcases can be treated together.

Let a and b be the two terminal nodes in T_n labeled by C_i and C_j, respectively. Suppose M is the root of the smallest subtree of T_n containing both nodes a and b. Let T_m denote this tree rooted at M, and T^a (T^b) the largest proper subtree of T_m containing node a (b). Then T^a and T^b are both ladder trees. Furthermore, the tree L obtained from T_m by replacing T^b by a simple node is also a ladder tree. Finally, let T^* be the tree obtained from T by replacing T_m by a single node.

Suppose A is an assignment on T. Then let A^m, A^L, and A^* be the assignments on T_m, L, and T^* obtained by restricting A to the terminal nodes on these respective subtrees. (In the cases of A^L and A^*, the new node is assumed to be labeled by C_i.) The set of players assigned to T^a by A, except C_i, is denoted by C^a. Then clearly,

$$\frac{P_i(T_n, A; \pi)}{P_j(T_n, A_{ij}; \pi)} = \frac{P_i(T_m, A^m; \pi)}{P_j(T_m, A_{ij}{}^m; \pi)} \cdot \frac{P_i(T^*, A^*; \pi)}{P_j(T^*, A_{ij}{}^*; \pi)} .$$

It is easy to show by induction that

$$P_i(T^*, A^*; \pi) \geq P_j(T^*, A_{ij}{}^*; \pi)$$

when π satisfies strong stochastic transitivity. Furthermore,

$$\begin{aligned} P_i(T_m, A^m; \pi) &= \prod_{C_k \in C^a} \frac{\pi_i}{\pi_i + \pi_k} P_i(L, A^L; \pi) \\ &\geq \prod_{C_k \in C^a} \frac{\pi_j}{\pi_j + \pi_k} P_j(L, A_{ij}{}^L; \pi) \\ &= P_j(T_m, A_{ij}{}^m; \pi) \text{ by Lemma 2(iii)} . \end{aligned}$$

By averaging over all possible assignments, we obtain:

Corollary: $\pi_i \geq \pi_j \Rightarrow P_i(T_n, \pi) \geq P_j(T_n, \pi)$.

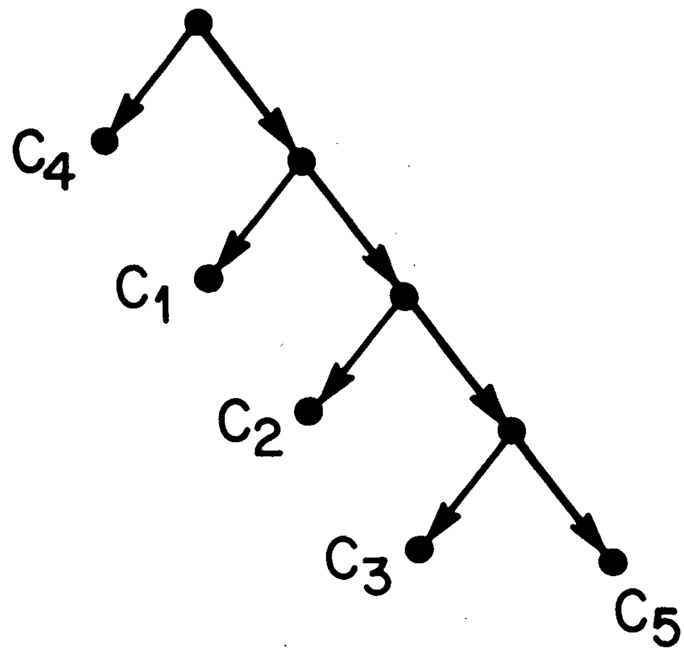
3. CONCLUSIONS

We proved the theorem and its corollary when the preference scheme is of the Bradley–Terry type, and we showed that they do not hold if only stochastic transitivity is assumed. We conjecture that the corollary to the theorem is still true if strong stochastic transitivity is assumed. However, the approach used in this paper does not extend easily to this case because:

(i) Lemma 2(iii) and the theorem are not true if only strong stochastic transitivity is assumed.
(ii) The left inequality of Lemma 1 does not hold even if π is assumed to be a linear preference scheme.

To see (i), consider the ladder tree L_5 and the assignment A in Figure B. Suppose π is the following matrix

B. The Tree L_5 and an Assignment



$\{\pi_{ij}\}$:

	1	2	3	4	5
1		$.5+\epsilon$	1	1	1
2			$.5+\epsilon$	$1-\epsilon$	$1-\epsilon$
3				$.5+\epsilon$	$1-\epsilon$
4					$.5+\epsilon$

where $\epsilon < .25$.

Then it can easily be shown that

$$P_4(L_5, A; \pi) - P_5(L_5, A_{45}; \pi) = -\epsilon(.5 - \epsilon)(.5 - 2\epsilon)^2 < 0 .$$

However, this is not a counterexample to the corollary.

To see (ii), consider the following counterexample supplied by a referee. Let $H(x) = \frac{1}{2} + x$ for $x \in [\frac{1}{2}, \frac{1}{2}]$; i.e., a uniform probability distribution over the prescribed interval. Then when $n = 3$ with $(M_1, M_1', M_2, M_3) = (\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4})$, it follows that

$$\frac{H(M_3 - M_1)}{H(M_3 - M_1')} = 2 < \frac{9}{4} = \frac{P_3(L_3, A; \pi)}{P_3(L_3, A; \pi')} ,$$

where A assigns C_i to v_i, $i = 1, 2, 3$. It is easily verified that this is not a counterexample to Lemma 2.

We summarize our results in the accompanying table.

A Summary of Proven Results

Property	*Left inequality of Lemma 1*	*Lemma 2(iii)*	*Theorem*	*Corollary*
ST[a]	false	false	false	false
SST[a]	false	false	false	unknown
LPS[b]	false	unknown	unknown	unknown
B-T	true	true	true	true

[a] (S)ST = (strong) stochastic transitivity.
[b] LPS = linear preference scheme.

The two counterexamples indicate the difficulty of the question posed in the title to this article. A fresh approach is needed to attack the problem in its full generality.

APPENDIX

Proof of Lemma 1: Lemma 1 is easily verified for $n = 2$. We prove the general case by induction on n.

Let τ_i denote the tournament obtained from (L_n, A) by replacing L_2 with root u_2 by a single terminal node labeled C_i, where $i = 1$ or 2. Then by induction,

$$\frac{P_n(L_n, A; \pi)}{P_n(L_n, A; \pi')} = \frac{\frac{\pi_1}{\pi_1+\pi_2} P_n(\tau_1; \pi) + \frac{\pi_2}{\pi_1+\pi_2} P_n(\tau_2; \pi)}{\frac{\pi_1'}{\pi_1'+\pi_2} P_n(\tau_1; \pi') + \frac{\pi_2}{\pi_1'+\pi_2} P_n(\tau_2; \pi')}$$

$$\leq \frac{\frac{\pi_1}{\pi_1+\pi_2} \cdot \frac{\pi_n+\pi_1'}{\pi_n+\pi_1} P_n(\tau_1; \pi') + \frac{\pi_2}{\pi_1+\pi_2} P_n(\tau_2; \pi)}{\frac{\pi_1'}{\pi_1'+\pi_2} P_n(\tau_1; \pi') + \frac{\pi_2}{\pi_1'+\pi_2} P_n(\tau_2; \pi')} . \quad \text{(A.1)}$$

Therefore it suffices to show that the right side of (A.1) is less than or equal to $(\pi_n + \pi_1')/(\pi_n + \pi_1)$. Since $P_n(\tau_2; \pi) = P_n(\tau_2; \pi')$, this is equivalent to

$$(\pi_n - \pi_2) P_n(\tau_2; \pi) \leq (\pi_n + \pi_1') P_n(\tau_1; \pi') . \quad \text{(A.2)}$$

When $\pi_1' \geq \pi_2$, or $\pi_1' < \pi_2$, then by induction

$$\frac{P_n(\tau_2; \pi)}{P_n(\tau_1; \pi')} \leq \frac{\pi_n + \pi_1'}{\pi_n + \pi_2} \quad \text{or} \quad \frac{P_n(\tau_1; \pi')}{P_n(\tau_2; \pi)} \geq \frac{\pi_n - \pi_2}{\pi_n + \pi_1'} ,$$

respectively. In either case, (A.2) follows immediately.

To prove the other inequality in Lemma 1, we use induction to obtain

$$\frac{P_n(L_n, A; \pi)}{P_n(L_n, A; \pi')} \geq \frac{\frac{\pi_1}{\pi_1+\pi_2} \cdot \frac{\pi_n-\pi_1'}{\pi_n+\pi_1} P_n(\tau_1; \pi') + \frac{\pi_2}{\pi_1+\pi_2} P_n(\tau_2; \pi)}{\frac{\pi_1'}{\pi_1'+\pi_2} P_n(\tau_1; \pi') + \frac{\pi_2}{\pi_1'+\pi_2} P_n(\tau_2; \pi')} . \quad \text{(A.3)}$$

Therefore it suffices to prove that the right side of (A.3) is greater than or equal to $(\pi_n - \pi_1')/(\pi_n + \pi_1)$, which is equivalent to

$$\{\pi_n(\pi_1' - \pi_1) + 2\pi_1\pi_1' + \pi_2(\pi_1 + \pi_1')\} P_n(\tau_2, \pi) \geq (\pi_n - \pi_1')(\pi_1' - \pi_1) P_n(\tau_1, \pi') . \quad \text{(A.4)}$$

When $\pi_1' \geq \pi_2$ or $\pi_1' < \pi_2$,

$$\frac{P_n(\tau_2; \pi)}{P_n(\tau_1; \pi')} \geq \frac{\pi_n - \pi_1'}{\pi_n + \pi_2} \quad \text{or} \quad \frac{P_n(\tau_1; \pi')}{P_n(\tau_2; \pi)} \leq \frac{\pi_n + \pi_2}{\pi_n + \pi_1} ,$$

respectively. Since

$$\pi_n(\pi_1' - \pi_1) + 2\pi_1\pi_1' + \pi_2(\pi_1 + \pi_1') \geq (\pi_n + \pi_2)(\pi_1' - \pi_1) ,$$

(A.4) follows immediately in either case.

[*Received December 1976. Revised February 1978.*]

REFERENCES

Bradley, Ralph A. (1976), "Science, Statistics and Paired Comparisons," *Biometrics*, 32, 213–232.

———, and Terry, Milton E. (1952), "Rank Analysis of Incomplete Block Designs I. The Method of Paired Comparisons," *Biometrika*, 39, 324–345.

David, H.A. (1963), *The Method of Paired Comparisons*, New York: Hafner Press.

Davidson, R.R., and Farquhar, P.H. (1976), "A Bibliography on the Method of Paired Comparisons," *Biometrics*, 32, 241–252.

Hartigan, J.A. (1966), "Probabilistic Completion of a Knockout Tournament," *Annals of Mathematical Statistics*, 37, 495–503.

Hwang, Frank K. (1977), "Several Problems on Knockout Tournaments," *Proceedings of the 8th Southeastern Conference on Combinatorics, Graph Theory and Computing*, Baton Rouge, 363–380.

Moon, John W. (1968), *Topics on Tournaments*, New York: Holt, Rinehart & Winston.