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A LOWER BOUND FOR THE STEINER TREE PROBLEM*

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Abstract. Let V be a finite set of points in the plane. In this paper, we show that the ratio of the lengths of Steiner minimal tree for V and the minimal spanning tree for V can never be less than

$$\frac{2\sqrt{3}+2-\sqrt{7+2\sqrt{3}}}{3}=.74309\cdots.$$

1. Introduction. Suppose we have a finite set of points $V=\{A_1, A_2, \cdots, A_n\}$ in a Euclidean space. A minimal spanning tree for V is a tree on V having minimal length, i.e., having the least possible sum of the lengths of all its lines. A Steiner tree for V is a tree which may have some extra points $S_1, \cdots, S_t$ (called *Steiner points*) in addition to the $A_1, \cdots, A_n$ (called *regular points*). It will be assumed that at least three lines go to any Steiner point. A Steiner tree with minimal length is called a *Steiner minimal tree*. Many properties of these trees are reviewed by Gilbert and Pollak [2]. For example, in a Steiner minimal tree, a Steiner point is incident to exactly three lines meeting each other at 120°. A Steiner minimal tree has at most $n-2$ Steiner points. A Steiner tree with $n-2$ Steiner points is called a *full* Steiner tree. Every Steiner minimal tree can be decomposed into a union of full Steiner trees. The terminology in this paper follows [2].

Let $L_s(V)$ and $L_m(V)$ denote the lengths of a Steiner minimal tree on V and a minimal spanning tree on V, respectively. Define

$$\sigma_D=\min_V \frac{L_s(V)}{L_m(V)}$$

where V ranges over all sets of points in D-dimensional Euclidean space. It has long been conjectured [2] that σ_2 is equal to $\sqrt{3}/2$. Recently, Chung and Gilbert [1] proved that

$$\lim_{D\to\infty} \sigma_D \leqq \frac{(3/2)^{1/2}}{2^{3/2}-1}=0.66984\cdots.$$

It is also conjectured in [1] that equality holds.

Graham and Hwang [3] have recently shown that $\sigma_D \geqq 1/\sqrt{3}=0.5771\cdots$. In this paper, we improve the lower bound for $D=2$ by showing $\sigma_2 \geqq 0.74309\cdots$ (see Main Theorem below).

2. An improved lower bound for σ_2. In the rest of the paper, we will assume $V=\{A_1, A_2, \cdots, A_n\}$ is a set of n points in the Euclidean plane. We will show the following.

MAIN THEOREM.

$$\sigma_2 \geqq \frac{2\sqrt{3}+2-\sqrt{7+2\sqrt{3}}}{3}=\rho=0.74309\cdots.$$

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In order to prove the Main Theorem, we first normalize and then prove a number of auxiliary lemmas from which the desired result will easily follow.

It is proved in [4] that

$$\frac{L_s(V)}{L_m(V)} \geqq \frac{\sqrt{3}}{2} \quad \text{for } n \leqq 4.$$

Thus, we may assume $n \geqq 5$. Furthermore, we may assume $L_s(V)/L_m(V) \geqq \rho$ for all V with at most $n-1$ points, and there is a Steiner minimal tree s for V which is a full tree. Thus, there exists a Steiner point, say S_1, which is adjacent to exactly two distinct regular points, say A_1 and A_2 (see Fig. 1).

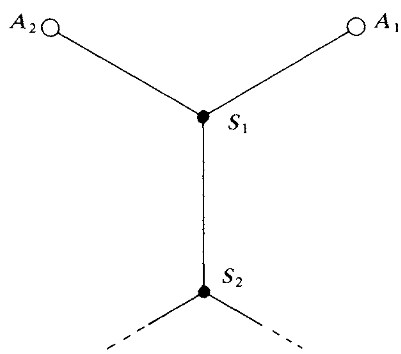

FIG. 1.

Without loss of generality, we may assume

$$d(A_1, S_1) \geqq d(A_2, S_1),$$

where $d(A_1, S_1)$ denotes the Euclidean distance between A_1 and S_1. We let $[A_1, S_1]$ denote the line connecting A_1 and S_1.

LEMMA 1. *Let A be a regular point which is adjacent to a Steiner point S. Let Y be any point on a line of the Steiner minimal tree s which is not on the line $[A, S]$. Then we have*

$$d(A, Y) \geqq d(A, S)$$

i.e., the open disc $\{X : d(A, X) < d(A, S)\}$ does not contain any point of $s - [A, S]$ (see Fig. 2).

Proof. If Lemma 1 is not true, then we can connect A to Y (instead of to S) and get a shorter tree. □

LEMMA 2. *Let A be a regular point which is adjacent to a Steiner point S. If there is a regular point different from A in the disc*

$$\left\{X: d(A, X) \leqq \frac{d(A, S)}{\rho}\right\},$$

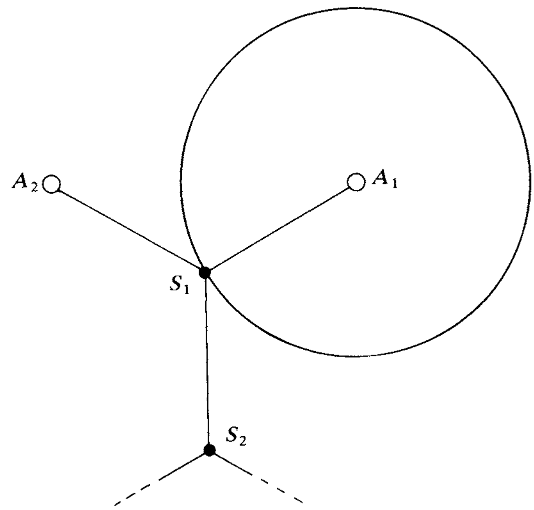

FIG. 2.

then we have

$$\frac{L_s(V)}{L_m(V)} \geqq \rho.$$

Proof. Let $B \in V - \{A\}$ with $d(B, A) \leqq d(A, S)/\rho$. We have

$$L_s(V) = d(A, S) + \sum_{\substack{[X,Y]\in s \\ [X,Y]\neq[A,S]}} d(X, Y) \geqq \rho d(B, A) + \rho L_m(V - \{A\}).$$

Since

$$d(B, A) + L_m(V - \{A\}) \geqq L_m(V)$$

we have

$$L_s(V) \geqq \rho L_m(V).$$

The lemma is proved. □

By Lemma 2, we may assume

$$d(A_1, S_1) < \rho d(A_i, A_1) \quad \text{for all } i \neq 1$$

and

$$d(A_2, S_1) < \rho d(A_i, A_2) \quad \text{for all } i \neq 2.$$

In particular,

$$d(A_1, S_1) < \rho d(A_1, A_2).$$

We also have

$$d(A_2, S_1) > \left(\frac{\sqrt{4-3\rho^2} - \rho}{\cdot 2}\right) d(A_1, A_2) = .3938 \cdots d(A_1, A_2)$$

since the lines $[S_1, A_1]$, $[S_1, A_2]$ meet at 120° so that

$$(d(A_1, S_1))^2 + (d(S_1, A_2))^2 + d(A_1, S_1)d(S_1, A_2) = (d(A_1, A_2))^2.$$

The Steiner point S_1 is incident to exactly three lines, namely $[S_1, A_1]$, $[S_1, A_2]$, and $[S_1, S_2]$ (since $n > 3$).

LEMMA 3.

$$d(S_1, S_2) \geqq \left(\frac{2}{\sqrt{3}} - 1\right) d(S_1, A_1).$$

Proof.

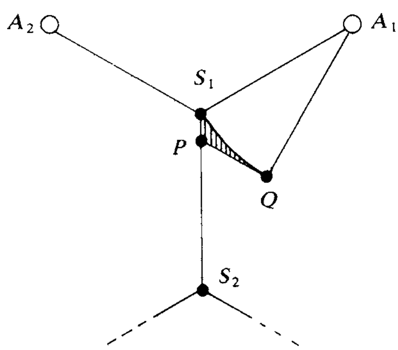

FIG. 3.

Let Q be the point with $d(A_1, Q) = d(S_1, A_1)$ such that the line passing through Q and perpendicular to the line $[A_1, Q]$ meets the line $[S_1, S_2]$ in the point P at an angle of 120° as indicated in Fig. 3. The shaded region in Fig. 3 is bounded by lines $[S_1, P]$, $[P, Q]$ and a portion of the circle with center A_1 and radius $d(S_1, A_1)$. It is easy to verify that $d(S_1, P) = (2/\sqrt{3} - 1)\, d(S_1, A_1)$.

By Lemma 2, the shaded region does not contain any regular points, since

$$d(A_1, P) = \sqrt{\frac{7 - 2\sqrt{3}}{3}}\, d(A_1, S_1) \leqq \frac{d(A_1, S_1)}{\rho}.$$

If S_2 falls in between S_1 and P, then S_2 has a line lying completely in the shaded region (Lemma 1 forbids the extension of this line from penetrating the sector A_1S_1Q). Either this line ends at a regular point, or at a Steiner point S_3 to which a similar argument applies, etc. (see Fig. 4). Therefore there must be a regular point in the shaded region, which is a contradiction.

Thus, we have shown

$$d(S_1, S_2) \geqq d(S_1, P) = \left(\frac{2}{\sqrt{3}} - 1\right) d(S_1, A_1)$$

and the lemma is proved. □

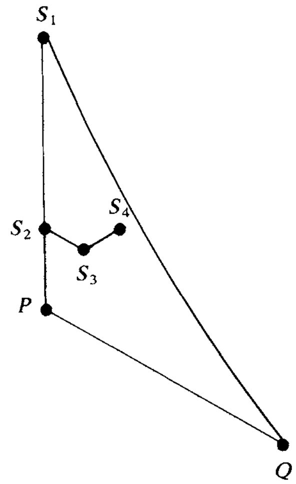

FIG. 4.

The Steiner point S_2 is incident to exactly three lines, namely $[S_1, S_2]$, $[S_2, Q]$, $[S_2, P]$, where P and Q are not both regular points of s (since $n \geqq 5$).

LEMMA 4. *If $d(S_2, P) \geqq d(A_2, S_1)$ and $d(S_2, Q) \geqq d(A_2, S_1)$ in Fig. 5, then we have*

$$\frac{L_s(V)}{L_m(V)} \geqq \rho.$$

Proof. It is known [2] that $d(S_1, S_2)$ must have length at least $(\sqrt{3}-1)L_0$, where L_0 is the shortest of the four lines $[S_1, A_1]$, $[S_1, A_2]$, $[S_2, P]$, $[S_2, Q]$. Hence we have $d(S_1, S_2) \geqq (\sqrt{3}-1)d(A_2, S_1)$. Choose P', Q' on the lines $[P, S_2]$, $[Q, S_2]$ respectively, with $d(S_2, P') = d(S_2, Q') = d(A_2, S_1)$.

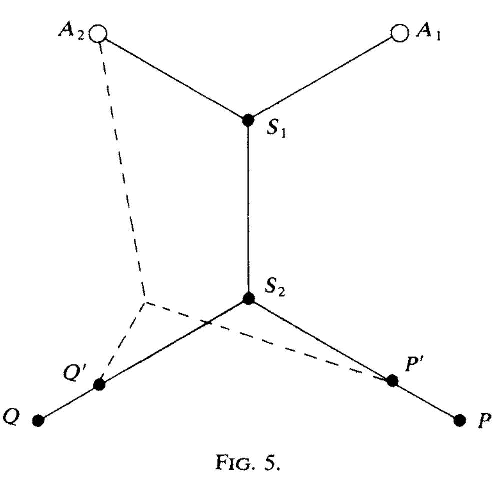

FIG. 5.

Let t denote the length of the Steiner minimal tree connecting the three points A_2, Q', P'. Then

$$L_s(V)-d(A_1,S_1)-d(A_2,S_1)-d(S_1,S_2)-d(S_2,Q')-d(S_2,P')+t \geqq L_s(V-\{A_1\}),$$

since the left-hand-side is the length of a tree on $V-\{A_1\}$.

It is easy to verify that

$$d(A_2,S_1)+d(S_1,S_2)+d(S_2,Q')+d(S_2,P')-t \geqq (2+\sqrt{3}-\sqrt{6+3\sqrt{3}})d(A_2,S_1) \geqq 0.38d(A_2,S_1).$$

Therefore

$$L_s(V) \geqq d(A_1,S_1)+0.38d(A_2,S_1)+L_s(V-\{A_1\}).$$

It suffices to show

$$d(A_1,S_1)+0.38d(A_2,S_1) \geqq \rho d(A_1,A_2),$$

since we have, by induction,

$$L_s(V-\{A_1\}) \geqq \rho L_m(V-\{A_1\})$$

and consequently,

$$L_s(V) \geqq \rho d(A_1,A_2)+\rho L_m(V-\{A_1\}) \geqq \rho L_m(V).$$

CLAIM.

$$d(A_1,S_1)+\alpha d(A_2,S_1) \geqq \rho d(A_1,A_2) \quad \textit{for}\ \frac{\sqrt{4-3\rho^2}+\rho}{4\rho} > \alpha \geqq \sqrt{3}\rho-1.$$

Proof. Let

$$x=\frac{d(A_1,S_1)}{d(A_1,A_2)}, \qquad y=\frac{d(A_2,S_1)}{d(A_1,A_2)}.$$

It follows from the 120° constraint on the lines of S_1 that

$$x^2+y^2+xy=1.$$

In order to minimize $x+\alpha y$, we consider

$$f(x,y)=x+\alpha y \qquad g(x,y)=x+\alpha y+\lambda(x^2+xy+y^2),$$

where

$$\rho \geqq x \geqq y \geqq \frac{\sqrt{4-3\rho^2}-\rho}{2} \quad \text{(by Lemma 2)}.$$

Now

$$\frac{\partial g}{\partial x}(x,y)=1+\lambda(2x+y)$$

and

$$\frac{\partial g}{\partial y}(x, y) = \alpha + \lambda(x + 2y).$$

$\partial g/\partial x$ and $\partial g/\partial y$ can both be zero only if

$$(1-2\alpha)x + (2-\alpha)y = 0.$$

Thus

$$1 \geqq -\left(\frac{1-2\alpha}{2-\alpha}\right) = \frac{y}{x} \geqq \frac{(\sqrt{4-3\rho^2}-\rho}{2\rho} = .53004\cdots.$$

Since $2 > \alpha$, it follows that

$$2-\alpha \geqq 2\alpha - 1 \geqq .5300435\cdots = \frac{(\sqrt{4-3\rho^2}-\rho}{2\rho},$$

i.e.,

$$1 \geqq \alpha \geqq .7650\cdots = \frac{\sqrt{4-3\rho^2}+\rho}{4\rho},$$

a contradiction to our assumption on α.

Hence, there is no local minimum. Therefore the minimum is assumed at a boundary point. It is easily verified that

$$f(x, y) \geqq f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1+\alpha}{\sqrt{3}} \geqq \rho$$

for $\alpha \geqq \sqrt{3}\rho - 1 = .28707\cdots$. Thus, the claim and Lemma 4 are proved. □

LEMMA 5. *If* $d(S_1, S_2) + d(S_2, Q) \geqq (2/\sqrt{3})d(S_1, A_2)$, *then we have*

$$\frac{L_s(V)}{L_m(V)} \geqq \rho.$$

Proof. In Fig. 6, choose Q' on the line $[S_2, Q]$ with

$$d(Q', S_2) + d(S_1, S_2) = \frac{2}{\sqrt{3}} d(A_2, S_1).$$

It is easily seen that

$$d(A_2, S_1) + d(S_1, S_2) + d(S_2, Q') - t$$
$$\geqq \left(\frac{\sqrt{3}+2-\sqrt{7+2\sqrt{3}}}{\sqrt{3}}\right) d(S_1, A_2) = (0.28707\cdots) d(S_1, A_2)$$

where t is the length of the Steiner minimal tree on A_2, Q' and S_2.

Similar to the proof of Lemma 4, it is easily verified that

$$d(A_1, S_1) + d(A_2, S_1) + d(S_1, S_2) + d(S_2, Q') - t \geqq \rho d(A_1, A_2).$$

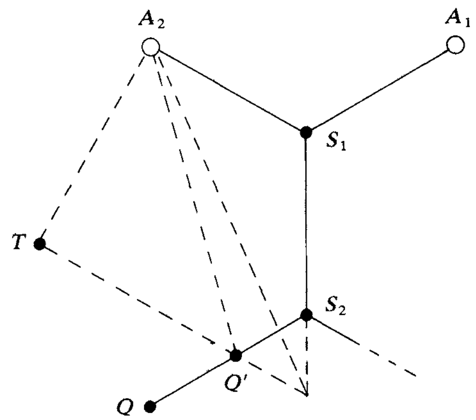

FIG. 6.

Thus, we have

$$L_s(V) \geqq d(A_1, S_1) + d(A_2, S_1) + d(S_1, S_2) + d(S_2, Q') - t + L_s(V - \{A_1\}).$$

Therefore

$$L_s(V) \geqq \rho d(A_1, A_2) + \rho L_m(V - \{A_1\}) \geqq \rho L_m(V). \quad \square$$

We may now assume

$$d(S_1, S_2) + d(S_2, Q) < \frac{2}{\sqrt{3}} d(S_1, A_2).$$

LEMMA 6. *If* $d(S_1, S_2) + d(S_2, Q) < (2/\sqrt{3}) d(S_1, A_2)$, *then we have*

$$\frac{L_s}{L_m} \geqq \rho.$$

Proof.

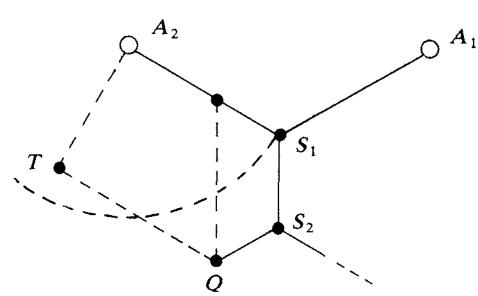

FIG. 7.

Let T denote the intersection of $[T, Q]$ and $[T, A_2]$ where the line $[T, A_2]$ is perpendicular to the line $[T, Q]$ and the line $[A_2, S_1]$ is parallel to the line $[T, Q]$

(see Fig. 7). Now, we have $d(T, A_2) < d(S_1, A_2)$ since

$$d(S_1, S_2) + d(Q, S_2) < \frac{2}{\sqrt{3}} d(S_1, A_2).$$

By arguments similar to those of Lemma 3, there exists a regular point, say R, in the pentagon $S_1S_2QTA_2$. The lemma will be proved if we can show

$$d(A_1, S_1) + d(A_2, S_1) + d(S_1, S_2) \geqq \rho(d(A_1, A_2) + d(A_2, R)),$$

since then,

$$\begin{aligned} L_s(V) &\geqq d(A_1, S_1) + d(A_2, S_1) + d(S_1, S_2) + L_s(V - \{A_1, A_2\}) \\ &\geqq \rho(d(A_1, A_2) + d(A_2, R)) + \rho L_m(V - \{A_1, A_2\}) \\ &\geqq \rho L_m(V). \end{aligned}$$

CLAIM.

$$d(A_1, S_1) + d(A_2, S_1) + d(S_1, S_2) \geqq \rho(d(A_1, A_2) + d(A_2, R)).$$

Proof. Let U be a point on $[A_2, S_1]$ such that $[Q, U]$ is parallel to $[S_1, S_2]$. It is easy to see that

$$d(A_2, R) \leqq \max(d(A_2, Q), d(A_2, S_2))$$

and

$$\begin{aligned} d(A_2, Q) \leqq d(A_2, U) + d(U, Q) &= (d(A_2, S_1) - d(Q, S_2)) + (d(S_1, S_2) + d(Q, S_2)) \\ &= d(A_2, S_1) + d(S_1, S_2). \end{aligned}$$

Furthermore,

$$d(A_2, S_2) \leqq d(A_2, S_1) + d(S_1, S_2).$$

Therefore,

$$d(A_2, R) \leqq d(A_2, S_1) + d(S_1, S_2).$$

Hence it suffices to show

$$d(A_1, S_1) + d(A_2, S_1) + d(S_1, S_2) \geqq \rho(d(A_1, A_2) + d(A_2, S_1) + d(S_1, S_2)),$$

i.e.,

$$d(A_1, S_1) + (1-\rho) d(A_2, S_1) + (1-\rho) d(S_1, S_2) \geqq \rho d(A_1, A_2).$$

From Lemma 3, we know that

$$d(S_1, S_2) \geqq \left(\frac{2}{\sqrt{3}} - 1\right) d(S_1, A_1) \geqq \left(\frac{2}{\sqrt{3}} - 1\right) d(A_2, S_1).$$

The claim is valid if we show

$$d(A_1, S_1) + \frac{2}{\sqrt{3}}(1-\rho) d(A_2, S_1) \geqq \rho d(A_1, A_2).$$

But this follows immediately from the claim in the proof of Lemma 4 since we have

$$\frac{\sqrt{4-3\rho^2}+\rho}{4\rho} \geqq \frac{2}{\sqrt{3}}(1-\rho) \geqq \sqrt{3}\rho - 1. \quad \square$$

The Main Theorem now follows immediately from Lemmas 5 and 6.

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