Improved separators from planar graphs

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ABSTRACT

The n vertices of a planar graph can be partitioned into three sets A, B, C, such that no edge connects a vertex in A with a vertex in B, A and B each have at most $n/2$ vertices, and C contains no more than $3\sqrt{6}\sqrt{n}$ vertices. The constant $3\sqrt{6}$ is an improvement over previous bounds in several papers by Lipton and Tarjan, Djidjev, and Venkatesan. A version for planar graphs with vertex-costs is also given.

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I. Introduction

Lipton and Tarjan [1] first proved the planar separator Theorem:

Theorem [LT1]: The n vertices of a planar graph can be partitioned into three sets, A, B, C, such that no vertex in A is adjacent to a vertex in B, A and B each have at most $n/2$ vertices, and C contains no more than $\sqrt{8}/(1-\sqrt{2/3})\sqrt{n}$ vertices.

Due to the great impact of this theorem in a wide range of areas, much attention has been focused upon improving the constants involved. The constant $\sqrt{8}/(1-\sqrt{2/3})$ (~ 15.413) was improved to $\sqrt{6}/(1-\sqrt{2/3})$ (~ 13.348) by Djidjev [D] and later to about 9.587 [D]. The best constant known so far is $7+1/\sqrt{3}$ (~ 7.587) by Venkatesan [V]. In this paper, we will improve the constant to $3\sqrt{6}$ (~ 7.348). Similar improvements can be made for planar graphs with vertex costs. Suppose G is an n-vertex planar graph with non-negative vertex cost sum to 1. Then G can be separated into two parts, each with total vertex cost not exceeding $\frac{1}{2}$, by removing $c\sqrt{n}$ vertices where the best known constant for c is $8+\frac{16}{81}\cdot(\sqrt{6}/(1-\sqrt{2/3}))$ (~ 10.636737) by Venkatesan [V]. In this note we will show that this constant can be improved to $\frac{108}{19}\sqrt{3}$ (~ 9.845).

We remark that if we only require A and B each have at most $2n/3$ vertices (instead of $n/2$ in the Theorem stated above), the best bound due to Gazit [G] for C is $7\sqrt{n}/3$, improving the original bound of $2\sqrt{2}\sqrt{n}$ in [LT1].

II. Preliminaries

The method we use here is based on the results of Lipton and Tarjan [LT1] [LT2], Djidjev [D] and Venkatesan [V]. The underlying approaches are very similar. Only the optimization formulation in the next section is somewhat different. Here we state the facts in [LT1], [D], [V] that we need here.

Lemma 1: [LT1] Let G be a planar graph of radius s, with non-negative vertex-cost sum to 1. Then the vertices of G can be partitioned into three parts, A, B, C, such that there is no edge between A and B, neither A nor B has total vertex cost exceeding $\frac{2}{3}$ and C contains at most $2s + 1$ vertices.

Lemma 2 [D]: Let G be an n-vertex planar graph of radius s. For any real number r, $\frac{1}{2} \le r \le 1$, there exists a set $S \subseteq V(G)$ with at most $3s + 1$ vertices such that by removing vertices in S from G the remaining graph is separated into three parts A, B, C such that A, B each contain at most $(1-r)n$ vertices, and C contains at most rn vertices.

Lemma 3 [V]: For any integer s, an n-vertex planar graph G contains a subgraph H of at least $n - n/s$ vertices so that any subgraph of H can be embedded unto another planar graph of radius $s - 1$.

III. Separation into two halves

Theorem 1: The n vertices of a planar graph can be partitioned into three parts A, B and C, such that no vertex in A is adjacent to a vertex in B, A and B each have at most $n/2$ vertices, and C contains at most $3\sqrt{6}\sqrt{n}$ vertices.

Proof: Let G denote a planar graph on n vertices. We will determine A, B and C iteratively as follows:

Step 0: Set $s = \lfloor \sqrt{\frac{n}{6}} \rfloor$ and use Lemma 3 to find a set S_0 with at most n/s vertices and embed $G - S_0$ onto a graph G' of radius $s - 1$. Set $A = B = \emptyset$, $C = S_0$ and $j = 1$.

In general, for $i = 1, 2, \ldots$, the step i can be described as follows:

Step i: Set $r = (\frac{n}{2} - |A|)/n'$ where $n' = |V(G')|$. For $j \le 2$, use Lemma 2 to find a separator S' containing at most $3(s-1)$ vertices which separate G' into A', B' and C' with $|A'| \le |B'| \le (1-r)n'$ and $|B'| \le |C'| \le r\,n'$. Set A to be the smaller one of $A \cup C'$ and $B \cup B'$; B to be the larger one of $A \cup C'$ and $B \cup B'$; and C to be $C \cup S'$. If $j < 2$, apply Lemma 3 to the induced subgraph of A' and form a new graph G' of radius $s - 1$, set $j = j + 1$ and repeat Step i. If $j = 2$, set i to be $i + 1$, j to be 1. Then set s to be $\sqrt{\frac{n'}{6}}$ when applying Lemma 3 to A' and form G' of radius $s - 1$, add n'/s vertice to C. Then go to the next step until G' is empty.

The correctness of this algorithm can be established by verifying the following facts in Step i. Suppose at the beginning of Step i A has p vertices, B has q vertices and G' has w vertices. The new A and B has no more than the maximum of $p + |C'|$ and $q + |B'|$. Since $p + |C'| \le p + rw \le p + (\frac{n}{2} - p) \le \frac{n}{2}$ and $q + |B'| \le q + (1-r)w \le q + w - \frac{n}{2} + p \le \frac{n}{2}$, the new A and B has no more than $n/2$ vertices. Furthermore $|A'| \le w/3$ since $|A'| \le |B'| \le |C'|$ and $|A'| + |B'| + |C'| \le w$. This implies the new G' has at more $w/3$ vertices. Because of our choice of j, for each i, Step i is repeated twice for $j = 1$ and 2 (except for possibly the last step). So altogether $|G'|$ is reduced by a factor of 9.

We can bound the separator $C = C(n)$ as follows:

$$C(n) \le 2\sqrt{6}\sqrt{n} + \sum_{i \ge 1} 2\sqrt{6}\sqrt{\frac{n}{9^i}}$$

$$\leq 2\sqrt{6}\sqrt{n} + |C(\frac{n}{9})|$$

By induction it can be easily shown that

$$|C(n)| \leq 3\sqrt{6}\sqrt{n}$$

This completes the proof for Theorem 1.

IV. Separating planar graphs with vertex costs

Theorem 2: Let G be an n-vertex planar graph with non-negative vertex costs sum to 1. Then the vertices of G can be partitioned into three sets A, B, C such that no vertex in A is adjacent to a vertex in B, neither A nor B has total cost exceeding $\frac{1}{2}$, and C contains no more than $\frac{108\sqrt{3}}{19}\sqrt{n}$ vertices.

Proof: We will use the following algorithm which is slightly different from that in Theorem 1.

Step 0: Set $s = \lfloor \sqrt{\frac{n}{12}} \rfloor$ and use Lemma 3 to find a set S_0 with at most n/s vertices and embed $G - S_0$ into a graph G' of radius $s-1$. Then use Lemma 1 to partition G' into A', B' and C' with $|c(A')| \leq |c(B')| \leq \frac{2}{3}$ and $|C'| \leq 2s+1$ where $c(A')$ denotes the total cost of the vertices in A'. Apply Lemma 3 to find a set S_0' with at most $|V(B')|/s$ vertices and set G' to be the graph of radius $s-1$ that $B' - S_0'$ can be embedded into. Apply Lemma 1 to partition B' into A'', B'' and C'' with $|c(A'')| \leq |c(B'')| < \frac{2}{3}|c(B')|$. Now choose A to be the smaller (in vertex cost) of A' and B'' and B to be the larger one. Set G' to be A'', C to be

$C' \cup C'' \cup S_0 \cup S_0''$ and $j = 3$.

In general for $i = 1, 2, \ldots$, Step i can be described as follows:

Step i: For $j \le 6$, using Lemma 1, the n' vertices of G' are partitioned into A', B' and C' with $|c(A')| \le |c(B')| \le \frac{2}{3} |c(G')|$, and $|C'| \le 2s - 1$ where $c(A')$ denotes the total cost of the vertices in A'. Set A to be $A \cup A'$. Then set A to be the smaller of A and B, and B to be the larger one. If $j < 6$, apply Lemma 3 to the induced subgraph of B' and form G' of radius $s - 1$. Set $j = j + 1$ and repeat Step i. If $j = 6$, set i to be $i + 1$ and j to be 1. Then set s to be $\sqrt{\frac{|V(G')|}{12}}$ when applying Lemma 3 on B' and form G' of radius $s - 1$. Add $|V(G')|/s$ vertices to C and repeat Step i until G' becomes empty.

To verify that A, B, C satisfy the required conditions we note that in Step i, the new A and B has vertex-cost no more than the sum of $c(A')$ and the vertex-cost of the old A. Since the old A has vertex-cost no more than old B and $c(A') \le c(B')$, the new A and B has vertex-cost no more than 1/2 and new G' has vertex cost at most 2/3 of the old $|G'|$. Because of our choice of j, Step i is repeated 6 times (except Step 1 for 4 times). The total size of $C = C(n)$ can be bounded as follows:

$$|C(n)| \le 2\sqrt{12n} + \sum_{i \ge 1} 2\sqrt{12\, n \left[\frac{2}{3}\right]^{6i}} \le 2\sqrt{12n} + \left|C\left(n\left(\frac{2}{3}\right)^6\right)\right|$$

It can be easily shown by induction that

$$|C(n)| \le \frac{108\sqrt{3}}{19}\sqrt{n}$$

This completes the proof of Theorem 2.

V. Concluding remarks

There exists an n-vertex planar graph for which any separator bisecting the graph contains at least $\frac{2}{\sqrt{\pi}} \sqrt{n}$ ($\sim 1.128 \sqrt{n}$). So the best constant for planar separators lies inbetween 1.128 and 7.348 for (unweighted) planar graphs; and inbetween 1.178 and 9.845 for the weighted version. There is still considerable room for further improvement.

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