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SOME RESULTS ON HOOK LENGTHS

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A *tableau* is a rectangular array of points with the property that, for all i, the number of points in the ith row is greater than or equal to the number of points in the $(i+1)$st row. The *hook length* h_{ij} is defined to be the total number of points which are either directly to the right or directly below the (i, j)-point together with the (i, j)-point itself. It was conjectured by Logan and Shepp that a tableau is always uniquely determined (up to reflection) by its set of hook lengths. In this paper, we give several families of counterexamples to this conjecture. However, by extending the definition of hook length, we show that a tableau is always uniquely determined (up to reflection) by its extended set of hook lengths.

1. Introduction

A *tableau* * is a rectangular array of points with the property that the length of the ith row is greater than or equal to the length of the $(i+1)$st row. We illustrate an example in Fig. 1.

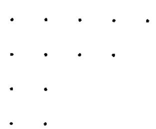

Fig. 1. A tableau λ.

We may denote the tableau $\lambda = \lambda(r_1 \geqslant r_2 \geqslant \cdots \geqslant r_t)$ by the set $\{(i, j)\colon 1 \leqslant j \leqslant r_i, 1 \leqslant i \leqslant t\}$ where r_i is the length of the ith row.

The *reflection* λ^* of the tableau λ is defined by

$$\lambda^* = \{(j, i) : (i, j) \in \lambda\}.$$

* The terminology and definitions follow that of [1, 6].

The reflection of the tableau in Fig. 1 is shown in Fig. 2.

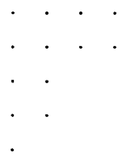

Fig. 2. The reflected tableau λ^*.

The *complement* $\bar{\lambda}$ of the tableau λ has row lengths $r_1 - r_i$, where $1 < i \leq t$ and $r_1 > r_i$, as shown in Fig. 3.

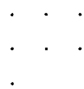

Fig. 3. The complementary tableau $\bar{\lambda}$.

The *hook length* h_{ij} of the (i, j)-point of the tableau λ is defined by

$$h_{ij} = (r_i - i) + (c_j - j) + 1$$

where c_j is the number of points in the jth column of the tableau λ. In other words, h_{ij} is the total number of points directly to the right or directly below the (i, j)-point in λ together with the (i, j)-point itself. As shown in Fig. 4, $h_{12} = 7$, the number of points in the shaded hook-shaped region.

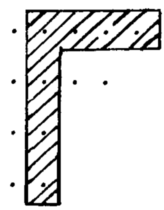

Fig. 4.

The *hook length* tableau of a tableau λ is a tableau with h_{ij} replacing the point at (i, j)-entry. The hook length tableau for the λ of Fig. 1 is given in Fig. 5.

8	7	4	3	1
6	5	2	1	
3	2			
2	1			

Fig. 5.

Let $H(\lambda)$ be the set of h_{ij}, $(i, j) \in \lambda$, counting multiplicity. We will use square brackets to indicate that repetitions are to be counted. ([1, 1, 3] is often denoted by $[1^2, 3]$.) Of course, $H(\lambda) = H(\lambda^*)$.

The set $H(\lambda)$ of hook lengths was first considered by Nakayama [5] in connection with modular representation theory, although it was implicit in the work of Young on the representation of the symmetric group [8, 9]. Since then, $H(\lambda)$ has been studied extensively and has played an important part in the development of group representation theory (see [2, 4, 7]).

A conjecture which was raised by Logan and Shepp in connection with their recent paper [3] is that λ is uniquely determined by $H(\lambda)$ up to reflection, i.e., $H(\lambda) = H(\lambda')$ implies $\lambda' = \lambda$ or $\lambda' = \lambda^*$. In this paper, we show that the conjecture is false. Counterexamples are given in Section 3.

Because of the fact that $h_{ij} = h_{1j} + h_{i1} - h_{11}$, $(i, j) \in \lambda$, (see Section 2), h_{ij} could be defined more generally to be

$$h_{ij} = h_{1j} + h_{i1} - h_{11}, \qquad 1 \leq i \leq r_1, \qquad 1 \leq j \leq c_1.$$

With this extended definition, h_{ij} is negative for $(i, j) \notin \lambda$ and positive otherwise. It can easily be seen that h_{ij} cannot be zero.

Define $\bar{H}(\lambda) = [h_{ij} : 1 \leq i \leq r_1,\ 1 \leq j \leq c_1]$.

We note that $\bar{H}(\lambda)$ is the union of the set of hook length $H(\lambda)$ and the set of negative values of hook lengths $H(\bar{\lambda})$.

We ask the following question: Is λ uniquely determined by $\bar{H}(\lambda)$ up to reflection? In section 4 we show that the answer is affirmative.

2. Some basic properties of $H(\lambda)$

First, we state some basic properties of hook lengths. The proofs can be found in [1].

Fact 1. For all $i < i'$, $j < j'$, we have $h_{ij} + h_{i'j'} = h_{ij'} + h_{i'j}$.

Fact 2. Given $(i, j) \in \lambda$, the sequence $(h_{i,j}, h_{i,j+1}, \ldots, h_{i,r_i}, h_{i,j} - h_{i+1,j}, h_{i,j} - h_{i+2,j}, \ldots, h_{i,j} - h_{c_j,j})$ is a permutation of the set of integers $\{1, 2, \ldots, h_{i,j}\}$. In particular, we have the following:

Fact 3. Let $h_{11} = n$. The sequence $(h_{11}, h_{12}, \ldots, h_{1,r_1}, n - h_{21}, n - h_{31}, \ldots, n - h_{c_1,1})$ is a permutation of $\{1, 2, \ldots, n\}$.

Fact 4. The hook length tableau is determined by the set $\{h_{11}, h_{12}, \ldots, h_{1,r_1}\} \subseteq \{1, 2, \ldots, n\}$, where $n = h_{11}$. Also, every subset of $\{1, 2, \ldots, n\}$ which contains n determines a hook length tableau with $h_{11} = n$.

Fact 5. Suppose we are given a subset $A \subseteq \{1, 2, \ldots, n\}$ with $n \in A$. Let λ_A be the tableau determined by A. Then we have

$$H(\lambda_A) = [a - \bar{a} : a \in A,\ \bar{a} \in \{0, 1, 2, \ldots, n\} \backslash A,\ a > \bar{a}],$$

$$\bar{H}(\lambda_A) = [a - \bar{a} : a \in A,\ \bar{a} \in \{0, 1, 2, \ldots, n\} \backslash A].$$

Fact 6. λ_{A^*} is the reflection of λ_A if and only if

$$A^* = \{n - \bar{a} : \bar{a} \in \{0, 1, 2, \ldots, n\} \backslash A\},$$

i.e.,

$$\lambda_A^* = \lambda_{A^*}.$$

In this paper, we will show that A is uniquely determined up to reflection by the set $\bar{H}(\lambda_A)$ but not by $H(\lambda_A)$.

3. Counterexamples

Let $n = 9 = h_{11}$ and consider the sets A and B given by

$$A = \{1, 2, 4, 8, 9\},$$

$$B = \{2, 4, 5, 9\}.$$

It is easily seen that $B \neq \{n - \bar{a} : \bar{a} \in \{0, 1, 2, \ldots, 9\} \setminus A\} = A^*$. Hence λ_A is not the reflection of λ_B. The hook length tableaus for λ_A, λ_B are shown in Fig. 6.

9	8	4	2	1
6	5	1		
4	3			
3	2			
2	1			

$H(\lambda_A)$
(a)

9	5	4	2
8	4	3	1
6	2	1	
3			
2			
1			

$H(\lambda_B)$
(b)

Fig. 6.

However, since $H(\lambda_A) = H(\lambda_B) = [1^3, 2^3, 3^2, 4^2, 5, 6, 8, 9]$, then λ does not determine $H(\lambda)$ up to reflection.

There are, as a matter of fact, infinitely many such pairs of tableaus with the same set of hook lengths. One such family is given as follows.

Let $n \geq 9$ and choose

$$A_n = \{1, 2, n-5, n-1, n\},$$

$$B_n = \{2, n-5, n-4, n\}.$$

A direct calculation shows that $A_n \neq B_n^*$ and

$$H(\lambda_{A_n}) = H(\lambda_{B_n}).$$

Another such family is given below.

Let $n \geq 12$ and choose

$$A_n = \{1, 2, 4, n-7, n-2, n\},$$

$$B_n = \{1, 4, n-7, n-5, n\}.$$

Similarly we have $A_n \neq B_n^*$ and

$$H(\lambda_{A_n}) = H(\lambda_{B_n}) \quad \text{for } n \geq 12.$$

It is not known how many such families exist.

4. The uniqueness of $\bar{H}(\lambda)$

Let A, B be subsets of $\{0,1,2,\ldots,n\}$ with $n \in A \cap B$ and $0 \notin A \cup B$. We now show that $\bar{H}(\lambda_B) = \bar{H}(\lambda_A)$ implies $B = A$ or $B = A^*$, i.e., A is uniquely determined by $\bar{H}(\lambda_A)$ up to reflection.

Let $p(X) = \sum_{i=0}^{n} x^i$. For any $S \subseteq \{0,1,2,\ldots,n\}$, we define

$$f_S(x) = \sum_{i \in S} x^i = \sum_{i=1}^{n} \delta_S(i) x^i \quad \text{where } \delta_S(i) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $f_{A^*}(x) = p(x) - x^n f_A(1/x)$.

From Facts 1 and 5, it is easily seen that

$$f_A(x) f_{A^*}(x) = \sum_{i=-n}^{n} \alpha_i x^{n+i},$$

where α_i is the number of times i occurs in $\bar{H}(\lambda_A)$. $\bar{H}(\lambda_A) = \bar{H}(\lambda_B)$ will then imply

$$f_A(x) f_{A^*}(x) = f_B(x) f_{B^*}(x).$$

In order to prove the main theorem we need the following lemma.

Lemma. *If $\bar{H}(\lambda_A) = \bar{H}(\lambda_B)$ then $i \in A \setminus B$ if and only if $n - i \in A \setminus B$.*

Proof. Since $f_A(x) f_{A^*}(x) = f_B(x) f_{B^*}(x)$, we have

$$f_A(x)\left(p(x) - x^n f_A\left(\frac{1}{x}\right)\right) = f_B(x)\left(p(x) - x^n f_B\left(\frac{1}{x}\right)\right) \tag{1}$$

i.e.,

$$p(x)(f_A(x) - f_B(x)) = x^n\left(f_A(x) f_A\left(\frac{1}{x}\right) - f_B(x) f_B\left(\frac{1}{x}\right)\right). \tag{2}$$

We note that the coefficients of x^i and x^{-i} are the same in the expansion of $f_A(x) f_A(1/x)$. Hence, the coefficients of x^{n+i} and x^{n-i} in $p(x)(f_A(x) - f_B(x))$ are also the same. Thus,

$$p(x)(f_A(x) - f_B(x)) = \sum_{i=0}^{n} x^i \left(\sum_{j=0}^{n} \delta_A(j) x^j - \sum_{j=0}^{n} \delta_B(j) x^j\right) = \sum_{i=0}^{n} \sum_{j=0}^{n} (\delta_A(j) - \delta_B(j)) x^{i+j}$$

$$= \sum_{k=n}^{2n} \left(\sum_{m=k-n}^{n} (\delta_A(m) - \delta_B(m)) \right) x^k$$
$$+ \sum_{k=0}^{n-1} \left(\sum_{m=0}^{k} (\delta_A(m) - \delta_B(m)) \right) x^k.$$

Hence we have

$$\sum_{j=i}^{n} (\delta_A(j) - \delta_B(j)) = \sum_{j=0}^{n-i} (\delta_A(j) - \delta_B(j)) \quad \text{for } i = 0, 1, \ldots, n.$$

Then

$$\sum_{j=i}^{n} (\delta_A(j) - \delta_B(j)) - \sum_{j=i+1}^{n} (\delta_A(j) - \delta_B(j)) = \sum_{j=0}^{n-i} (\delta_A(j) - \delta_B(j)) - \sum_{j=0}^{n-i-1} (\delta_A(j) - \delta_B(j))$$

i.e.,

$$\delta_A(i) - \delta_B(i) = \delta_A(n-i) - \delta_B(n-i) \quad \text{for } i = 0, 1, \ldots, n.$$

This implies immediately that $i \in A \setminus B$ if and only if $n - i \in A \setminus B$. This proves the lemma.

Now we let $A' = A \setminus B$, $B' = B \setminus A$ and $X = A \cap B$. We can rewrite the equality (1) as

$$(f_X(x) + f_{A'}(x)) \left(p(x) - x^n \left(f_X\left(\frac{1}{x}\right) + f_{A'}\left(\frac{1}{x}\right) \right) \right) =$$
$$= (f_X(x) + f_{B'}(x)) \left(p(x) - x^n \left(f_X\left(\frac{1}{x}\right) + f_{B'}\left(\frac{1}{x}\right) \right) \right).$$

Then

$$f_{A'}(x) \left(p(x) - x^n \left(f_X\left(\frac{1}{x}\right) + f_{A'}\left(\frac{1}{x}\right) \right) \right) - f_X(x) f_{A'}\left(\frac{1}{x}\right) x^n$$
$$= f_{B'}(x) \left(p(x) - x^n \left(f_X\left(\frac{1}{x}\right) + f_{B'}\left(\frac{1}{x}\right) \right) \right) - f_X(x) f_{B'}\left(\frac{1}{x}\right) x^n. \tag{4}$$

From the lemma we know that

$$x^n f_{A'}\left(\frac{1}{x}\right) = f_{A'}(x)$$

and

$$x^n f_{B'}\left(\frac{1}{x}\right) = f_{B'}(x).$$

Hence (4) is equivalent to the following equation:

$$f_{A'}(x) \left(p(x) - x^n f_X\left(\frac{1}{x}\right) - f_{A'}(x) - f_X(x) \right)$$
$$= f_{B'}(x) \left(p(x) - x^n f_X\left(\frac{1}{x}\right) - f_{B'}(x) - f_X(x) \right),$$

i.e.,

$$(f_{A'}(x)-f_{B'}(x))\left(p(x)-x^nf_X\left(\frac{1}{x}\right)-f_X(x)-f_{A'}(x)-f_{B'}(x)\right)=0.$$

Assume $A \neq B$. Then $f_{A'}(x) \neq f_{B'}(x)$. We will have

$$p(x)-x^nf_X\left(\frac{1}{x}\right)-f_X(x)-f_{A'}(x)-f_{B'}(x)=0$$

$$p(x)-x^nf_X\left(\frac{1}{x}\right)-f_{A'}(x)=f_X(x)+f_{B'}(x),$$

i.e.,

$$p(x)-x^n\left(f_X\left(\frac{1}{x}\right)+f_{A'}\left(\frac{1}{x}\right)\right)=f_B(x),$$

$$p(x)-x^nf_A\left(\frac{1}{x}\right)=f_B(x),$$

$$f_{A^*}(x)=f_B(x),$$

$$A^*=B.$$

λ_B is then the reflection of λ_A. Hence we have proved the following theorem.

Theorem. *The hook length set λ is uniquely determined by $\bar{H}(\lambda)$ up to reflection.*

5. Some related problems

The preceding results suggest a number of related problems, which we now mention:

(i) What are the necessary and sufficent conditions for λ to be uniquely determined by $H(\lambda)$ up to reflection?

(ii) For a given tableau λ, how many tableaus have the same hook length set as $H(\lambda)$?

(iii) In Section 3, we illustrated two families which contain infinitely many pairs of tableaus with the same set of hook lengths. Can the structure of such families be characterized? In the given examples, we use one parameter. Are there such families with two or more parameters?

(iv) Let τ_n be the number of tableaus which are not uniquely determined by the hook length set $H(\lambda)$ with the largest hook length being n, i.e.,
$\tau_n = |\{\{\lambda, \lambda^*\}$: There exists some λ' with $H(\lambda)=H(\lambda')$, $\lambda' \neq \lambda$, $\lambda' \neq \lambda^*$ and the largest number in $H(\lambda)$ is $n\}|$.

In the Table 1, we list the value of τ_n for some small values of n. What is τ_n in general?

Table 1

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
τ_n	0	0	0	0	0	0	0	0	2	4	8	10	14	26	...

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