

# Isoperimetric inequalities for cartesian products of graphs

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## Abstract

We give a characterization for isoperimetric invariants, including the Cheeger constant and the isoperimetric number of a graph. This leads to an isoperimetric inequality for the cartesian products of graphs.

## 1 Introduction

For a graph  $G$  and a subset  $S$  of vertex set  $V(G)$  of  $G$ , the *edge-boundary*  $\partial S$  of  $S$  consists of all edges with exactly one endpoint in  $S$ :

$$\partial S = \{\{u, v\} \in E(G) : u \in S \text{ and } v \notin S\}$$

Let  $\bar{S}$  denote the complement of  $S$ , i.e.,  $\bar{S} = V - S$ . Clearly,  $\partial S = \partial \bar{S} = E(S, \bar{S})$  where  $E(A, B)$  denotes the set of edges with one endpoint in  $A$  and one endpoint in  $B$ .

There are two types of isoperimetric invariants which are often mentioned in the literature:

- (1) The *Cheeger constant* of  $G$  is defined (see [5, 6]) to be

$$h(G) = \inf_S \frac{|E(S, \bar{S})|}{\min(\text{vol } S, \text{vol } \bar{S})}. \quad (1)$$

where the *volume* of  $S$ , denoted by  $\text{vol } S$ , is the sum of all degrees  $d_x$  for  $x \in S$ .

- (2) The *isoperimetric number* of  $G$  is defined (see [11]) to be

$$i(G) = \inf_S \frac{|E(S, \bar{S})|}{\min(|S|, |\bar{S}|)}. \quad (2)$$

For a weighted graph  $G$  with vertex-weight  $w(v)$ , for  $v \in V(G)$  and edge-weight  $w(u, v) = w(v, u)$ , we can define the isoperimetric invariant  $h(G, w)$ :

$$h(G, w) = \inf_S \frac{\sum_{\{u,v\} \in E(S, \bar{S})} w(u, v)}{\min(\sum_{u \in S} w(u), \sum_{v \in \bar{S}} w(v))}. \quad (3)$$

We say the weight function  $w$  is *consistent* if

$$\sum_u w(u, v) = w(v)$$

For example, the Cheeger constant is obtained by using the weight function  $w_0(u, v) = 1$  for any edge  $\{u, v\}$  and  $w_0(v) = d_v$  for any vertex  $v$ . Clearly,  $w_0$  is consistent. On the other hand, the isoperimetric number is just  $i(G) = h(G, w_1)$  where the weight function  $w_1$  satisfies  $w_1(u, v) = 1$  for any edge  $\{u, v\}$  and  $w_1(v) = 1$  for any vertex  $v$ . Of course,  $w_1$  is not consistent. We note that graphs with consistent weight functions correspond in a natural way to random walks and reversible Markov chains. Namely, for a graph with a consistent weight function  $w$ , we can define the random walk with transition probability of moving from a vertex  $u$  to each of its neighbors  $v$  to be

$$P(u, v) = \frac{w(u, v)}{w(u)} \quad (4)$$

For further discussions, the reader is referred to [1, 6].

First, we will establish the following characterizations for  $h(G, w)$ .

**Theorem 1** *For a graph  $G$  with weight function  $w$ , the isoperimetric invariant  $h(G, w)$  of a graph  $G$  satisfies*

$$h(G, w) = \inf_{f \neq 0} \sup_{c \in R} \frac{\sum_{x \sim y} |f(x) - f(y)| w(x, y)}{\sum_{x \in V} |f(x) - c| w(x)} \quad (5)$$

where  $f$  ranges over all  $f : V \rightarrow R$  which are not identically zero.

As an immediate consequence of Theorem 1, we have the following characterization for the Cheeger constant and the isoperimetric number.

$$h(G) = \inf_{f \neq 0} \sup_{c \in R} \frac{\sum_{x \sim y} |f(x) - f(y)|}{\sum_{x \in V} |f(x) - c| d_x} \quad (6)$$

where  $f$  ranges over all  $f : V \rightarrow R$  which are not identically zero. The isoperimetric number  $i(G)$  of a graph  $G$  satisfies

$$i(G) = \inf_{f \neq 0} \sup_{c \in R} \frac{\sum_{x \sim y} |f(x) - f(y)|}{\sum_{x \in V} |f(x) - c|} \quad (7)$$

where  $f$  ranges over all  $f : V \rightarrow R$  which are not identically zero.

For two graphs  $G$  and  $H$ , the *cartesian product*  $GH$  has vertex set  $V(G) \times V(H)$  with  $(u, v)$  adjacent to  $(u', v')$  if and only if  $u = u'$  and  $v$  is adjacent to  $v'$  in  $H$ , or  $v = v'$  and  $u$  is adjacent to  $u'$  in  $G$ . There are many research papers examining the isoperimetric number for various graphs. For example, Wang and Wang [12] and Bollobas and Leader [3, 4] studied the isoperimetric number for grids (which are cartesian products of paths) and torus (which are cartesian products of cycles). In particular, there is a large literature examining the isoperimetric problems for  $n$ -cube which is just the  $n$ -fold cartesian product of a single edge (see Harper [7], Lindsey [10], Bernstein [2] and Hart [8]). Also, Mohar [11] considered the special case of the cartesian product of a graph and a complete graph. In this paper, we consider cartesian products of general graphs. We will establish the relationship between the isoperimetric invariants of graphs and their cartesian products. The proof is by using a variation of the above characterization in Theorem 1.

**Theorem 2** *The isoperimetric number of the cartesian product of  $G_1, G_2, \dots, G_k$  satisfies*

$$\begin{aligned} \min(i(G_1), i(G_2), \dots, i(G_k)) &\geq i(G_1 G_2 \cdots G_k) \\ &\geq \frac{1}{2} \min(i(G_1), i(G_2), \dots, i(G_k)) \end{aligned}$$

We remark that a Markov chains analog of Theorem 2 was proved in [9] with a somewhat complicated proof and a weaker constant ( $2/9$  instead of  $1/2$ ).

We will consider a cartesian product of weighted graphs with consistent weight functions. For two weighted graphs  $G$  and  $G'$ , with weight functions  $w, w'$ , respectively, the weighted cartesian product  $G \otimes G'$  has vertex set  $V(G) \times V(G')$  with weight function  $w \otimes w'$  defined as follows: For an edge  $\{u, v\}$  in  $E(G)$ , we define  $w \otimes w'((u, v'), (v, v')) = w(u, v)w'(v')$  and for an edge  $\{u', v'\}$  in  $E(G')$ , we define  $w \otimes w'((u, u'), (u, v')) = w(u)w'(u', v')$ . We require  $w \otimes w'$  to be consistent. Clearly, for a vertex  $x = (u, v)$  in  $G \otimes G'$ , the weight of  $x$  in  $G \otimes G'$  is exactly  $2w(u)w'(v)$ .

In general, for graphs  $G_i$  with consistent weight functions  $w_i, i = 1, \dots, k$ , the weighted cartesian product  $G_1 \otimes \cdots \otimes G_k$  has vertex set  $V(G) \otimes \cdots \otimes V(G_k)$  with a consistent weight function  $w_1 \otimes \cdots \otimes w_k$

defined as follows: For an edge  $\{u, v\}$  in  $E(G_i)$ , the edges joining  $(v_1, \dots, v_{i-1}, u, v_{i+1}, \dots, v_k)$  and  $(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$  has weight  $w_1(v_1) \cdots w_{i-1}(v_{i-1})w_i(u, v)w_{i+1}(v_{i+1}), \dots, w_k(v_k)$ .

For a graph  $G$ , the natural consistent weight function associated with  $G$  is with edge weight 1 and vertex weight  $d_x$  for any vertex  $x$ . (Here, we consider a graph as a weighted graph with the natural consistent weight function.)

**Theorem 3** *The Cheeger constant of a weighted cartesian product of  $G_1, G_2, \dots, G_k$  satisfies*

$$\begin{aligned} \min_j (h(G_j)) &\geq h(G_1 \otimes G_2 \otimes \cdots \otimes G_k) \\ &\geq \frac{1}{2k} \min_j (h(G_j)) \end{aligned}$$

We remark that the weighted cartesian product of graphs corresponds to the cartesian product of random walks on graphs. Suppose  $G_1, \dots, G_k$  are weighted graphs with the vertex set  $V(G_i)$ . Each  $G_i$  associates with a random walk with transition probability  $P_i$  as defined as in 4. The cartesian product of the random walks can be defined as follows: At the vertex  $(v_1, \dots, v_k)$ , first choose a random “direction”  $i$ , between 1 and  $k$ , each with probability  $1/k$ . Then move to the vertex  $(v_1, \dots, v_{i-1}, u_i, v_{i+1}, \dots, v_k)$  according to  $P_i$ . In other words,

$$P((v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, u_k), (v_1, \dots, v_{i-1}, u_i, v_{i+1}, \dots, v_k)) = \frac{1}{k} P(v_i, u_i)$$

## 2 A characterization of the isoperimetric invariant

In this section, we will give the proof of Theorem 1 for a graph  $G$  with a weight function  $w$  by showing:

$$h(G, w) = \inf_{f \neq 0} \sup_{c \in \mathbb{R}} \frac{\sum_{x \sim y} |f(x) - f(y)| w(x, y)}{\sum_{x \in V} |f(x) - c| w(x)}$$

*Proof of Theorem 1:* We choose  $c$  such that

$$\sum_{x: f(x) < c} w(x) \leq \sum_{x: f(x) \geq c} w(x)$$

and

$$\sum_{x: f(x) \leq c} w(x) \leq \sum_{x: f(x) > c} w(x)$$

Let  $g = f - c$ . We consider

$$\tilde{g}(\sigma) = \sum_{g(x) \leq \sigma < g(y)} w(x, y)$$

Then we have

$$\begin{aligned} \sum_{x \sim y} |f(x) - f(y)| w(x, y) &= \int_{-\infty}^{\infty} \tilde{g}(\sigma) d\sigma \\ &= \int_{-\infty}^0 d\sigma \frac{\tilde{g}(\sigma)}{\sum_{g(x) < \sigma} w(x)} \sum_{g(x) < \sigma} w(x) \\ &\quad + \int_0^{\infty} d\sigma \frac{\tilde{g}(\sigma)}{\sum_{g(x) > \sigma} w(x)} \sum_{g(x) > \sigma} w(x) \\ &\geq h(G, w) \left( \int_{-\infty}^0 d\sigma \sum_{g(x) < \sigma} w(x) + \int_0^{\infty} d\sigma \sum_{g(x) > \sigma} w(x) \right) \\ &= h(G, w) \sum_{x \in V} |f(x) - c| w(x) \end{aligned}$$

In the opposite direction, suppose  $X$  is a subset of  $V$  satisfying

$$h(G, w) = \frac{\sum_{\{x, y\} \in E(X, \bar{X})} w(x, y)}{\sum_{x \in X} w(x)}$$

We consider the following *characteristic* function  $\chi$  of  $X$ :

$$\chi(x) = \begin{cases} 1 & \text{if } x \in X \\ -1 & \text{otherwise} \end{cases}$$

Then we have, for any  $C$ ,

$$\sup_C \frac{\sum_{x \sim y} |\chi(x) - \chi(y)| w(x, y)}{\sum_{x \in V} |\chi(x) - C| w(x)} = \sup_C \frac{\sum_{\{x, y\} \in E(X, \bar{X})} 2w(x, y)}{(1 - C) \sum_{x \in X} w(x) + (1 + C) \sum_{x \in \bar{X}} w(x)}$$

The supremum is achieved when  $C = -1$ , and we have

$$\sup_C \frac{\sum_{x \sim y} |\chi(x) - \chi(y)| w(x, y)}{\sum_{x \in V} |\chi(x) - C| w(x)} = h(G, w)$$

Therefore, we have

$$h(G, w) \geq \inf_{f \neq 0} \sup_{c \in R} \frac{\sum_{x \sim y} |f(x) - f(y)| w(x, y)}{\sum_{x \in V} |f(x) - c| w(x)}$$

and Theorem 1 is proved.

To derive the isoperimetric relationship between graphs and their cartesian products, we need the following variation of Theorem 1:

**Corollary 1** *For a graph  $G$ , we have*

$$h(G, w) \geq \inf_f \frac{\sum_{x \sim y} |f(x) - f(y)| w(x, y)}{\sum_{x \in V} |f(x)| w(x)} \geq \frac{1}{2} h(G, w)$$

where  $f : V(G) \rightarrow R$  satisfies

$$\sum_{x \in V} f(x) w(x) = 0. \tag{8}$$

*Proof:* From Theorem 1, we already have

$$h(G, w) \geq \inf_f \frac{\sum_{x \sim y} |f(x) - f(y)| w(x, y)}{\sum_{x \in V} |f(x)| w(x)}$$

for  $f$  satisfies (8). It remains to prove the second part of the inequality. Suppose we define  $c$  as in the proof of Theorem 1. If  $c \leq 0$ , then we have

$$\begin{aligned} \sum_x |f(x) - c| w(x) &\geq \sum_{x: f(x) \geq 0} |f(x) - c| w(x) \\ &\geq \sum_{x: f(x) \geq 0} |f(x)| w(x) \\ &\geq \frac{1}{2} \sum_x |f(x)| w(x) \quad \text{by (7)}. \end{aligned}$$

Similarly, if  $c \geq 0$ , then we have

$$\begin{aligned} \sum_x |f(x) - c| w(x) &\geq \sum_{x: f(x) \leq 0} |f(x) - c| w(x) \\ &\geq \sum_{x: f(x) \leq 0} |f(x)| w(x) \\ &\geq \frac{1}{2} \sum_x |f(x)| w(x) \quad \text{by (7)}. \end{aligned}$$

Therefore we have

$$\inf_f \frac{\sum_{x \sim y} |f(x) - f(y)| w(x, y)}{\sum_{x \in V} |f(x)| w(x)} \geq \frac{1}{2} h(G, w)$$

This completes the proof of the corollary.

### 3 An isoperimetric inequality for Cartesian products

We first derive Theorem 2 concerning the isoperimetric number of the cartesian product of graphs  $G_j$ ,  $j = 1, \dots, k$ . (This is the case with the weight function  $w(v) = 1$  for each vertex  $v$ .)

$$\begin{aligned} \min(i(G_1), i(G_2), \dots, i(G_k)) &\geq i(G_1 G_2 \cdots G_k) \\ &\geq \frac{1}{2} \min(i(G_1), i(G_2), \dots, i(G_k)) \end{aligned}$$

*Proof of Theorem 2:* Without loss of generality, we assume that

$$i(G_1) \leq i(G_j), \text{ for } j = 1, \dots, k.$$

It is easy to see that

$$i(G_1 G_2 \cdots G_k) \leq i(G_1)$$

since we can choose a function  $h : V(G) \times V(H) \rightarrow R$  by setting

$$h(v_1, \dots, v_k) = f(v_1)$$

where  $f : V(G) \rightarrow R$  is a function achieving  $i(G_1)$  in (7).

To prove

$$i(G_1 G_2 \cdots G_k) \geq \frac{1}{2} \min(i(G_1), i(G_2), \dots, i(G_k))$$

we consider a function  $g : V(G_1) \times \cdots \times V(G_k) \rightarrow R$  which achieves the value  $i(G_1 G_2 \cdots G_k)$  in (7).

For vertices  $v_1 \in V(G_1), \dots, v_k \in V(G_k)$ , we consider functions  $\bar{g}_j$  for  $j = 1, \dots, k$ .

$$\bar{g}_j(v_1, \dots, v_k) = \frac{\sum_{u_1, \dots, u_j} g(u_1, \dots, u_j, v_{j+1}, \dots, v_k)}{|V(G_1)| \cdots |V(G_j)|}$$

Note that

$$\bar{g}_j(v_1, \dots, v_k) = \frac{1}{|V(G_j)|} \sum_{u_j} \bar{g}_{j-1}(v_1, \dots, v_{j-1}, u_j, v_{j+1}, \dots, v_k) \quad (9)$$

In particular,  $\bar{g}_k$  is a constant function, namely, the average of all values of  $g$ . We have

$$i(G_1 G_2 \cdots G_k) \geq \frac{I_1 + \cdots + I_k}{\sum_{v_1, \dots, v_k} |g(v_1, \dots, v_k) - \bar{g}_k|}$$

where

$$I_1 = \sum_{v_2, \dots, v_k} \sum_{v_1 \sim v'_1} |g(v_1, v_2, \dots, v_k) - g(v'_1, v_2, \dots, v_k)|$$

and

$$I_j = \sum_{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k} \sum_{v_j \sim v'_j} |g(v_1, \dots, v_{j-1}, v_j, v_{j+1}, \dots, v_k) - g(v_1, \dots, v_{j-1}, v'_j, v_{j+1}, \dots, v_k)|$$

Using Corollary 1, we have

$$I_1 \geq \frac{1}{2} \sum_{v_2, \dots, v_k} i(G_1) \sum_{v_1} |g(v_1, v_2, \dots, v_k) - \bar{g}_1(v_1, v_2, \dots, v_k)|$$

By using the definition of  $\bar{g}_{j-1}$  and the triangle inequality, we have, for  $j \geq 2$ ,

$$I_j \geq |V(G_1)| \cdots |V(G_{j-1})| \sum_{v_{j+1}, \dots, v_k} \sum_{v_j \sim v'_j} |\bar{g}_{j-1}(v_1, \dots, v_{j-1}, v_j, v_{j+1}, \dots, v_k) - \bar{g}_{j-1}(v_1, \dots, v_{j-1}, v'_j, v_{j+1}, \dots, v_k)|$$

Now, applying the second part of the inequality in Corollary 1 for each copy of  $G_j$ , and using (8), we get

$$\begin{aligned} I_j &\geq \frac{1}{2} i(G_j) |V(G_1)| \cdots |V(G_{j-1})| \sum_{v_{j+1}, \dots, v_k} \sum_{v_j} |\bar{g}_{j-1}(v_1, \dots, v_k) - \bar{g}_j(v_1, \dots, v_k)| \\ &= \frac{1}{2} i(G_j) \sum_{v_1, \dots, v_k} |\bar{g}_{j-1}(v_1, \dots, v_k) - \bar{g}_j(v_1, \dots, v_k)| \end{aligned}$$

where we used the fact that  $\bar{g}_{j-1}$  and  $\bar{g}_j$  do not depend on the particular choice of the first  $j-1$  variables.

We also note that

$$\begin{aligned} \sum_{v_1, \dots, v_k} |g(v_1, \dots, v_k) - g_k| &\leq \sum_{v_1, \dots, v_k} |g(v_1, \dots, v_k) - g_1(v_1, \dots, v_k)| + \cdots \\ &\quad + |g_{k-1}(v_1, \dots, v_k) - g_k| \end{aligned}$$

So, by comparing term by term, Theorem 2 is proved.



*Sketch of Theorem 3:*

The analogous version for weighted cartesian product can be proved in essentially the same way (with the extra complexity of notation). Here, we will describe part of the proof which illustrates the qualitative difference from the proof of Theorem 2 and omit the rest of the proof of Theorem 3.

Without loss of generality, we assume  $h(G_1) = h(G_j)$  for all  $j$ . Then it follows from the proof above that

$$I_j = \sum_{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k} \sum_{v_j \sim v'_j} |g(v_1, \dots, v_{j-1}, v_j, v_{j+1}, \dots, v_k) - g(v_1, \dots, v_{j-1}, v'_j, v_{j+1}, \dots, v_k)| d_{v_1} \dots d_{v_{i-1}} w(v_j, v'_j) d_{v_{j+1}} \dots d_{v_k}$$

Therefore, we have

$$\begin{aligned} h(G_1 \otimes \dots \otimes G_k) &\geq \frac{I_1 + I_2 + \dots + I_k}{\sum_{v_1, \dots, v_k} |g(v_1, \dots, v_k) - \bar{g}_k| d_{(v_1, \dots, v_k)}} \\ &\geq \frac{\frac{1}{2} h(G_1) \sum_{j=1}^k \sum_{v_1, \dots, v_k} |\bar{g}_{j-1}(v_1, \dots, v_k) - \bar{g}_j(v_1, \dots, v_k)| d_{v_1} \dots d_{v_j}}{\sum_{v_1, \dots, v_k} |g(v_1, \dots, v_k) - \bar{g}_k| d_{(v_1, \dots, v_k)}} \end{aligned}$$

where  $\bar{g}_0 = g$ , and  $d_{(v_1, \dots, v_k)}$  represents the degree of the vertex  $(v_1, \dots, v_k)$  in the product graph.

Thus

$$\begin{aligned} h(G_1 \otimes \dots \otimes G_k) &\geq \frac{1}{2} h(G_1) \frac{\sum_{j=1}^k \sum_{v_1, \dots, v_k} |\bar{g}_{j-1}(v_1, \dots, v_k) - \bar{g}_j(v_1, \dots, v_k)| d_{v_1} \dots d_{v_k}}{\sum_{v_1, \dots, v_k} |g(v_1, \dots, v_k) - \bar{g}_k| d_{v_1} \dots d_{v_k}} \\ &\geq \frac{1}{2k} h(G_1), \end{aligned}$$

by, again, the triangle inequality.

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