

# An upper bound for the Turán number $t_3(n, 4)^*$

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## Abstract

Let  $t_r(n, r+1)$  denote the smallest integer  $m$  such that every  $r$ -uniform hypergraph on  $n$  vertices with  $m+1$  edges must contain a complete graph on  $r+1$  vertices. In this paper, we prove that

$$\lim_{n \rightarrow \infty} \frac{t_3(n, 4)}{\binom{n}{3}} \leq \frac{3 + \sqrt{17}}{12} = .593592 \dots$$

## 1 Introduction

For an  $r$ -uniform hypergraph  $H$  (or  $r$ -graph, for short), we denote by  $t_r(n, H)$  the smallest integer  $m$  such that every  $r$ -graph on  $n$  vertices with  $m+1$  edges must contain  $H$  as a subgraph. When  $H$  is a complete graph on  $k$  vertices, we write  $t_r(n, k) = t_r(n, H)$ . In 1941, Turán [10] determined the Turán number  $t_2(n, k)$  for 2-graphs and he asked the problem of determining the limit

$$\lim_{n \rightarrow \infty} \frac{t_r(n, k)}{\binom{n}{r}},$$

for  $2 < r < k$ . For this problem, Erdős offered \$1000 in honor of Paul Turán, (see [1] and [10]). Since 1941, the above problem has remained open, even for the first non-trivial case of  $r = 3$  and  $k = 4$ . The exact value for Turán number  $t_3(n, 4)$  is conjectured [10] as follows:

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*Conjecture:*

$$t_3(n, 4) = \begin{cases} k^2(5k - 3)/2 & \text{if } n = 3k, \\ k(5k^2 + 2k - 1)/2 & \text{if } n = 3k + 1, \\ k(k + 1)(5k + 2)/2 & \text{if } n = 3k + 2, \end{cases}$$

For small values of  $n$ , the conjectured values for  $t_3(n, 4)$ ,  $n \leq 13$ , have been verified [8]. The above conjecture, if true, would give

$$\lim_{n \rightarrow \infty} \frac{t_3(n, 4)}{\binom{n}{3}} = \frac{5}{9}.$$

For the lower bound, Kostochka [5] gave several different constructions which achieve the conjectured value for  $t_3(n, 4)$ . For the upper bound for  $t_3(n, 4)/\binom{n}{3}$ , de Caen [3] gave an upper bound of  $0.6213 \dots$  which is the real root of  $9x^3 - 33x^2 + 46x - 18$ . The best upper bound is due to Giraud (unpublished, see [3]) who proved

$$\lim_{n \rightarrow \infty} \frac{t_3(n, 4)}{\binom{n}{3}} \leq \frac{\sqrt{21} - 1}{6} = .5971 \dots$$

We will show the following:

**Theorem 1**

$$\lim_{n \rightarrow \infty} \frac{t_3(n, 4)}{\binom{n}{3}} \leq \frac{3 + \sqrt{17}}{12} = .5936 \dots$$

For general  $r$ -graphs, de Caen [2], Sidorenko [6], Tazawa and Shirakura [9] proved

$$\lim_{n \rightarrow \infty} \frac{t_r(n, r + 1)}{\binom{n}{r}} \leq 1 - \frac{1}{r}$$

Giraud (unpublished) improved this upper bound to

$$\lim_{n \rightarrow \infty} \frac{t_r(n, r + 1)}{\binom{n}{r}} \leq 1 - \frac{2}{r(1 + \sqrt{\frac{r}{r+4}})} \quad \text{for odd } r.$$

By similar methods as in the proof of Theorem 1, we have the following improvement:

**Theorem 2** *For any odd  $r \geq 3$ , we have*

$$\lim_{n \rightarrow \infty} \frac{t_r(n, r + 1)}{\binom{n}{r}} \leq 1 - \frac{5r + 12 - \sqrt{9r^2 + 24r}}{2r(r + 3)}.$$

We remark that the best lower bound was due to A. Sidorenko [7]:

$$\lim_{n \rightarrow \infty} \frac{t_r(n, r+1)}{\binom{n}{r}} \geq 1 - (0.5 + o_r(1)) \frac{\log r}{r}$$

where  $o_r(1)$  denotes a quantity that goes to 0 as  $r$  approaches infinity. For smaller even  $r = 2s$ , several constructions (due to Giraud, D. de Caen, D. L. Kreher and J. Wiseman, see [4]) gave better lower bounds. They showed

$$\lim_{n \rightarrow \infty} \frac{t_{2s}(n, 2s+1)}{\binom{n}{2s}} \geq 1 - \frac{1}{4} - 4^{-s}, \text{ for all } s \geq 1.$$

## 2 Preliminaries

In this section, we consider a 3-graph  $G = (V, E)$ . We consider a 3-graph  $G = (V, E)$  which contains no complete 3-graph on 4 vertices as a subgraph. We say that  $G$  is  $\Delta_4$ -free. Let  $e$  be the number of edges and  $\bar{e}$  be the number of non-edges in  $G$ . We first introduce some definitions and notations.

Let  $d_{ij}$  denote the number of edges (triples) which are incident with both vertices  $i$  and  $j$ . Similarly, we denote by  $\bar{d}_{ij}$  the number of non-edges which are incident with both vertices  $i$  and  $j$ . Clearly  $d_{ij} + \bar{d}_{ij} = n - 2$ . Here are two basic equations.

$$(1) \quad 3e = \sum_{\{i,j\}} d_{ij}$$

$$(2) \quad 3\bar{e} = \sum_{\{i,j\}} \bar{d}_{ij}$$

where the summation is over all the subsets of unordered two-elements of  $V(G)$ . Now we define

$$\epsilon_{ijk}^a = \begin{cases} 1 & \text{if } \{a, i, j\}, \{a, i, k\} \text{ and } \{a, j, k\} \text{ are edges} \\ & \text{while } \{i, j, k\} \text{ is non-edge.} \\ 0 & \text{otherwise.} \end{cases}$$

We associate each unordered pair  $\{i, j\}$  with the weight  $w_{ij}$  defined as follows:

$$w_{ij} = \frac{\sum_{a,k=1}^n \epsilon_{ijk}^a}{\bar{d}_{ij}}$$

We remark that in the extremal constructions of 3-graphs achieving the Turán number  $t_3(n, 4)$ , the weights  $w_{ij}$  are approximately equal for all  $i, j$ . However, the degrees  $d_{ij}$ 's are not. This illustrates the difficulties for tightening the bounds by using Cauchy-Schwarz inequalities. The main idea of our improved bounds in this paper is by utilizing the weights  $w_{ij}$  together with Giraud's bounds.

We will prove several useful facts about the relations between  $d_{ij}$ ,  $w_{ij}$  and  $e$ .

**Lemma 1** *For a  $\Delta_4$ -free 3-graph  $G$ , we have*

$$(3) \quad \sum_{\{i,j\}} \bar{d}_{ij}(d_{ij} + w_{ij}) \geq 2(n-3)e$$

**Proof** For  $i = 1, 2, 3, 4$ , let  $\delta_i$  denote the number of the induced subgraphs of  $G$  which are isomorphic to the unique 3-graph (denoted by  $\Delta_i$ ) on 4 vertices with  $i$  triples. We have two basic equations:

$$(4) \quad (n-3)e = \delta_1 + 2\delta_2 + 3\delta_3$$

$$(5) \quad \sum_{\{i,j\}} d_{ij}\bar{d}_{ij} = 3\delta_1 + 4\delta_2 + 3\delta_3$$

By the definition of  $w_{ij}$ , we have

$$(6) \quad \sum_{\{i,j\}} \bar{d}_{ij}w_{ij} = 3\delta_3$$

From equations (4), (5) and (6), we get

$$\sum_{\{i,j\}} \bar{d}_{ij}(d_{ij} + w_{ij}) = 2(n-3)e + \delta_1$$

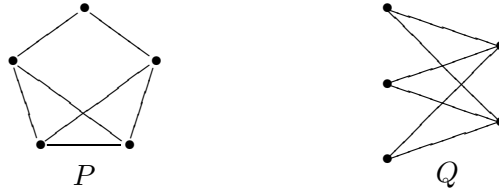
The proof of inequality (3) then follows. □

The next lemma involves further structures in a  $\Delta_4$ -free 3-graph and it is particularly useful later.

**Lemma 2** *In a  $\Delta_4$ -free 3-graph  $G$ , we have*

$$(7) \quad \sum_{\{i,j\}} d_{ij} \bar{d}_{ij} (d_{ij} + w_{ij} - 1) \geq \sum_{\{i,j\}} \bar{d}_{ij} w_{ij} (4w_{ij} - 3)$$

**Proof** Every 3-graph  $H$  on 5 vertices is in one-to-one correspondence to a 2-graph  $F$  on 5 vertices as following. We connect two vertices in  $F$  if the other 3 vertices form an edge in the 3-graph  $H$ . Among all 3-graphs on 5 vertices, we are particularly interested in two of them,  $P$  and  $Q$ , described below:



Let  $p$  (or  $q$ ) denote the number of the induced sub-3-graphs on 5 vertices of  $G$  which are isomorphic to  $P$  (or  $Q$ ). For every non-edge  $\{i, j, k\}$  in  $G$ , we choose two vertices  $a$  and  $b$  from the subset consisting of vertices  $x \in V(G)$  so that the induced sub-3-graph of  $G$  on vertices  $\{i, j, k, x\}$  is isomorphic to  $\Delta_3$ . Since  $G$  is  $\Delta_4$ -free, the induced graph on vertices  $\{i, j, k, a, b\}$  is isomorphic to either  $P$  or  $Q$ . By careful counting, we get

$$(8) \quad \begin{aligned} p + q &= \sum_{\text{non-edge}\{i,j,k\}} \binom{\sum_a \epsilon_{ijk}^a}{2} \\ &= \frac{1}{6} \sum_{\{i,j\}} \sum_k ((\sum_a \epsilon_{ijk}^a)^2 - \sum_a \epsilon_{ijk}^a) \\ &\geq \frac{1}{6} \sum_{\{i,j\}} (\bar{d}_{ij} w_{ij}^2 - \bar{d}_{ij} w_{ij}) \end{aligned}$$

and

$$(9) \quad \sum_{\{i,j\}} \binom{d_{ij}}{2} \bar{d}_{ij} \geq 7p + 9q$$

$$(10) \quad \sum_{\{i,j\}} (d_{ij} - 1) \bar{d}_{ij} w_{ij} \geq 10p + 6q$$

By summing inequalities (9) twice and (10), we can use (8) to derive the inequality (7). $\square$

We will further manipulate the inequalities in Lemma 1 and Lemma 2 to derive the following result

**Lemma 3** *Suppose  $e \geq \bar{e}$ . Then for any  $n \geq 4$ , we have*

$$(11) \quad \sum_{\{i,j\}} (d_{ij} + \frac{1}{2})^2 \bar{d}_{ij} \geq \frac{49 - 9\sqrt{17}}{32} \sum_{\{i,j\}} \bar{d}_{ij} (d_{ij} + w_{ij})^2$$

**Proof** We rewrite inequality (7) as

$$(12) \quad \begin{aligned} \frac{9}{4} \sum_{\{i,j\}} (d_{ij} + \frac{1}{2}) \bar{d}_{ij} (d_{ij} + w_{ij}) \\ \geq \sum_{\{i,j\}} (d_{ij} + \frac{1}{2})^2 \bar{d}_{ij} + \sum_{\{i,j\}} \bar{d}_{ij} (d_{ij} + w_{ij})^2 \\ + (\frac{3}{8} \sum_{\{i,j\}} \bar{d}_{ij} (d_{ij} + w_{ij}) - \frac{1}{4} \sum_{\{i,j\}} \bar{d}_{ij}) \end{aligned}$$

By the assumptions, inequality (3) in Lemma 3 and equation (2), we see that the last term is always non-negative.

$$\begin{aligned} \frac{3}{8} \sum_{\{i,j\}} \bar{d}_{ij} (d_{ij} + w_{ij}) - \frac{1}{4} \sum_{\{i,j\}} \bar{d}_{ij} \\ \geq \frac{3}{8} 2(n-3)e - \frac{1}{4} 3\bar{e} \\ \geq (\frac{3}{4}(n-3) - \frac{3}{4})\bar{e} \\ = \frac{3}{4}(n-4)\bar{e} \\ \geq 0 \end{aligned}$$

Now we use the Cauchy-Schwarz inequality

$$(13) \quad (\sum_{\{i,j\}} A_{ij}^2) (\sum_{\{i,j\}} B_{ij}^2) \geq (\sum_{\{i,j\}} A_{ij} B_{ij})^2$$

where  $A_{ij} = (d_{ij} + \frac{1}{2})\sqrt{\bar{d}_{ij}}$  and  $B_{ij} = \sqrt{\bar{d}_{ij}}(d_{ij} + w_{ij})$ .

Inequality (12) can be rewritten as follows:

$$\frac{9}{4} \sum_{\{i,j\}} A_{ij} B_{ij} \geq \sum_{\{i,j\}} A_{ij}^2 + \sum_{\{i,j\}} B_{ij}^2.$$

By combining the above two inequalities, we have

$$\left( \sum_{\{i,j\}} A_{ij}^2 \right) \left( \sum_{\{i,j\}} B_{ij}^2 \right) \geq \frac{16}{81} \left( \sum_{\{i,j\}} A_{ij}^2 + \sum_{\{i,j\}} B_{ij}^2 \right)^2.$$

After solving this quadratic inequality, we get

$$\sum_{\{i,j\}} A_{ij}^2 \geq \frac{49 - 9\sqrt{17}}{32} \sum_{\{i,j\}} B_{ij}^2.$$

The proof of Lemma 3 is complete.  $\square$

The next lemma is due to Giraud. The original version has the constant  $c = 0$ . Here it is modified for latter usage.

**Lemma 4** For a constant  $c$  with  $d_{ij} \geq c$  for all pairs  $\{i, j\}$ , if  $e \geq \frac{1}{3}(\frac{n}{2} + \frac{c}{2} - 1)\binom{n}{2}$ , we have

$$(14) \quad 3\bar{e} \left( \frac{3e}{\binom{n}{2}} - c \right)^2 \geq \sum_{\{i,j\}} (d_{ij} - c)^2 \bar{d}_{ij}.$$

**Proof** Observe the fact that the following function  $f(x)$  is convex,

$$f(x) = \begin{cases} (\frac{n}{2} - \frac{c}{2} - 1)^2(x - c) & \text{if } c \leq x \leq \frac{n}{2} + \frac{c}{2} - 1 \\ (x - c)^2(n - 2 - x) & \text{if } \frac{n}{2} + \frac{c}{2} - 1 \leq x \leq n - 2, \end{cases}$$

and that  $f(x) \geq (x - c)^2(n - 2 - x)$ , for all  $c \leq x \leq n - 2$ . Since the average  $\frac{\sum_{\{i,j\}} d_{ij}}{\binom{n}{2}} \geq \frac{n}{2} + \frac{c}{2} - 1$ , the proof of this lemma follows from the convexity of  $f(x)$ .  $\square$

### 3 The main theorems

We are now ready to prove the main theorem.

**The proof of Theorem 1:** Let  $G$  be a  $\Delta_4$ -free 3-graph with the maximum number of triples.

If  $e \leq \frac{1}{2}\binom{n}{3}$ , then we are done since  $\frac{1}{2} < \frac{3+\sqrt{17}}{12}$ . From now on we may assume that  $e \geq \frac{1}{2}\binom{n}{3}$ . Hence  $e \geq \bar{e}$ . All assumptions of Lemma 3 are satisfied. We can use inequality (11).

Let  $c = -\frac{1}{2}$ . Since  $n \geq 4$ , we have  $e \geq \frac{1}{2}\binom{n}{3} \geq \frac{1}{3}(\frac{n}{2} + \frac{c}{2} - 1)\binom{n}{2}$ . By inequality (14) in Lemma 4, we get

$$(15) \quad 3\bar{e}\left(\frac{3e}{\binom{n}{2}} + \frac{1}{2}\right)^2 \geq \sum_{\{i,j\}} (d_{ij} + \frac{1}{2})^2 \bar{d}_{ij}.$$

On the other hand, we can use the Cauchy-Schwartz inequality

$$(16) \quad \left(\sum_{\{i,j\}} \bar{d}_{ij}\right) \left(\sum_{\{i,j\}} \bar{d}_{ij} (d_{ij} + w_{ij})^2\right) \geq \left(\sum_{\{i,j\}} \bar{d}_{ij} (d_{ij} + w_{ij})\right)^2$$

Combining inequalities (15) and (16) as well as inequality (3) of Lemma 1 and inequality (11) of Lemma 3, we have

$$(17) \quad 9\bar{e}^2 \left(\frac{3e}{\binom{n}{2}} + \frac{1}{2}\right)^2 \geq \frac{49 - 9\sqrt{17}}{32} (2(n-3)e - \bar{e})^2$$

By taking square root on both sides, we get

$$(18) \quad 3\bar{e} \left(\frac{3e}{\binom{n}{2}} + \frac{1}{2}\right) \geq \frac{9 - \sqrt{17}}{8} (2(n-3)e - \bar{e})$$

Let  $x$  denote  $\lim_{n \rightarrow \infty} \frac{e}{\binom{n}{3}}$ . We divide by  $(n-2)\binom{n}{2}$  on both sides of (18) and let  $n$  approach infinity. Then

$$3(1-x)x \geq \frac{9 - \sqrt{17}}{8} 2x$$



Hence, we have  $x \leq \frac{3+\sqrt{17}}{12}$  and therefore

$$\lim_{n \rightarrow \infty} \frac{t_3(n, 4)}{\binom{n}{3}} \leq \frac{3 + \sqrt{17}}{12}.$$

We have proved Theorem 1. □

## 4 Concluding remarks

The result in Theorem 1 can be generalized for any odd  $r \geq 3$  by using the same technique. Although the formulation is more complicated, the application is quite straightforward. We will omit the proof here.

**Theorem 3** *For any odd  $r \geq 3$ , we have*

$$\lim_{n \rightarrow \infty} \frac{t_r(n, r+1)}{\binom{n}{r}} \leq 1 - \frac{5r + 12 - \sqrt{9r^2 + 24r}}{2r(r+3)}.$$

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