A chip-firing game and Dirichlet eigenvalues

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Dedicated to Dan Kleitman in honor of his sixty-fifth birthday

Abstract

We consider a variation of the chip-firing game in a induced subgraph S of a graph G. Starting from a given chip configuration, if a vertex v has at least as many chips as its degree, we can fire v by sending one chip along each edge from v to its neighbors. The game continues until no vertex can be fired. We will give an upper bound, in terms of Dirichlet eigenvalues, for the number of firings needed before a game terminates. We also examine the relations among three equinumerous families, the set of spanning forests on S with roots in the boundary of S, a set of "critical" configurations of chips, and a coset group, called the sandpile group associated with S.

1 Introduction

Chip-firing is a game played on a graph G. Each vertex of G contains an integral number of chips. A vertex may be *fired* provided that it has at least as many chips as its degree, and upon firing it sends one chip along each edge to the other vertex incident to the edge. The game proceeds by firing a sequence of vertices in succession, whereby firing leads from one *configuration* of the game to another. Chip-firing has been studied previously in terms of classification of legal game sequences [9, 10], critical configurations [4], chromatic polynomials [6], and the Tutte polynomial [7, 20]. Algorithmic aspects of chip-firing are discussed in [19, 21]. An early version of chip-firing appears in a work by Engel [17]. Chip-firing is closely related to the *abelian sandpile model*, introduced by D. Dhar [14, 15], and discussed by Cori [13]. Related topics include *self-organized criticality* [2, 3] and *avalanche models* [18].

In this paper we consider a new variant of the chip-firing game, in which chips are removed from the game when they are fired across a boundary. This modified chip-firing game is motivated in part by communication network models in which the chips represent packets or jobs and the boundary nodes represent processors with unlimited computational power. We will refer to this variant as the chip-firing game with *Dirichlet boundary conditions*, and hereafter simply refer to it as the "Dirichlet game" unless otherwise specified. For this game, of importance are the *Dirichlet eigenvalues*, which are the eigenvalues of the Laplacian of the graph with rows and columns of boundary vertices deleted. After preliminary definitions in Section 2, in Section 3 we obtain a bound on the length of the Dirichlet game in terms of the number of chips and the Dirichlet eigenvalues of the graph. In Sections 4 - 6 we consider three families of structures associated with an induced subgraph of G on a subset S of vertices:

(1) The set of spanning forests on S with roots on the boundary of S;
(2) A set of "critical configurations" that are special distributions of chips (detailed definition to be given later in Section 5);
(3) A coset group, that is often called the "sandpile" group.

As it turns out, all three families have the same cardinalities. We will discuss the bijections among these three families. Some questions and remarks are included in Section 7.

2 Preliminaries

The chip-firing game takes place in the setting of a simple loopless connected graph G with vertex set $V(G)$. Let S denote a subset of $V(G)$. For our purposes, the induced subgraph $G(S)$ must be connected. The boundary of S, denoted by δS, consists of all vertices $y \notin S$ that are adjacent to some vertex in S. For simplicity, we will assume that $V(G) = S \cup \delta S$. Any vertex $v \in S$ is *ready* to be fired if it has at least as many chips as its degree. If the firing of one vertex causes a second vertex to go from not ready to ready, then we say the first *primes* the second, or the second is *primed*. Chips fired from a vertex in S to a vertex in δS are instantly processed and removed from the game. Thus a configuration c of the Dirichlet game is a vector $c : V(G) \to \mathbb{Z}^+ \cup \{0\}$ which satisfies Dirichlet boundary condition $c(v) = 0$ for all $v \in \delta S$. A configuration is *stable* if no vertex $v \in S$ is ready. Given the initial configuration of a Dirichlet game, c_i, we may fire vertices in succession provided that they are ready at the time of their firing, yielding a firing sequence $\mathcal{F} : \mathbb{N} \to S$. Again, if the firing sequence $\mathcal{F}$ is finite, then we say c_i yields c_e under $\mathcal{F}$. The *score* of a Dirichlet game is the vector $f : S \to \mathbb{Z}^+ \cup \{0\}$ defined by $f(v) = \left|\mathcal{F}^{-1}(\{v\})\right|$, where $f(v)$ may be interpreted as the number of times the vertex $v \in S$ is fired during the Dirichlet game. The *length* of the Dirichlet game may thus be defined as the total number of firings, $\sum_v f(v)$. A configuration c is *Dirichlet-critical* provided that c is stable and *recurrent*; i.e., chips can be added to S to form a new configuration c' which yields c under a finite firing sequence $\mathcal{F}$.

The combinatorial Laplacian L of a graph G is a $|V(G)| \times |V(G)|$ matrix indexed by the vertices of G and defined by

$$L(u,v) = \begin{cases} d_v & \text{if } u = v \\ -1 & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{otherwise,} \end{cases}$$

where d_v is the degree of v in G. Alternatively, we may define L by its operation on a vector $f \in \mathbb{Z}^{|V(G)|}$:

$$Lf(u) = \sum_{v \sim u} f(u) - f(y).$$

Let x_v be the standard basis vector $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{V(G)}$ corresponding to v. Firing a vertex $v \in S$ which has no neighbors in δS at the configuration c_j to obtain the configuration c_{j+1} may be expressed as

$$c_j(u) - c_{j+1}(u) = Lx_v(u),$$

but in general for the Dirichlet game, this must be expressed as

$$c_j(u) - c_{j+1}(u) = \begin{cases} Lx_v(u) & \text{if } u \in S \\ 0 & \text{if } u \in \delta S, \end{cases}$$

More generally, if f is the score of a Dirichlet game, then $Lf = c_i - c_e$ on S.

The eigenvalues of L, or of its normalized version, are of considerable interest in the study of graph diameters, routings, random walks, expanders, and many other topics. The reader is referred to [11] for undefined terminology. In this paper we are interested in the eigenvalues of a particular restriction of L. If the vertex set of G is divided into S and a boundary set δS, with empty intersection, then we define L_S to be the Laplacian of G with rows and columns corresponding to δS deleted. If we identify a vector $f \in \mathbb{Z}^{|S|}$ with a function $g \in \mathbb{Z}^{|S \cup \delta S|}$ satisfying $g(v) = f(v)$ for $v \in S$ and

$$g(u) = 0 \tag{1}$$

for $u \in \delta S$, then we have

$$L_S f(v) = Lg(v)$$

for $v \in S$. For g satisfying (1), we say that g satisfies the Dirichlet boundary condition. The *Dirichlet eigenvalues* of G with respect to vertex set S and boundary set δS are the eigenvalues of L_S. It is not hard to show that when G is connected and the boundary δS is nonempty, then all the Dirichlet eigenvalues are positive (see [11]). We will write these eigenvalues as

$$0 < \sigma_1 \le \sigma_2 \le \cdots \le \sigma_{|S|},$$

In particular, σ_1 satisfies the following:

$$\begin{aligned} \sigma_1 &= \inf_{f \neq 0} \frac{\langle f, L_S f \rangle}{\langle f, f \rangle} \\ &= \inf_f \frac{\sum\limits_{x \sim y} (f(x) - f(y))^2}{\sum_{x \in S} f^2(x)} \end{aligned} \tag{2}$$

where the "$\inf_f$" ranges over all f satisfying the Dirichlet boundary condition and the "$\sum_{x \sim y}$" is ranging over all unordered pairs of vertices x and y so that x is adjacent to y and at least one of x and y is in S.

3 Convergence bounds for chip-firing games with Dirichlet boundary conditions

Given the setting of a chip-firing game with Dirichlet boundary conditions, we may wish to determine the length of a game based on its initial configuration. Starting from an initial configuration c_i, we fire vertices successively for as long as possible. A game terminates when it reaches a stable configuration, where no vertex $v \in S$ may be fired. That the game must terminate when G is connected and $\delta S \neq \emptyset$ is a minor variant on Lemma 3.1 of [9]:

Lemma 1 *Every chip-firing game with Dirichlet boundary conditions terminates in a finite number of firings.*

Proof: Letting G be the connected graph with vertex set $S \cup \delta S$, recall that only vertices in S may be fired, and that vertices in δS immediately remove any chips that are sent to them. Let $N = \sum_v c_i(v)$ be the total number of chips at the start of the game. Now suppose to the contrary that a game does not terminate. Then there is a vertex $v_1 \in S$ that is fired infinitely often. Let $P = v_1, \ldots, v_k$ be a simple path from v_1 to some vertex $v_k \in \delta S$, with all vertices except for v_k in S. For each $i \in \{1, \ldots, k-1\}$, if vertex v_i is fired infinitely often, then vertex v_{i+1} receives infinitely many chips, and must also be fired infinitely often if it is in S. This is because each vertex may have no more than N chips at a single time. Therefore infinitely many chips are removed from the game, which is a contradiction. This completes the proof of the lemma. □

A fascinating result on the characterizations of score vectors of games is that the score vector depends only on the initial configuration and not on the firing sequence used to go from the initial configuration to the final configuration. As long as two such distinct firing sequences are legal, they will have the same score. This fact is obtained by pushing through Theorem 2.1 in [9] with the Dirichlet game variant. In fact, if c_i and c_e are the initial and final configurations, respectively, of a terminating game, then the score vector f is uniquely defined by

$$L_S f \quad = \quad c_i - c_e. \tag{3}$$

This expression of the score vector in terms of the Laplacian leads us to obtain a bound on f using Dirichlet eigenvalues.

Theorem 1 *Let f be the score vector of a chip-firing game with Dirichlet boundary conditions. Then the number of firings in the game is bounded as follows:*

$$\sum_{x \in S} f(x) \leq D \cdot N n^{3/2},$$

where N is the total number of chips initially, $n = |S|$ is the size of the vertex set, and D is the diameter of the graph.

Proof: We may assume that the graph is connected. (Otherwise, D is infinity and Theorem 1 holds.) Let c_i, c_e be the initial and final configurations, respectively, of the game; thus $N = \sum_{x \in S} c_i(x)$. Let $0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$ be the Dirichlet eigenvalues of the Laplacian of S in G, and let $\phi_1, \ldots, \phi_{|S|}$ be the respective normalized eigenvectors. Accordingly, write $f = \sum a_i \sigma_i$. We may bound the $a_i's$ using (3).

$$\begin{aligned} a_i &= \langle f, \phi_i \rangle = \frac{1}{\sigma_i} \langle f, \sigma_i \phi_i \rangle \\ &= \frac{1}{\sigma_i} \langle f, L_S \phi_i \rangle = \frac{1}{\sigma_i} \langle L_S f, \phi_i \rangle = \frac{1}{\sigma_i} \langle c_i - c_e, \phi_i \rangle . \end{aligned} \tag{4}$$

The number of firings in the game is simply the sum of the entries of f; i.e.,

$$\sum_{x \in S} f(x) \quad = \quad \langle 1_S, f \rangle . \tag{5}$$

Putting (4) and (5) together, we obtain

$$\sum_{x \in S} f(x) \quad = \quad \left\langle 1_S, \sum_i a_i \phi_i \right\rangle = \left\langle 1_S, \sum_i \frac{1}{\sigma_i} \langle c_i - c_e, \phi_i \rangle \phi_i \right\rangle$$

$$
\begin{aligned}
&= \sum_i \frac{1}{\sigma_i} \langle c_i - c_e, \phi_i \rangle \langle 1_S, \phi_i \rangle \\
&\le \frac{1}{\sigma_i} \left(\sum_i \langle c_i - c_e, \phi_i \rangle^2 \right)^{1/2} \left(\sum_i \langle 1_S, \phi_i \rangle^2 \right)^{1/2} \qquad \text{(by Cauchy-Schwarz)} \\
&= \frac{1}{\sigma_i} ||c_i - c_e||_2 ||1_S||_2 \\
&\le \frac{1}{\sigma_1} ||c_i - c_e||_2 \sqrt{|S|}. \qquad (6)
\end{aligned}
$$

$||c_i - c_e||_2$ is bounded above by the number N of chips at the start of the game, and $|S| = n$. In order to bound σ_1, we consider the eigenvector f achieving σ_1. Let x_0 denote the vertex with $|f(x_0)| = max_{x \in S} |f(x)|$. Let P denote a shortest path with vertices $x_0, x_1, \ldots, x_k$, where x_k is in δS and x_i is adjacent to x_j in G. Clearly, k is no more than the diameter D of G. From (2), we have

$$
\begin{aligned}
\sigma_1 &= \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_{x \in S} f^2(x)} \\
&\ge \frac{\sum_{i=1}^{k} (f(x_{i-1}) - f(x_i))^2}{\sum_{x \in S} f^2(x)} \\
&\ge \frac{\left\{ \sum_{i=1}^{k} (f(x_{i-1}) - f(x_i)) \right\}^2 / k}{\sum_{x \in S} f^2(x)} \qquad \text{by Cauchy-Schwarz} \\
&\ge \frac{f(x_0)^2 / k}{n f^2(x_0)} \\
&\ge \frac{1}{nD}
\end{aligned}
$$

Combining this and (6), we obtain

$$
\begin{aligned}
\sum_{x \in S} f(x) &\le \frac{1}{\sigma_1} N \sqrt{n} \\
&\le D \cdot N n^{3/2}.
\end{aligned}
$$

This completes the proof of the theorem. □

4 A matrix-tree theorem for induced subgraphs with Dirichlet boundary conditions

For an induced subgraph on S with non-empty boundary in a graph G, we define a rooted spanning forest of S to be subgraph F satisfying

(1) F is an acyclic subgraph of G,
(2) F has vertex set $S \cup \delta S$,
(3) Each connected component of F contains exactly one vertex in δS.

The following theorem relates the product of Dirichlet eigenvalues of S with the enumeration of rooted spanning forests of S. The proof method is quite similar to the original proof of the matrix-tree theorem as well as the proof in [12]. For completeness, we will briefly sketch the proof here.

Theorem 2 *For an induced subgraph on S in a graph G with $\delta S \neq \emptyset$, the number $\tau(S)$ of rooted spanning forests of S is*

$$\tau(S) = \prod_{i=1}^{|S|} \sigma_i$$

where σ_i are the Dirichlet eigenvalues of the Laplacian of S in G.

Proof: We consider the incidence matrix B with rows indexed by vertices in S and columns indexed by edges in S' as follows:

$$B(x,e) = \begin{cases} 1 & \text{if } e = \{x,y\}, x < y \\ -1 & \text{if } e = \{x,y\}, x > y \\ 0 & \text{otherwise} \end{cases}$$

We have

$$L_S = B\ B^* \tag{7}$$

where B^* denotes the transpose of B.

We have

$$\begin{aligned} \prod_{i=1}^{|S|} \sigma_i &= \det L_S \\ &= \det B\ B^* \\ &= \sum_X \det B_X \det B_X^* \end{aligned}$$

where X ranges over all possible choices of $s-1$ edges and B_X denotes the square submatrix of B whose $s-1$ columns correspond to the edges in X.

Claim 1: If the subgraph with vertex set $S \cup \delta S$ and edge set X contains a cycle, then $\det B_X = 0$. The proof is similar to that in the classical matrix-tree and will be omitted.

Claim 2: If the subgraph formed by edge set X contains a connected component having two vertices in δS, then $\det B_X = 0$.
Proof: Let Y denote a connected component of the subgraph formed by X. If Y contains more than one vertex in δS, then Y has no more than $|E(Y)| - 1$ vertices in S. The submatrix formed by the columns corresponding to edges in Y has rank at most $|E(Y)| - 1$. Consequently, $\det B_X = 0$.

Claim 3: If the subgraph formed by X is a rooted forest of S with roots δS, then $|\det B_X| = 1$.

Combining Claims 1-3, we have

$$\begin{aligned}\prod_{i=1}^{|S|}\sigma_i &= \det L_S = \sum_X \det B_X \det B_X^* \\ &= |\{\text{rooted spanning forests of } S\}|\end{aligned}$$

□

We remark that the usual matrix-tree theorem can be viewed as a special case of Theorem 2. Namely, for a graph G, we apply Theorem 2 to an induced subgraph H on $V(G) - \{v\}$ for some vertex v in G. The rooted spanning forests are all trees on G.

5 Dirichlet-critical configurations and rooted spanning forests

The Dirichlet-critical configurations of a Dirichlet game, as defined in Section 2, have several surprising properties, and have the same cardinality as the number of spanning forests of G rooted in δS. In fact, a bijection between the two sets may be obtained algorithmically by playing a chip-firing game using the Dirichlet-critical configuration as a starting point. This bijection will be described after we state and discuss the following useful facts on Dirichlet-critical configurations.

Lemma 2 *Let c be a stable configuration of the Dirichlet game. Then the following are equivalent:*
(a) *c is Dirichlet critical.*
(b) *There exists a configuration b and a firing sequence $\mathcal{F}$ such that b yields c under $\mathcal{F}$ and each vertex in S appears at least once in $\mathcal{F}$.*
(c) *There exists a configuration b and a firing sequence $\mathcal{F}$ such that b yields c under $\mathcal{F}$ and each vertex in S appears exactly once in $\mathcal{F}$.*
(d) *Starting with c, if one chip is added at every vertex v for each edge crossing into δS to obtain a second configuration b, then there is a firing sequence $\mathcal{F}$ which is a permutation of S such that b yields c under $\mathcal{F}$.*

Instead of giving a detailed proof, we here discuss these equivalent formulations of a Dirichlet-critical configuration. In (b) we see that every Dirichlet-critical configuration may be obtained by adding enough chips to the graph and firing enough vertices to eventually return to the Dirichlet-critical configuration. This condition is apparently strengthened but is actually equivalent to the condition in (c). These two conditions are discussed for a similar variant of chip-firing (equivalent to taking $|\delta S| = 1$) in Theorem 3.6 of [19]. The chips added to vertices in S in condition (d) are those that would be added if the vertices in δS were allowed to fire. The idea is that if all the vertices in δS are "fired", then there is a permutation of the vertices of S in which they may be legally fired to return to the original Dirichlet-critical configuration. This has been proved for $|\delta S| = 1$ as Theorem 3.1 in [20]. Note that the configuration b created from c by adding chips in (4) may be taken as the one which satisfies (c). One way of obtaining the results for the general case from the case for $|\delta S| = 1$ is as follows. Beginning with the Dirichlet game on G, create G_q by attaching a distinguished vertex q by a single edge to each vertex in δS. Now take $S_q = S \cup \delta S$ and δS_q to be only this vertex q. For a critical configuration c from the original game, define the configuration c_q

on the new game by

$$c_q(v) = \begin{cases} c(v) & \text{if } v \in S, \\ \deg_{G_q}(v) - 1 & \text{if } v \in \delta S, \\ 0 & \text{if } v = q. \end{cases} \tag{8}$$

All of the necessary information about the Dirichlet-critical configuration c may be obtained by using the existing theorems on c_q. This leads us to the main theorem of the section.

Theorem 3 *The number of Dirichlet-critical configurations of the Dirichlet game on G is the same as the number of spanning forests of G rooted in δS.*

Proof: In the case of $|\delta S| = 1$, Biggs and Winkler have proved the theorem for a related chip-firing variant in Theorems 1-3 of [6]. For completeness, we sketch the proof in the language of the Dirichlet game as follows. Consider the Dirichlet game on G with boundary $\delta S = \{q\}$. Note that spanning forests of G rooted in δS can be viewed as spanning trees of G. Fix once and for all a total ordering on the edges of G. Let c be a Dirichlet-critical configuration of G. We require a bijective mapping

$$\theta : \{\text{Dirichlet-critical configurations}\} \to \{\text{spanning trees of } G\}.$$

Define $\theta(c) = T$ as follows.

Algorithm A

1. Initialize $T = \{\}$, the tree to be constructed.
2. Add chips to the game as if q were fired. The number of chips at q remains 0. Add $\{q, u\}$ to T for each u adjacent to q which becomes ready.
3. Fire a vertex v that is ready. Ties are broken by firing the vertex v where the shortest path from q to v has an edgelist which is least possible in the lexicographic ordering on edges. If v primes any vertex u, add edge $\{u, v\}$ to T.
4. Repeat 3. until all vertices have been fired.

That this process is well-defined and completes with T a spanning tree of G is a result of part (d) of Lemma 2. The details of proving that θ is a bijection may be viewed in [6]. This completes the proof for $|\delta S| = 1$.

Now, let G be a general Dirichlet game with boundary δS. We require a bijective mapping

$$\theta_q : \{\text{Dirichlet-critical configurations on } G\} \to \{\text{spanning forests of } G \text{ rooted in } \delta S\}.$$

Define θ_q as follows. Convert G to a Dirichlet game with boundary of size 1 by constructing G_q from G as previously described. Let c be a Dirichlet-critical configuration of G. Define a configuration c_q of G_q according to (8). Use c_q to construct a spanning tree T of G_q using Algorithm A. Remove the edges $\{q, v\}$ for all $v \in \delta S$ from T to obtain a spanning forest F rooted in δS. Let $\theta_q(c_q) = F$. We must show that θ_q defined in this way is a bijection.

First we show that θ_q is well-defined. To show that c_q is Dirichlet-critical for the game on G_q, note that c_q is stable. Also, c_q must be recurrent. Adding one chip to each vertex adjacent to q in

G_q primes each vertex in δS. Firing each vertex of δS in succession causes one chip to be added at each vertex $v \in S$ for each edge crossing into δS. Then by Lemma 2(d) for the Dirichlet game on G, there is a permutation in which the vertices of S may be legally fired. Every vertex $v \neq q$ has now been fired, yielding the original configuration c_q. By Lemma 2(d) for the Dirichlet game on G_q, c_q is critical. Also, we must show that F is a spanning forest of G rooted in δS. But all that is required for this is that the tree T produced in Algorithm A contain all edges $\{q, v\}$ for $v \in \delta S$. This is true since Step 2 of the algorithm primes every vertex in δS (recall that $c_q(v) = \deg_{G_q}(v)$ for all $v \in \delta S$). Therefore F is a spanning forest of G rooted in δS and θ_q is well-defined.

Now we must show that θ_q is one-to-one. The preparatory mapping from c to c_q is one-to-one because the values of c on S are preserved. The Biggs-Winkler bijection θ gives a one-to-one mapping from the Dirichlet-critical configurations on G_q to the spanning trees T of G_q. In going from T to F, the exact same edges, $\{\{q, v\} | v \in \delta S\}$, are removed from T in each case, so this step is also a one-to-one mapping. Thus θ_q as the composition of three one-to-one mappings is one-to-one.

Finally, we must show that θ_q is onto. Let F be a spanning forest of G rooted in δS. Construct T in G_q by adding all edges $\{\{q, v\} | v \in \delta S\}$. From the Biggs-Winkler bijection θ, we obtain the Dirichlet-critical configuration c_q for G_q which corresponds to this T. Because all edges $\{\{q, v\} | v \in \delta S\}$ are in T, Step 2 of Algorithm A must prime all vertices in δS, and thus $c_q(v) = \deg_{G_q}(v - 1)$ for all $v \in \delta S$. Define c on S by restricting c_q to S. Since c_q is Dirichlet-critical, after adding chips to all vertices adjacent to q there is a permutation in which all the other vertices may be legally fired. This implies that in the Dirichlet-game on G, after adding one chip to $v \in S$ for each edge incident to a vertex in δS, there is a permutation of S which is a legal firing sequence and yields the same original configuration, c. Thus c is Dirichlet-critical for G. Therefore $\theta_q(c) = F$, completing the proof of Theorem 3. □

6 The sandpile group and rooted forests

The sandpile group originated in the study of modeling the behavior of grains of sand piled onto the nodes of a structure ([14, 15]). Once the number of grains of sand at a particular node exceeds a threshold condition, the sand topples down from this more saturated node, possibly causing sand in adjacent nodes to exceed stability thresholds as well (thus the notion of avalanches). On a graph, the threshold at a vertex is exceeded when the vertex gets a number of chips equal to its degree. The sandpile group of a graph models the allowable transitions which may occur when vertices topple in succession. A starting sandpile configuration is a member of one of the cosets of the sandpile group. As we wish to view the toppling of sand as leaving the underlying structure or dynamics of the sandpile unchanged, toppling is modeled by traveling to various other members of the same coset via the allowable transitions.

The sandpile group of a graph is defined as follows. Let $V(G) = \{1, \ldots, n\}$, and root the graph G at vertex n. Consider $\mathbb{Z}^n$ as a group under addition, and associate each vertex i with the standard basis vector $x_i \in \mathbb{Z}^n$. Define

$$\Delta_i = \deg_G(i) x_i - \sum_{j=1}^{n} A(i, j) x_j,$$

where $\deg_G(i)$ is the degree of i in G and A is the adjacency matrix. Δ_i may be interpreted as the

i^{th} row vector of the Laplacian of G. Then the sandpile group $\mathrm{SP}(G)$ of G is the group

$$\mathrm{SP}(G) = \mathbb{Z}^n / \langle \Delta_1, \ldots, \Delta_n, x_n \rangle .$$

The order of $\mathrm{SP}(G)$ is the number of spanning trees of G; this is a restatement of the *Matrix-Tree Theorem*. In fact, a group structure may be imposed on the Dirichlet-critical configurations of the Dirichlet game with boundary $|\delta S| = 1$ which yields a group isomorphic to $\mathrm{SP}(G)$. This is done for an equivalent chip-firing variation in [4] from the point of view of critical configurations, and again in [13] from the perspective of the sandpile group.

We now define a more general sandpile group which is related to the Dirichlet-critical configurations of the Dirichlet game. Let $V(G) = \{1, \ldots, n, n+1, \ldots, n+m\}$ with $S = \{1, \ldots, n\}$ and $\delta S = \{n+1, \ldots, n+m\}$, by relabeling if necessary. Define

$$\mathrm{SP}_S(G) = \mathbb{Z}^{m+n} / \langle \Delta_1, \ldots, \Delta_{n+m}, x_{n+1}, \ldots, x_{n+m} \rangle .$$

The motivation for constructing $\mathrm{SP}_S(G)$ is to encode the firing rule for vertex $i \in S$ with the Δ_i and the processing of chips in δS by $x_{n+1}, \ldots, x_{n+m}$. As a result, two configurations of the Dirichlet game are in the same coset of the coset group $\mathrm{SP}_S(G)$ if one can be reached from the other by firing a sequence of vertices. The size of $SP_S(G)$ is given by the next theorem.

Theorem 4 *Let $SP_S(G)$ be the generalized sandpile group on G with specified vertex set S and boundary set δS. Then*

$$|SP_S(G)| = det\ L_S = \tau(S),$$

where L_S is the restricted Laplacian of G, and $\tau(S)$ is the number of spanning forests of G rooted in δS.

Theorem 4 is a restatement of Theorem 2, the generalized Matrix-Tree theorem for rooted spanning forests. Thus we know that the set of Dirichlet-critical configurations has the same size as the order of $\mathrm{SP}_S(G)$, which leads us to desire a meaningful bijection between the two sets. We now state the main theorem of the section, which Cori and Rossin proved ([13]) for an equivalent game for the case $|\delta S| = 1$.

Theorem 5 *$SP_S(G)$ is isomorphic to the set of Dirichlet-critical configurations of the chip-firing game with Dirichlet boundary $\delta S = \{n+1, \ldots, n+m\}$.*

Proof: The proof may be had as an extension of the proof of Theorem 1 of [13]. We now outline that extension. A configuration u of the Dirichlet game may be viewed as an element of $\mathrm{SP}_S(G)$ simply by extending u to be 0 on δS. For future reference, call this extension $\phi(u)$. Thus adding two configurations of the game corresponds to adding vectors in the group. We equip the set of Dirichlet-critical configurations with a candidate group operation $\oplus$ by defining $x \oplus y$ to be the unique Dirichlet-critical configuration obtained as the final configuration of the Dirichlet game played with initial configuration $x+y$. In order to prove Theorem 5, by showing that $\oplus$ is a group operation for which ϕ is an isomorphism, it is sufficient to show that for any configuration u in the Dirichlet game there exists a unique Dirichlet-critical configuration v such that

$$u - v \in \Delta_S = \langle \Delta_1, \ldots, \Delta_{n+m}, x_{n+1}, \ldots, x_{n+m} \rangle . \tag{9}$$

That is, every Dirichlet-critical configuration matches with exactly one coset of the general sandpile group for G. For convenience, define $\Delta_S = \langle \Delta_1, \ldots, \Delta_{n+m}, x_{n+1}, \ldots, x_{n+m} \rangle$. In order to show existence of v in (9), there must be a way of starting with any configuration u and obtaining a Dirichlet-critical configuration in the same coset. This is achieved by adding u together with a cleverly chosen configuration $u' \in \Delta_S$ such that $u + u'$ yields a critical configuration v under a firing sequence. Since $u' \in \Delta_S$, u and $u + u'$ are in the same coset and the firing rules are encoded by elements of Δ_S, u and v are in the same coset. Thus the existence of v is ascertained. Uniqueness of v in (9) is shown by proving that if u and v are both Dirichlet-critical configurations with $u - v \in \Delta_S$, then $u = v$. □

Corollary 1 *The mapping ϕ from Dirichlet-critical configurations of G to $SP_S(G)$, where $\phi(c)$ is defined by extending c to be 0 on δS, is a bijection from Dirichlet-critical configurations to cosets of the general sandpile group $SP_S(G)$.*

7 Problems and remarks

There are several related versions of the Dirichlet game that we will mention here.

1. The sandpile game is different in that it has no boundary vertices to process chips; therefore games may proceed indefinitely provided that there are enough chips in the right configuration (see Theorem 3.3, [9]). A directed version of the sandpile game may be found in [10].
2. The special case of the Dirichlet game of $|\delta S| = 1$ is of particular interest. This version of the game is nearly equivalent to the dollar game introduced in [4]. Note that in the dollar game, chips are not processed, but rather the single boundary vertex is fired only when all other vertices are stable. It has been mentioned ([4]) that the dollar game can be used to model an economy in which the government infuses the system with money only when the economy is stuck. Consideration of the dollar game variant has led to many results on critical configurations ([4, 5, 6, 7, 13, 19, 20]).
3. The sandpile game, dollar game, Dirichlet game, and other chip-firing variants have the special property that the resulting configuration of a game with a certain score vector (defined in Section 2) is independent of the order in which the vertices appear in the firing sequence. This has led to parallel chip firing games ([8]) in which all the vertices that are ready at one stage are fired simultaneously. A succession of configurations in a parallel chip-firing game will also be configurations in the corresponding non-parallel game, but in general not vice-versa. An infinite parallel chip-firing game will eventually stabilize with the same subsequence of configurations repeated over and over again, which leads naturally to questions concerning the periodicity of such recurrences.
4. Chip-firing on the infinite path (infinite in both directions) has been studied extensively in [1]. The initial configuration considered is a finite number of chips placed on a single vertex. Every vertex may be fired. Results include the characterization of the possible final configurations and bounds on the number of firings required.
5. Sandpile models on finite dimensional lattices have been studied in detail, especially from the point of view of self-organized criticality. Computational complexity of sandpiles on lattices,

and more specifically, the inherent complexity of computing stable and recurrent states, is treated in [21]. In fact, it is shown that any problem solvable with a polynomial time algorithm may be reduced to determining the final state of a sandpile game on a finite lattice. The reader is referred to the bibliography of this paper for references to work done on sandpile model variations of interest in physics.

We remark that natural extensions of most of these results to the Dirichlet game are likely to be obtained without great difficulty. There are numerous open questions concerning chip-firing remained unsolved. Here we describe some of these problems and mention associated remarks:

- Of interest is to have an intuitive bijection between spanning forests rooted in δS and Dirichlet critical configurations that does not depend on a total ordering of edges (cf. Theorem 3).
- Currently, we are able to bound the number of steps required to obtain a stable configuration from an arbitrary configuration (cf. Theorem 1). It would be desirable to compute directly the stable configuration in the same coset of the sandpile group as the original configuration, bypassing the firing sequence.
- A continuous version of the dollar game, called the *oil game*, has been introduced in [19]. In the oil game, chips are converted to quantities of oil which flow continuously between the vertices at rates modeled on the firing rules in the original game. Analysis of the oil game shows that the critical configuration corresponding to an initial configuration can be computed in a polynomial number of steps. In order to preserve the purely algebraic character of the sandpile model, an alternate method needs to be examined. Such a method for the case of a 1-dimensional lattice was shown in [21].
- There is an interesting connection between critical configurations and the Tutte polynomial by an 1-dimensional grading of critical configurations in terms of the number of chips ([20]). In fact, it may be possible to obtain other or finer (e.g., 2-dimensional) gradings using the Tutte polynomial. Further research in this direction can be found in [16].
- Chip-firing games can be used to model several aspects of Internet computing, in particular, in connection with routing and fault tolerance. Numerous directions remain to be explored.

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