THE CHOW RING OF THE MODULI SPACE OF CURVES OF GENUS 5

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Introduction

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g over an algebraically closed field (of characteristic different from 2 and 3) and let $\overline{\mathcal{M}}_g$ be its compactification by Deligne-Mumford stable curves. The Chow rings of $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ have attracted much attention:

In characteristic 0, D. Mumford (see [17], part I) defined an intersection product on the Chow group of $\overline{\mathcal{M}}_g$ with rational coefficients. He used the fact that $\overline{\mathcal{M}}_g$ is locally the quotient of a smooth variety by a finite group and globally the quotient of a Cohen-Macaulay variety by a finite group. Recently, E. Looijenga (see [15], Theorem) showed that, over $\mathbb{C}$, $\overline{\mathcal{M}}_g$ is globally the quotient of a smooth variety by a finite group. Then M. Pikaart and J. de Jong (see [19], Theorem 3.1.1) extended this result to positive characteristic. This provides a new and simpler way to define an intersection product on the Chow group of $\overline{\mathcal{M}}_g$ using the intersection product on the Chow group of the smooth variety: if $\overline{\mathcal{M}}_g$ is the quotient of X by the action of the finite group G, then, by [10] (Examples 1.7.6 and 8.3.12), $A_*(\overline{\mathcal{M}}_g) = A_*(X)^G$ where A_* is the Chow group with rational coefficients and $A_*(X)^G$ is the subring of invariants of $A_*(X)$ for the action of G.

The Chow ring $A_*(\overline{\mathcal{M}}_g)$ being defined, one would like to know more about it. For instance, one would like to write down naturally occuring elements, generators and relations, etc. Ideally, one would like a presentation and a multiplication table with a "nice" description of the generators. With the case of the grassmannians as a model, Mumford defined ([17], page 299) some classes in $A_*(\overline{\mathcal{M}}_g)$ called the tautological classes and wrote some relations between them, using, as a main tool, the Riemann-Roch theorem. He then showed that the Chow ring of $\overline{\mathcal{M}}_2$ is generated by classes coming from the boundary and wrote a complete set of relations between tautological and boundary classes together with a multiplication table.

Continuing this work, C. Faber showed that the Chow ring of $\overline{\mathcal{M}}_3$ is generated by λ (one of Mumford's codimension one tautological classes) and boundary classes. He also wrote a complete set of relations between the tautological and boundary classes and showed that $A_*(\mathcal{M}_3) \cong \mathbb{Q}[\lambda]/(\lambda^2)$. He then showed that $A_*(\mathcal{M}_4) \cong \mathbb{Q}[\lambda]/(\lambda^3)$ and wrote a set of generators and relations for the codimension 1 and 2 Chow groups of $\overline{\mathcal{M}}_4$: these are again generated by tautological and boundary classes (see [9], the principal results of the paper are valid in positive characteristic $\neq 2, 3$, see the end of section 4 below).

C. Faber also wrote a set of generators for the codimension one Chow group of $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ (in characteristic zero, see [8]).

Let us assume for a moment that the base field is $\mathbb{C}$. We would like to mention some topological results. In [12] (see the Introduction and page 238), J. Harer showed that $H^2(\mathcal{M}_g, \mathbb{Z}) \cong \mathbb{Z}$ (for $g \geq 5$). This implies that the Picard group of $\mathcal{M}_g$ is also isomorphic to $\mathbb{Z}$ (for $g \geq 5$, for $g \leq 4$, the result is true up to torsion by the above-mentioned results of Mumford and Faber). Later, Arbarello and Cornalba (see [1]) showed that, if $\delta_0, \delta_1, \dots, \delta_{[g/2]}$ are the classes of the codimension

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one irreducible boundary components, then $\lambda, \delta_0, \delta_1, \ldots, \delta_{[g/2]}$ freely generate the Picard group of the moduli functor of $\overline{\mathcal{M}}_g$. It follows, in particular, that $\lambda, \delta_0, \delta_1, \ldots, \delta_{[g/2]}$ give a basis of $A_{3g-4}(\overline{\mathcal{M}}_g)$.

Then, in [11], Harer showed that $H^4(\mathcal{M}_g, \mathbb{Q})$ has dimension 2 when $g \geq 12$. D. Edidin [7] used this to give a basis for $H_{2(3g-3)-4}(\overline{\mathcal{M}}_g, \mathbb{Q})$ consisting of tautological and boundary classes for $g \geq 12$. More precisely, he showed the homological independence of two of the tautological classes (λ^2 and κ_2) together with the boundary classes for $g \geq 6$. Harer's result then permitted him to conclude that these were also generators for $g \geq 12$.

It was conjectured by Mumford (see [17] page 272) that if k is small compared to g, then $H^{2k}(\mathcal{M}_g, \mathbb{Q})$ is generated by tautological classes and $H^{2k+1}(\mathcal{M}_g, \mathbb{Q}) = 0$ (by results of Harer, $H^k(\mathcal{M}_g, \mathbb{Q})$ is independent of g for g large enough).

In this paper we continue the study of Chow rings of moduli spaces of low genus curves. Specifically, we consider the case $g = 5$. We write down a geometric stratification of $\mathcal{M}_5$ such that the strata have trivial Chow groups. The classes of the closures of the strata will thus form a set of generators for the Chow ring of $\mathcal{M}_5$. We show that our generators can be expressed as polynomials in Mumford's tautological classes. It thus follows

Theorem The Chow ring of $\mathcal{M}_5$ (with coefficients in $\mathbb{Q}$) is generated by the tautological classes.

C. Faber indicated (private communication) that he has shown that in genus 5 the ring generated by the tautological classes is generated by λ and the relation $\lambda^4 = 0$ holds in it. Combining our result with the result of Faber, we obtain that the Chow ring of $\mathcal{M}_5$ is a quotient of $\mathbb{Q}[\lambda]/(\lambda^4)$. We therefore have the following:

$$
\begin{aligned}
A_*(\mathcal{M}_2) &\cong \mathbb{Q} \\
A_*(\mathcal{M}_3) &\cong \mathbb{Q}[\lambda]/(\lambda^2) \\
A_*(\mathcal{M}_4) &\cong \mathbb{Q}[\lambda]/(\lambda^3) \\
A_*(\mathcal{M}_5) &\longleftarrow \mathbb{Q}[\lambda]/(\lambda^4)
\end{aligned}
$$

with the last map onto. We do not know whether λ^3 is nonzero in $A_*(\mathcal{M}_5)$. Note that if one could show the existence of a complete 3-dimensional subvariety of $\mathcal{M}_5$, then it would follow that λ^3 is nonzero since λ is ample on $\mathcal{M}_g$ (It has been shown by Diaz that the dimension of any complete subvariety of $\mathcal{M}_g$ is at most $g-2$ (see [5] Theorem 4 page 407, his proof works in case the characteristic is zero or greater than g).) Also note that, for $g \geq 6$, the Chow ring of $\mathcal{M}_g$ cannot be generated by λ anymore (at least in characteristic zero) since, by Edidin's result, $A_{3g-5}(\mathcal{M}_g)$ has dimension at least 2.

The paper is organized as follows:

In the first section we introduce a preliminary stratification of $\mathcal{M}_5$ into well-known subvarieties (U, B, T and H) and reduce the proof of our theorem (roughly) to showing that the Chow rings of U, $V := U \cup B$ and $W := U \cup T \cup B$ are generated by tautological classes. In the second section we gather some preliminary results about quartic Del Pezzo surfaces and canonical curves of genus 5. In the third section we show that the Chow ring of the "biggest" stratum U of our preliminary stratification is generated by tautological classes. In the last two sections we show that the Chow rings of V and W are generated by tautological classes.

Acknowledgments: Thanks are due to E. Arbarello: it was after a stimulating conversation with him that I became seriously interested in studying Chow rings of moduli spaces of curves. Thanks are also due to C. Faber for much of this work was inspired by his beautiful paper "Chow rings of moduli spaces of curves" ([9]). This work was begun while the author was a research fellow at the Mathematical Sciences Research Institute (MSRI) at Berkeley and supported by MSRI and the NSF grant number DMS-9204266.

Notation and Conventions

We denote by ω_X the canonical sheaf of a smooth curve X of genus 5 and let K_X be an arbitrary element of the linear system $|\omega_X|$. We let g^r_n denote an arbitrary complete linear system of degree n and (projective) dimension r on X. We identify a non-hyperelliptic curve X with its canonical model. For a divisor D on X, we will denote by $\langle D \rangle$ its linear span in the canonical space of X. For two points s and t on X, we will denote by $\pi_{st} : |K_X|^* \cong \mathbb{P}^4 \longrightarrow |K_X - s - t|^* \cong \mathbb{P}^2$ the projection from $\langle s + t \rangle$ and by $X^{st} \subset \mathbb{P}^2$ the image of X by π_{st}. By a bielliptic curve we mean a curve which is a (ramified) double cover of an elliptic curve.

We will call the point of $\mathcal{M}_g$ corresponding to X "the moduli point of X" and we will denote this point by m_X.

Finally, by the Chow ring or Chow group $A_*(M)$ of a scheme M we always mean the Chow ring or Chow group with coefficients in $\mathbb{Q}$. For a subscheme P of M, we will denote by $[P]_M$ the (usual fundamental) class of P in $A_*(M)$. By a node of M we mean an ordinary double point.

1. A preliminary stratification

We will first write $\mathcal{M}_5$ as a disjoint union of well-known locally closed subvarieties:

H: closed subvariety of $\mathcal{M}_5$ parametrizing hyperelliptic curves.
T: locally closed subvariety of $\mathcal{M}_5$ parametrizing trigonal (non-hyperelliptic) curves.
B: closed subvariety of $\mathcal{M}_5$ parametrizing bielliptic curves.
U: $= \mathcal{M}_5 \setminus (T \cup B \cup H)$

Note that the closure of T in $\mathcal{M}_5$ is the union of T and H and does not intersect B and that a curve with moduli point in T has a unique g^1_3 (see [2] page 366 Exercises C-1 and C-2, the result holds in characteristic $\neq 2, 3$ since for us $d_1, d_2 = 2$ or 3). By [10] page 21, for any closed subscheme Z of a scheme Y and for any $k \in \mathbb{Z}$ the sequence

$$A_k(Z) \xrightarrow{i_*} A_k(Y) \xrightarrow{j^*} A_k(Y \setminus Z) \longrightarrow 0$$

where i is the embedding $Z \hookrightarrow Y$ and j is the embedding $(Y \setminus Z) \hookrightarrow Z$, is exact. Therefore to prove our theorem it suffices to show that

1. $A_*(U)$ is generated by tautological classes
2. the image of $A_*(B)$ in $A_*(U \cup B)$ is generated by tautological classes
3. the image of $A_*(T)$ in $A_*(U \cup T \cup B)$ is generated by tautological classes
4. the image of $A_*(H)$ in $A_*(\mathcal{M}_5)$ is generated by tautological classes

Note that Mumford showed in [17] page 314 that $[H]_{\mathcal{M}_5}$ is a combination of tautological classes. Also, it is shown in [9] Lemma 1.4 that $A_*(H)$ is trivial. So the last step has already been done. (The proofs remain valid in characteristic $\neq 2$.)

To prove the other steps, we will stratify each of U, B and T separately. We will start with U and B. The stratification is based on the fact that for a given quartic Del Pezzo surface (with rational double points and determined up to projecitve equivalence) there is a locally closed subset of $V = U \cup B$ parametrizing curves whose canonical model embeds into the surface. For these two stratifications we need some preliminaries which we gather in the next section.

2. Quartic Del Pezzo surfaces and non-trigonal curves of genus 5

The canonical model of a smooth non-trigonal (and non-hyperelliptic) curve X of genus 5 is the base locus of a net of quadrics in $\mathbb{P}^4$. Let $\Pi \cong \mathbb{P}^2$ parametrize the net of quadrics containing X. Let $Q \subset \Pi$ be the plane quintic parametrizing the singular quadrics. It is well-known that Q has at worst ordinary double points as singularities and that these singularities correspond exactly to the quadrics of rank 3 containing X (cf. [3], pages 321 and 361-362). Let l be a pencil of quadrics containing X. Suppose that l is not contained in Q and let $q_1, ..., q_5$ be the points of intersection of Q with l. Then we have

Lemma 2.1. *Suppose that $q_1, ..., q_5$ are distinct. Then the base locus of l is a smooth quartic Del Pezzo surface $S(l)$. The 16 lines in $S(l)$ are all secant to X. Let $\langle s+t\rangle$ be one of the lines in $S(l)$, with $s, t \in X$. Then X^{st} is a plane sextic with 5 distinct double points, say $p_1, ..., p_5$. The points $p_1, ..., p_5$ are at worst ordinary cusps and no three of them are on a line. The conic through $p_1, ..., p_5$ also contains $\overline{s} := \pi_{st}(s)$ and $\overline{t} := \pi_{st}(t)$. The double point p_i is the image of the vertex s_i of q_i. The surface $S(l)$ is isomorphic to the blow up of $\mathbb{P}^2$ at the points $p_1, ..., p_5$.*

Proof: If the base locus $S(l)$ of l has a singular point p, then there is an element of l, say q, which is singular at p. Since l intersects Q in 5 distinct points, all these points are smooth points of Q hence correspond to quadrics of rank 4. So q has rank 4. In the space parametrizing the quadrics in $\mathbb{P}^4$ (isomorphic to $\mathbb{P}^{14}$), the locus of quadrics of rank ≤ 4 is an irreducible hypersurface G of degree 5 (defined by the vanishing of the determinant of the 5 by 5 matrix associated to a quadric). An elementary computation shows that the projectivized Zariski tangent space to G at q is the set of quadrics containing the vertex of q. Hence l is tangent to G at q. It follows that the number of singular quadrics in l is less than 5 which contradicts the assumption.

So the base locus of l is smooth and it is well-known in that case that it is a quartic Del Pezzo surface and contains exactly 16 distinct lines which are all secant to X since X is the complete intersection of $S(l)$ with a quadric. Since, by the above, none of the s_i's is on $S(l)$, for each i, the line $\langle s+t\rangle$ does not contain s_i. So $\langle s+t\rangle$ is contained in exactly one ruling of q_i which cuts a g^1_4, say g_i, on X such that $h^0(g_i - s - t) > 0$ (see e.g. [16] pages 342-343). Conversely, any g^1_4 such that $h^0(g^1_4 - s - t) > 0$ is cut by a ruling of a quadric q of rank ≤ 4 which must contain $S(l)$ since its intersection with $S(l)$ contains $X \cup \langle s+t\rangle$ (curve of degree $9 > 2.4 = \deg(q). \deg(S(l))$). Hence there are exactly 5 distinct lines $\langle p'_i + p''_i\rangle$ ($p'_i + p''_i \equiv g_i - s - t$) in $S(l)$ intersecting $\langle s+t\rangle$ and they project to five distinct double points $p_1, ..., p_5$ of X^{st}. The point s_i belongs to the plane $\langle s+t+p'_i+p''_i\rangle$ so that p_i is the image of s_i by the projection $\pi_{st} : \mathbb{P}^4 \longrightarrow \mathbb{P}^2$ from $\langle s+t\rangle$.

It is an easy exercise in intersection theory to check that the restriction of π_{st} to $S(l)$ is an isomorphism on the complement of the lines $\langle p'_i + p''_i\rangle$ (computation of the residual intersection of a plane containing $\langle s+t\rangle$ with $S(l)$, see e.g. [10] Example 9.1.1). Therefore the surface $S(l)$ is isomorphic to the blow up of $\mathbb{P}^2$ at $p_1, ..., p_5$ and, by adjunction, it is the image of this blow up in $\mathbb{P}^4$ by the linear system of strict transforms of cubics through $p_1, ..., p_5$. Hence, no three of the points $p_1, ..., p_5$ are collinear because $S(l)$ is smooth. Finally, $s+t \neq p'_i + p''_i$ for all i because this is the case for X, s, t general so that if for some X, s, t, $s+t = p'_i + p''_i$ for some i, then $S(l)$ can't be smooth because the number of lines in $S(l)$ would be less than 16. Finally, by adjunction, $s+t \equiv 2(K_X - s - t) - \sum_{i=1}^5 p'_i + p''_i$. Therefore $p_1, ..., p_5, \overline{s}$ and $\overline{t}$ lie on a conic. Q.E.D.

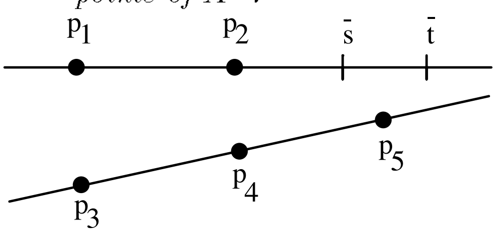
We now proceed to determine $S(l)$ for each $l \subset \Pi$ which is neither contained in Q nor transverse to it. In what follows, we call p_i an infinitely near double point of X^{st} if p_i is a double point of the strict transform of X^{st} in some blow up $\widetilde{\mathbb{P}}^2$ of $\mathbb{P}^2 \supset X^{st}$. We say that p_j is infinitely near of order 1 to p_i if p_j is an element of the exceptional divisor of the blow up of $\widetilde{\mathbb{P}}^2$ at p_i. In the set of double points of X^{st} we include those that are infinitely near. In this way we always have 5 distinct double points for X^{st} (note that the morphism $X \longrightarrow X^{st}$ is birational because l is not contained in Q). The list of anti-canonical Del Pezzo surfaces that we present in the lemma below is the same as that in [4] page 38. We need to present these from a different point of view for our purposes. The essence of the lemma is that the data of the configuration of double points of X^{st} is equivalent to the singularities and number of lines of $S(l)$ which is in turn equivalent to the data of the "type" of $l \cap Q$, where "type" is defined as follows

Definition 2.2. *To each point of intersection of l with Q we can associate two numbers: the first is the multiplicity of that point on Q and the second the order of contact of l with Q at that point. In this way we associate to $l \cap Q$ an unordered sequence of ordered pairs of positive integers of length ≤ 5. We call this sequence the* type *of $l \cap Q$ and denote it by* $\mathrm{type}(l \cap Q)$.

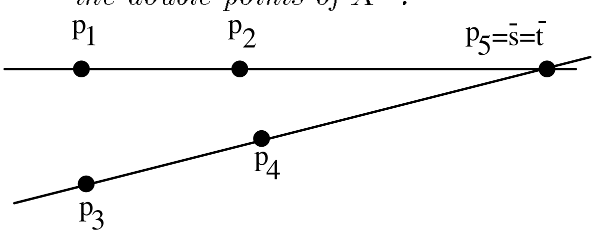
For instance, $\mathrm{type}(l \cap Q) = \{(1,1),(1,1),(1,1),(1,1),(1,1)\}$ if and only if $S(l)$ is smooth.

Lemma 2.3. *Suppose that $l \not\subset Q$ and that $q_1, \dots, q_5$ are* not *distinct. Whenever $S(l)$ has a double point we can and will assume that $\langle s+t \rangle$ contains it. If $S(l)$ has two double points (or more) and contains the line through them we can and will assume that $\langle s+t \rangle$ is that line. Then, up to a permutation of the q_i's, we have the following possibilities (in the following, the conic through $\overline{s}, \overline{t}, p_1, ..., p_5$ (see 2.1) has degenerated to the union of two lines or to twice a line: these are the lines represented on the pictures):*

1. $\mathrm{type}(l \cap Q) = \{(1,2),(1,1),(1,1),(1,1)\}$
   The line l is simply tangent to Q at $q_1 = q_2$ and is transverse to Q at q_3, q_4, q_5. This is equivalent to: $S(l)$ has only one node (i.e., an ordinary double point) and contains 12 *lines (4 of which pass through its node). Equivalently, we have the following picture for the double points of X^{st}:*

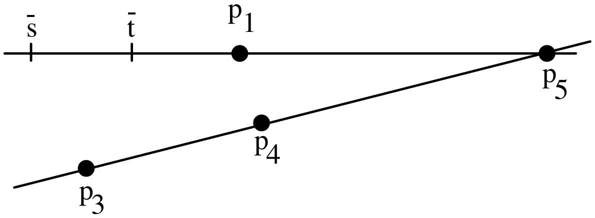


2. $\mathrm{type}(l \cap Q) = \{(1,2),(1,2),(1,1)\}$
   The line l is simply tangent to Q at two smooth points $q_1 = q_2$ and $q_3 = q_4$. This is equivalent to: $S(l)$ has two nodes and contains 9 lines. It is also equivalent to the following picture for the double points of X^{st}.



   In this case $S(l)$ contains the line through its nodes and there are 2 *other lines in $S(l)$ through each of the nodes.*

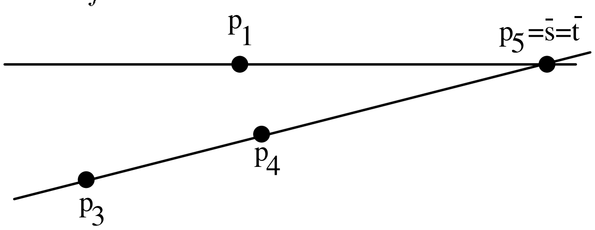
3. $\mathrm{type}(l \cap Q) = \{(1,3),(1,1),(1,1)\}$
   The line l has contact of order 3 *with Q at the smooth point $q_3 = q_4 = q_5$ and meets Q transversely at $q_1 \neq q_2$. This is equivalent to: $S(l)$ has one double point of type A_2 and contains* 8 *lines. It is also equivalent to the following picture for the double points of X^{st}*



with p_2 infinitely near of order 1 to p_5. In this case 4 of the lines in $S(l)$ pass through its double point.

4. $\mathrm{type}(l \cap Q) = \{(1,3),(1,2)\}$

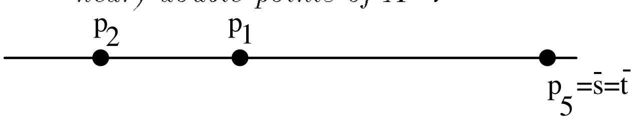
   The line l has contact of order 3 with Q at the smooth point $q_3 = q_4 = q_5$ and is tangent to Q at the smooth point $q_1 = q_2 \neq q_3$. Equivalently, $S(l)$ has one node and one double point of type A_2 and contains 6 lines. Equivalently, we have the following picture for the double points of X^{st}



   where p_2 is infinitely near of order 1 to p_5 and the line $\langle p_1 + p_5 \rangle$ is tangent to the branch(es) of X^{st} at p_5. In this case one of the lines in $S(l)$ passes through the two double points, one other passes through the node and two others through the A_2-double point.

5. $\mathrm{type}(l \cap Q) = \{(1,4),(1,1)\}$

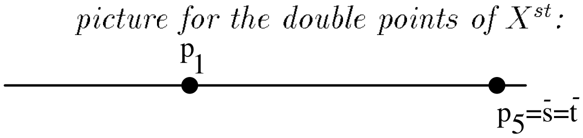
   The line l has contact of order 4 with Q at the smooth point $q_1 = q_2 = q_3 = q_4 (\neq q_5)$. Equivalently, the surface $S(l)$ has one double point of type A_3 and contains 5 lines. Equivalently, we can choose $\langle s + t \rangle$ in such a way that we have the following picture for the (non-infinitely near) double points of X^{st}:



   with p_3 infinitely near of order 1 to p_1 and p_4 infinitely near of order 1 to p_2. In this case 3 of the lines in $S(l)$ pass through the double point.

6. $\mathrm{type}(l \cap Q) = \{(1,5)\}$

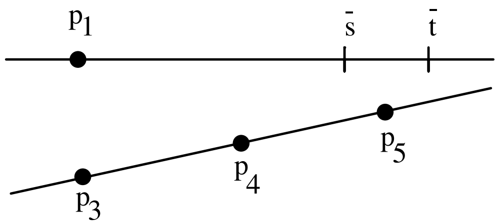
   The line l has contact of order 5 with Q at the smooth point $q_1 = q_2 = q_3 = q_4 = q_5$. Equivalently, $S(l)$ has one double point of type A_4 and contains 3 lines (2 of which pass through the double point). Equivalently, we can choose $\langle s+t \rangle$ in such a way that we have the following picture for the double points of X^{st}:



   with p_3 infinitely near of order 1 to p_1, p_4 infinitely near of order 1 to p_2, p_2 infinitely near of order 1 to p_5 and the line $\langle p_1 + p_5 \rangle$ tangent to the branch(es) of X^{st} at p_5.

7. $\mathrm{type}(l \cap Q) = \{(2,2),(1,1),(1,1),(1,1)\}$

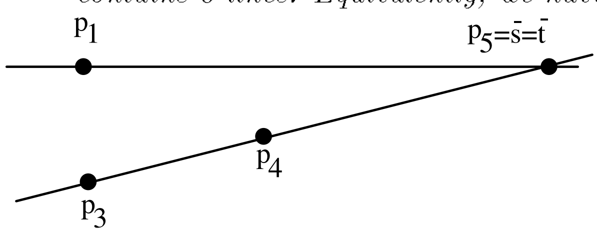
   The line l contains a node $q_1 = q_2$ of Q, is not tangent to any of the branches of Q at q_1 and is elsewhere transverse to Q. Equivalently, $S(l)$ has two nodes and contains 8 lines. Equivalently, we have the following picture for the double points of X^{st}:



with p_2 infinitely near of order 1 to p_1 and the line $\langle p_1+\overline{s}\rangle$ tangent to the branch(es) of X^{st} at p_1. In this case the nodes of $S(l)$ are located on the singular locus $Sing(q_1)$ of q_1 which is not contained in $S(l)$ and 4 lines (in $S(l)$) pass through each node.

8. type$(l\cap Q)=\{(2,2),(1,2),(1,1)\}$

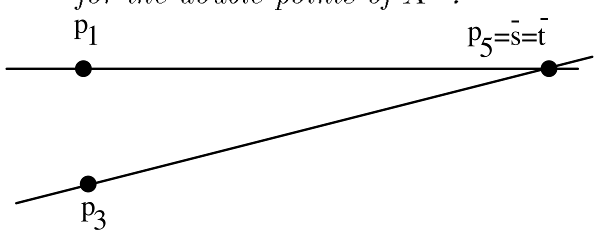
The line l contains a node $q_1=q_2$ of Q, is not tangent to any of the branches of Q at q_1 and is simply tangent to Q at a smooth point $q_3=q_4$. Equivalently, $S(l)$ has three nodes and contains 6 lines. Equivalently, we have the following picture for the double points of X^{st}:



with p_2 infinitely near of order 1 to p_1 and the line $\langle p_1+p_5\rangle$ tangent to the branch(es) of X^{st} at p_1. In this case two of the nodes of $S(l)$, say o_1 and o_2, are on $Sing(q_1)$ and the other is $Sing(q_3)$. The surface $S(l)$ contains the lines $\langle o_i, Sing(q_3)\rangle$ but does not contain $Sing(q_1)(=\langle o_1,o_2\rangle)$.

9. type$(l\cap Q)=\{(2,2),(1,3)\}$

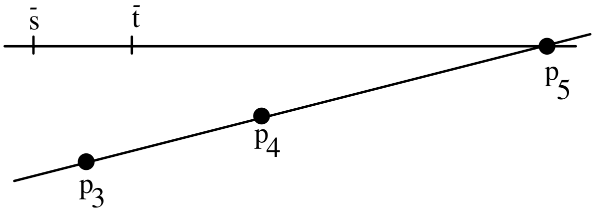
The line l contains a node $q_1=q_2$ and has contact of order 3 with Q at a smooth point $q_3=q_4=q_5$. Equivalently, $S(l)$ has two nodes and one double point of type A_2 and contains 4 lines (2 of which, as before, are $\langle o_i, Sing(q_3)\rangle$). Equivalently, we have the following picture for the double points of X^{st}:



with p_2 infinitely near of order 1 to p_1, p_4 infinitely near of order 1 to p_5, the line $\langle p_1+p_5\rangle$ tangent to the branch(es) of X^{st} at p_1 and the line $\langle p_3+p_5\rangle$ tangent to the branch(es) of X^{st} at p_5.

10. type$(l\cap Q)=\{(2,3),(1,1),(1,1)\}$

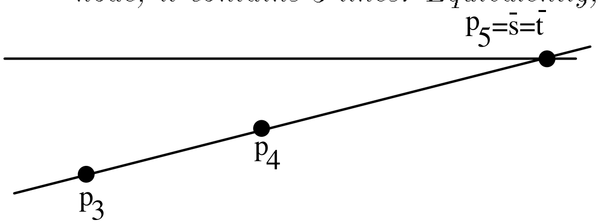
The line l contains a node $q_1=q_2$, is simply tangent to one of the branches of Q at q_1 and is elsewhere transverse to Q. Equivalently, $S(l)$ has one double point of type A_3 and contains 4 lines (all through the double point). Equivalently, the picture for the double points of X^{st} is as follows:



with p_2 infinitely near of order 1 *to p_1, p_1 infinitely near of order* 1 *to p_5 and the line $\langle p_5 + \overline{s} \rangle$ tangent to the branch(es) of X^{st} at p_5.*

11. $\mathrm{type}(l \cap Q) = \{(2,3),(1,2)\}$

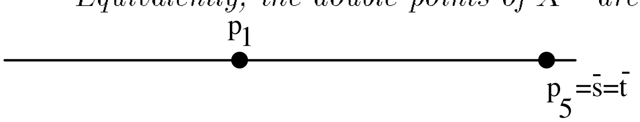
The line l contains a node $q_1 = q_2$, is simply tangent to one of the branches of Q at q_1 and is tangent to Q at a smooth point. Equivalently, $S(l)$ has one double point of type A_3 and one node; it contains 3 *lines. Equivalently, the double points of X^{st} are as follows:*



with p_2 infinitely near of order 1 *to p_1 and p_1 infinitely near of order* 1 *to p_5. The horizontal line in the picture is tangent to the branch(es) of X^{st} at p_1 and it intersects X^{st} only at p_1.*

12. $\mathrm{type}(l \cap Q) = \{(2,4),(1,1)\}$

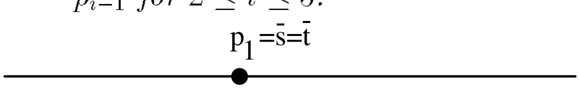
The line l contains a node $q_1 = q_2$ and has contact of order 3 *with one of the branches of Q at q_1. This is equivalent to: $S(l)$ has one double point of type D_4 and contains* 2 *lines. Equivalently, the double points of X^{st} are:*



with p_4 infinitely near of order 1 *to p_3, p_3 infinitely near of order* 1 *to p_2 and p_2 infinitely near of order* 1 *to p_1. The line $\langle p_1 + p_5 \rangle$ is tangent to the branch(es) of X^{st} at p_1.*

13. $\mathrm{type}(l \cap Q) = \{(2,5)\}$

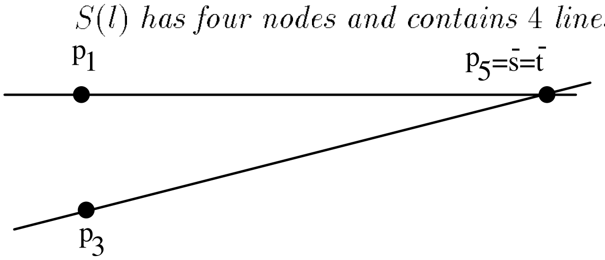
The line l contains a node of Q and has contact of order 4 *with one of the branches of Q there. Equivalently, $S(l)$ has one double point of type D_5 and contains* 1 *line. Equivalently, X^{st} has only one non-infinitely near double point, the point p_i is infinitely near of order* 1 *to p_{i-1} for $2 \leq i \leq 5$:*



The line in the picture is tangent to the branch(es) of X^{st} at p_1 and it intersects X^{st} only at p_1.

14. $\mathrm{type}(l \cap Q) = \{(2,2),(2,2),(1,1)\}$

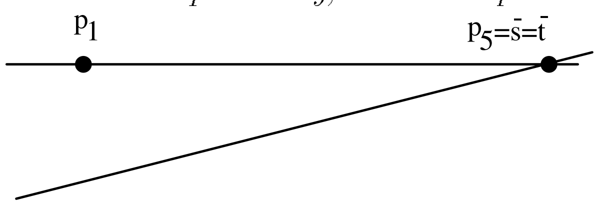
The line l passes through two nodes of Q and contains a smooth point of Q. Equivalently, $S(l)$ has four nodes and contains 4 *lines. Equivalently, the double points of X^{st} are as follows:*



with p_2 infinitely near of order 1 to p_1 and p_4 infinitely near of order 1 to p_3. The line $\langle p_i + p_5\rangle$ is tangent to the branch(es) of X^{st} at p_i for $i = 1$ or 3.

15. type$(l \cap Q) = \{(2,3),(2,2)\}$

The line l passes through two nodes of Q and is tangent to one of the branches of Q at one of the nodes. Equivalently, $S(l)$ has two nodes and one double point of type A_3; it contains 2 lines. Equivalently, the double points of X^{st} are as in the picture:



with p_4 infinitely near of order 1 to p_3, p_3 infinitely near of order 1 to p_5 and p_2 infinitely near of order 1 to p_1. The line $\langle p_1 + p_5\rangle$ is tangent to the branch(es) of X^{st} at p_1. The slanted line in the picture is tangent to the branch(es) of X^{st} at p_5 and it intersects X^{st} only at p_5.

Proof: The surface $S(l)$ is irreducible and generically reduced: otherwise X must be contained in a plane, a quadric surface or a cubic surface in $\mathbb{P}^4$. The first two cases are impossible because X is nondegenerate. The third case is excluded because X is neither trigonal nor hyperelliptic: By [18] page 365, any irreducible nondegenerate cubic surface must be a rational normal scroll in which case X is either hyperelliptic or trigonal since, by Riemann-Roch, the ruling of the rational normal scroll cuts either a g^1_2 or a g^1_3 or a g^2_4 on X. Since $S(l)$ is a complete intersection, it is Cohen-Macaulay. Since X is the complete intersection of $S(l)$ with a quadric and X is smooth, the surface $S(l)$ is smooth in codimension 1 and therefore normal.

Therefore, by, for instance, [18] Theorem 8 pages 366-367, (see also [4] page 34) the surface $S(l)$ is either the projection of a rational normal quartic scroll in $\mathbb{P}^5$ or an anticanonical Del Pezzo surface or a cone over a normal elliptic curve in $\mathbb{P}^3$. The first case is ruled out as before since the pencil of lines in $S(l)$ would cut a g^1_2 or a g^1_3 or a g^2_4 on X. In the third case every quadric of l is singular at the vertex of $S(l)$, hence is of rank at most 4 and $l \subset Q$: this contradicts our hypothesis. Therefore $S(l)$ is an anti-canonical Del Pezzo surface of degree 4. It is the image in $\mathbb{P}^4$ of the blow up of $\mathbb{P}^2$ at the double points (including the infinitely near ones) of X^{st} by the linear system of strict transforms of cubics passing through these double points (this follows, for instance, from the fact that the image of the strict transform of X^{st} in $\mathbb{P}^4$ is X which is smooth).

We will now show that $S(l)$ has one double point for each quadric of rank 4 appearing with multiplicity 2 in $l \cap Q$ and one or two double points for each quadric of rank 3 in l (note that we already saw in Lemma 2.1 that $S(l)$ is smooth if l contains 5 distinct quadrics of rank 4, i.e., if l is transverse to the locus of quadrics of rank ≤ 4):

If l is tangent to Q at a quadric q_1 of rank 4, then it is easily seen that all the quadrics of l contain the vertex s_1 of q_1 and hence $S(l)$ has a double point (since $S(l)$ does not have points of multiplicity ≥ 3) at s_1.

If l contains a quadric q of rank 3, then the only quadric of Π containing $Sing(q)$ is q: Containing $Sing(q)$ is a linear condition and if two distinct quadrics of l contain $Sing(q)$ then this is true for all the quadrics of l. If this is the case, then $S(l)$ is singular along $Sing(q)$. However, we saw above that $S(l)$ is normal. All the quadrics of $l \setminus \{q\}$ have the same intersection with $Sing(q)$ which is then either 2 distinct points or 1 point counted with multiplicity 2. It follows that $S(l)$ has 2 or 1 double points on $Sing(q)$.

To prove the beginning of the lemma, we show that every double point of $S(l)$ is on a secant of X. Since any double point is on the singular locus of some quadric parametrized by Q, we show that these singular loci are contained in the secant variety of X.

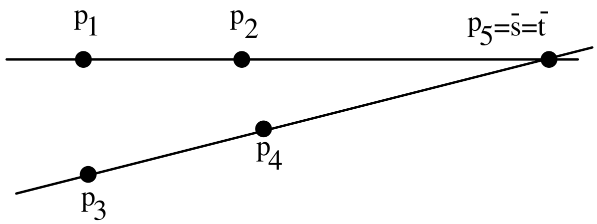
Let $s : Q \longrightarrow |K_X|^*$ be the rational map which sends a quadric $q \in Q$ of rank 4 to its vertex. Let N be the normalization of Q and $s_N : N \longrightarrow |K_X|^*$ the morphism induced by s. Let $s_N(N)$ be the image of N by this morphism.

Suppose for a moment that X is general. Then a general tangent line l to Q is also a general tangent line to G (the hypersurface parametrizing singular quadrics in $|\mathcal{O}_{\mathbb{P}^4}(2)|$, see the proof of 2.1). It is then easily seen that $S(l)$ has one node and contains 12 lines, 4 of which contain the node of $S(l)$. Since all the lines in $S(l)$ are secant to X (X is the complete intersection of $S(l)$ with a quadric), it follows that $s_N(N)$ is contained in the secant variety of X when X is general and hence for all X.

Now we show that the singular loci of the quadrics of rank 3 containing X are contained in the secant variety of X. Suppose for a moment that X is a general curve with a vanishing theta-null (this means that Q is a general plane quintic with a node). Then take l to be a pencil of quadrics of Π containing the quadric q of rank 3 and general for this property. Then l is also a line in $|\mathcal{O}_{\mathbb{P}^4}(2)|$ which intersects the subvariety parametrizing quadrics of rank ≤ 3 and l is general for this property. It is easily seen that $S(l)$ has 2 nodes which are the intersection of $S(l)$ with $Sing(q)$ and contains 4 lines through each node. Again, all the lines in $S(l)$ are secant to X. So it is enough to show that the two nodes of $S(l)$ move on $Sing(q)$ as l varies. The two nodes are the intersection with $Sing(q)$ of any quadric q' of l distinct from q. So their sum is the divisor of degree 2 associated to q' by the restriction map $\Pi \longrightarrow |\mathcal{O}_{Sing(q)}(2)|$. Hence it is sufficient to observe that Π induces a pencil on $Sing(q)$: by [3] page 364, the projectivized tangent cone to Q at q is the union of the two pencils of quadrics tangent to $Sing(q)$ at one of the two elements o_1, o_2 of $s_N(N) \cap Sing(q)$ so that $\Pi|_{Sing(q)}$ contains the two divisors $2o_1, 2o_2$ (so the pencil is in fact *base point free*!). So the singular locus of a quadric of rank 3 in Π is contained in the secant variety of X for X general with a vanishing theta-null and hence for all X with vanishing theta-nulls because this is a closed condition.

In the case by case analysis below we will prove that the type of $l \cap Q$ determines the configuration of double points of X^{st}. The assertions about the singularities of $S(l)$ and the number and configuration of lines in $S(l)$ are easily seen to be equivalent to those about the configuration of the double points of X^{st} with our choices of $\langle s+t \rangle$ by using the fact that $S(l)$ is the image of the blow up of $\mathbb{P}^2$ at the double points (including the infinitely near ones) of X^{st} by the linear system of strict transforms of cubics passing through these double points. Since for different types of $l \cap Q$, we obtain different kinds of singularities and numbers of lines for $S(l)$, we obtain that the kind of singularities and number of lines in $S(l)$ determines the type of $l \cap Q$.

1. In this case, $q_1 = q_2$ has rank 4 and (we are supposing) $s_1 = s_2 \in \langle s+t \rangle$. Hence $\langle s+t \rangle$ is contained in both rulings of q_1 and $2(s+t) + p_1' + p_1'' + p_2' + p_2'' \in |\omega_X|$. It follows that $p_1' + p_1'' \neq p_2' + p_2''$ because q_1 has rank 4 and its rulings cut two distinct g^1_4 on X. So p_1 and p_2 are not infinitely near double points for X^{st} and $p_1, p_2, \overline{s}$ and $\overline{t}$ are collinear. Since $\overline{s}, \overline{t}$ and the p_i's are on a conic (see Lemma 2.1), p_3, p_4, p_5 are also collinear. Since q_1, q_3, q_4, q_5 are distinct, it follows that the p_i's are not infinitely near to each other. The configuration of the points $p_1, \ldots, p_5, \overline{s}, \overline{t}$ is as asserted unless $s + t = p_j' + p_j''$ for some j. If this is the case, then, since X^{st} does not have infinitely near double points, we have $j = 3$ or 4 or 5, say $j = 5$. We would then have the following picture for the double points of X^{st} and $\overline{s}, \overline{t}$:



However, this would imply that $S(l)$ has 2 double points which is not possible by the preliminary discussion above.

An observation which will be useful below is that the 4 lines $\langle s+t\rangle$, $\langle p_3'+p_3''\rangle$, $\langle p_4'+p_4''\rangle$, $\langle p_5'+p_5''\rangle$ all pass through s_1 (which is the node of $S(l)$). This is because

- these lines intersect in a point because they intersect two by two (and no three of them are coplanar): The line $\langle s+t\rangle$ intersects the others since it is coplanar with each of them. Since p_3, p_4, p_5 are collinear, we have $s+t+p_3'+p_3''+p_4'+p_4''+p_5'+p_5'' \in |\omega_X|$. Hence, by Riemann Roch, since $|s+t+p_i'+p_i''|$ is a g^1_4, so is $|p_j'+p_j''+p_k'+p_k''|$ with $\{i,j,k\}=\{3,4,5\}$.
- the point of intersection of the four lines is singular on $S(l)$ since the projectivized Zariski tangent space to $S(l)$ at this point contains all four lines.

2. In this case $s_1 = s_2$ and $s_3 = s_4$. We first claim that we can suppose $s_1, s_3 \in \langle s+t\rangle$. We already know that we can suppose $s_1 \in \langle s+t\rangle$. If $s_3 \notin \langle s+t\rangle$, then $\langle s+t\rangle$ is contained in only one ruling of q_3 and $p_4'+p_4''=p_3'+p_3''$. Then $s+t+2p_3'+2p_3''+p_5'+p_5'' \in |\omega_X|$ and $s_3 \in \langle p_3'+p_3''\rangle$. As this case is a limit of case 1, the line $\langle p_3'+p_3''\rangle$ contains s_1. So we can replace $s+t$ by $p_3'+p_3''$ and suppose that $s_1, s_3 \in \langle s+t\rangle$.

   Since $\langle s+t\rangle$ contains s_1 and s_3, it still intersects 5 *distinct* secants to X (see the proof of case 1). So the p_i's are not infinitely near to each other. Now, $\overline{s}$ and $\overline{t}$ are on the two *distinct* lines $\langle p_1, p_2\rangle$ and $\langle p_3, p_4\rangle$. Hence $\overline{s}=\overline{t}=p_5$ since any line cuts a divisor of degree 6 on X^{st}.
3. In this case we see that $\langle s+t\rangle$ intersects exactly 4 distinct secants to X (since $\langle s+t\rangle \ni s_1$, see the proof of case 1). So X^{st} has four non-infinitely near double points, say p_1, p_3, p_4, p_5 and, p_2 is, for instance, infinitely near to p_5 (which is then either a tacnode or a cusp of higher order of X^{st}). Since this is a degeneration of case 1, we see that p_3, p_4, p_5 are collinear as well as $\overline{s}, \overline{t}, p_1, p_5$. Since X^{st} has degree 6, neither of the two lines $\langle p_1, p_5\rangle$ and $\langle p_3, p_4\rangle$ is tangent to any branch of X^{st} at any point.
4. In this case $s_1 = s_2 \neq s_3 = s_4 = s_5$. As in case 2, we can suppose that $s_1, s_3 \in \langle s+t\rangle$. Hence, we have 4 non-infinitely near p_i's (the is similar to the proof of case 1) and, for instance, p_2 is infinitely near of order 1 to p_5 which is then a tacnode or cusp of higher order of X^{st}. That the configuration of double points of X^{st} is as asserted is now an easy consequence of the fact that this case is a degeneration of case 2.
5. In this case $s_1 = s_2 = s_3 = s_4 \neq s_5$. Supposing again $s_1 \in \langle s+t\rangle$, we have three non-infinitely near double points for X^{st}, say p_1, p_2, p_5 which are collinear (see the proof of case 1). Since this is a degeneration of case 2 in which q_1 and q_3 have come together, we have, for instance, p_3 is infinitely near of order 1 to p_1 and p_4 is infinitely near of order 1 to p_2.
6. This case is analogous to the previous case: it is a degeneration of case 4 in which q_1 and q_3 have come together.
7. This case is a degeneration of case 1 in which the quadric q_1 has rank 3. So p_3, p_4, p_5 are collinear and $p_1'+p_1''=p_2'+p_2''$. For instance, p_2 is infinitely near to p_1 which is a then tacnode or cusp of higher order. Also, since $2(s+t+p_1'+p_1'')$ is a canonical divisor, it spans a hyperplane in $\mathbb{P}^4$ and the tangent line to the branch(es) of X^{st} at p_1 also contains $\overline{s}$ and $\overline{t}$.

Cases $8, 10, 11, 12, 13$ are degenerations, respectively, of cases $2, 3, 4, 5, 6$ in which q_1 has rank 3. After exchanging (p_1, p_2) with (p_3, p_4) in cases 9 and 15, cases $9, 14, 15$ are degenerations of cases $4, 8, 11$ in which q_4 has rank 3. All these cases are similar to case 7. Q.E.D.

Remark 2.4. *Since the only configurations of double points (of X^{st}) in Lemma 2.3 that have moduli are 1 and 7, we see that, if the number of distinct points of intersection of Q with l is at most 3, then the projective equivalence class of $S(l)$ only depends on the* type *of $Q \cap l$ (see definition 2.2).*

3. The Chow group of U

We can now proceed to cut U into a union of locally closed subvarieties with trivial Chow groups. The classes of the closures of these locally closed subvarieties will then generate $A_*(U)$. We will show that these classes are polynomials in the tautological classes, henceforth proving that $A_*(U)$ is generated by tautological classes. We will then use this to show that, in fact, $A_*(U) = 0$.

Below we define our strata. For each subvariety P of U that we define, we denote by $\overline{P}$ the closure of P in U.

U_0: open subvariety of U parametrizing curves whose associated quintic Q is smooth, has no hyperflexes and no flex bitangents.

U_1: locally closed subvariety of U parametrizing curves such that Q is smooth, has a flex bitangent and has no hyperflex of contact 5.

V_1: locally closed subvariety of U parametrizing curves such that Q is smooth, has a hyperflex of contact 4 and has no hyperflex of contact 5.

V_6': closed subvariety of V_1 parametrizing curves such that for some $s+t$ such that X^{st} has the configuration of double points in Lemma 2.3,5, we have $p_1'' = p_1' = u_1$, $p_2'' = p_2' = u_2$ and $s = t$.

V_5': closed subvariety of $V_1 \setminus V_6'$ parametrizing curves such that for some $s+t$ such that X^{st} has the configuration of double points in Lemma 2.3,5, we have $p_1'' = p_1' = u_1$ and $p_2'' = p_2' = u_2$ (where $u_i + v_i \equiv K_X - s - t - 2p_i' - 2p_i''$ for $i = 1, 2$).

V_4': closed subvariety of $V_1 \setminus \overline{V}_5'$ parametrizing curves such that for some $s+t$ such that X^{st} has the configuration of double points in Lemma 2.3,5, we have $p_1'' = p_1' = u_1$ and p_2'' or $p_2' = u_2$.

V_3': closed subvariety of $V_1 \setminus \overline{V}_4'$ parametrizing curves such that for some $s+t$ such that X^{st} has the configuration of double points in Lemma 2.3,5, we have $p_i'' = p_i' = u_i$ for $i = 1$ or 2.

V_2': closed subvariety of $V_1 \setminus \overline{V}_3'$ parametrizing curves such that, for some $s+t$ such that X^{st} has the configuration of double points in Lemma 2.3,5, we have $p_i' = u_i$ for $i = 1$ or 2.

V_1': $= V_1 \setminus \overline{V}_2'$.

U_2: locally closed subvariety of U parametrizing curves such that Q is smooth, has a hyperflex of contact 5 and $p_1', p_1'' \neq u_1, v_1$ for all $s+t$ such that X^{st} has the configuration of double points in Lemma 2.3,6.

U_4: locally closed subvariety of $\overline{U}_2$ parametrizing curves such that Q is smooth and $p_1' = p_1'' = u_1$ for some $s+t$ such that X^{st} has the configuration of double points in Lemma 2.3,6.

U_3: locally closed subvariety of $\overline{U}_2 \setminus \overline{U}_4$ parametrizing curves such that Q is smooth and $p_1'' \neq p_1' = u_1$ for some $s+t$ such that X^{st} has the configuration of double points in Lemma 2.3,6.

U_{null}: closed subvariety of U parametrizing curves with a vanishing theta-null, i.e., with Q singular.

V_3: locally closed subvariety of U_{null} parametrizing curves with two vanishing theta-nulls or more, such that (at least) one of the lines through two of the nodes of Q is tangent to one of the branches of Q at one of the nodes and $s \neq t$ for all $s+t$ such that X^{st} has the configuration of double points in Lemma 2.3,15.

V_5: closed subvariety of $\overline{V}_3$ parametrizing curves such that $p'_1 = p''_1$ and $s = t$ for some $s + t$ such that X^{st} has the configuration of double points in Lemma 2.3,15.

V_4: closed subvariety of $\overline{V}_3 \setminus \overline{V}_5$ parametrizing curves such that $s = t$ for some $s + t$ such that X^{st} has the configuration of double points in Lemma 2.3,15.

Z_2: closed subvariety of $U_{null} \setminus \overline{V}_3$ parametrizing curves with two vanishing theta-nulls or more.

W_3: locally closed subvariety of U_{null} parametrizing curves such that the line tangent to one of the branches of Q at its node has contact of order 4 with that branch and $s \neq t$ on X for all $s + t$ such that X^{st} has the configuration in Lemma 2.3,13.

W_4: closed subvariety of $\overline{W}_3$ such that $s = t$ for some $s + t$ as in the previous case.

Y_2: closed subvariety of $U_{null} \setminus \overline{W}_3$ parametrizing curves such that the line tangent to one of the branches of Q at its node has contact of order 3 with that branch.

W_2: closed subvariety of $U_{null} \setminus (\overline{Z}_2 \cup \overline{W}_3)$ parametrizing curves such that the line tangent to one of the branches of Q at its node is tangent to Q at a smooth point.

V_2: closed subvariety of $U_{null} \setminus (\overline{V}_3 \cup \overline{W}_3)$ parametrizing curves such that a flex line of Q (at a smooth point) passes through a node of Q.

W_1: $= U_{null} \setminus (\overline{V}_2 \cup \overline{Y}_2 \cup \overline{W}_2 \cup \overline{Z}_2)$.

Note that in the above list the subscript of any letter is the codimension of the corresponding subvariety. We will now use what we proved earlier to show:

Theorem 3.1. *The varieties $U_i, V'_i, V_2, V_3, V_4, W_i, Y_i, Z_2$ are irreducible and have trivial Chow groups.*

Proof: In this preliminary part of the proof only, we will denote by P any of the subvarieties in the statement of the theorem. To prove the theorem, we will produce a nonempty open subset $\widehat{P}$ of some projective space which maps onto P by a finite morphism. The irreducibility of P will immediately follow from that of $\widehat{P}$. The complement of $\widehat{P}$ in its projective space will have codimension 1 so we can deduce that the Chow ring with rational coefficients of $\widehat{P}$ is trivial. It then follows from Lemma A, page 332 in [9], that the Chow ring of P is trivial as well.

By Remark 2.4, there is a *unique* (up to projective equivalence) quartic Del Pezzo surface $S(P)$ such that, for every X such that m_X (the moduli point of X, see "Notation and Conventions") is in P, the canonical model of X embeds in $S(P)$. (For instance: if $P = U_0$, then $S(P)$ is the base locus of a pencil of quadrics parametrized by a line which is bitangent to Q ; if $P = U_1$, then $S(P)$ is the base locus of a pencil of quadrics parametrized by a line which is a flex bitangent of Q.) We let $l(P)$ denote a line in $S(P)$ such that by projecting from it we obtain the corresponding configuration of points in Lemma 2.3. Fix a projective embedding of $S(P)$. Let $L(P)$ denote the pencil of quadrics containing $S(P)$ and let $\mathbb{P}(P)$ be the quotient of the linear system $|\mathcal{O}_{\mathbb{P}^4}(2)|$ of quadrics in $\mathbb{P}^4$ by $L(P)$. Then $\mathbb{P}(P)$ is a projective space of dimension 12 and its elements can be identified with nets of quadrics containing $L(P)$. Let $\widetilde{P}$ be the open subset of $\mathbb{P}(P)$ parametrizing nets of quadrics whose base locus is a smooth curve with moduli point in P. Then we have a canonical morphism $\widetilde{P} \longrightarrow P$ which to each net of quadrics associates (the isomorphism class of) its base locus. The variety $\widehat{P}$ (defined below and mentioned above) is the intersection of $\widetilde{P}$ with a linear subspace of $\mathbb{P}(P)$. In each case, the morphism $\widehat{P} \longrightarrow P$ is finite because it is quasi-finite and proper. The quasi-finiteness will follow from the fact that the elements of $\widehat{P}$ have finite stabilizers in the automorphism group of $S(P)$ and the properness will follow from the fact that the only time the cardinality of the fibers of $\widehat{P} \longrightarrow P$ goes down by specialization is when $\widehat{P} \longrightarrow P$ ramifies. We give a complete proof of the finiteness of the restriction morphism $\widehat{P} \longrightarrow P$ only in the case $P = U_0$ since all the other cases are analogous.

We now proceed with the proof case by case of the theorem.

U_0: The surface $S(U_0)$ is described in Lemma 2.3,2. The canonical morphism $\widetilde{U}_0 \longrightarrow U_0$ is finite (so $\widehat{U}_0 = \widetilde{U}_0$). i.e., quasi-finite and proper.

To show that it is quasi-finite we show that for every curve X in $S(U_0)$ there are at most a finite number of curves Y in $S(U_0)$ which are projectively equivalent to X. Let g_Y be the projective transformation sending Y to X. Then either g_Y preserves $S(U_0)$ or g_Y sends $S(U_0)$ to another Del Pezzo surface containing X. In the latter case $g_Y(S(U_0))$ is the base locus of a pencil of quadrics containing X which is distinct from $L(U_0)$ and is simply bitangent to Q by Lemma 2.3. Since Q has a finite number of bitangents, we have only a finite number of curves Y such that g_Y does not preserve $S(U_0)$. There are also a finite number of curves Y such that g_Y does preserve $S(U_0)$ because the automorphism group of $S(U_0)$ is finite (easy to check: an automorphism of $S(U_0)$ comes from an automorphism of $\mathbb{P}^2$ which fixes p_5 and preserves the sets $\{p_1, p_2\}$ and $\{p_3, p_4\}$ in Lemma 2.3,2).

To show that the morphism is proper, we observe that it is separated and of finite type and use the valuative criterion of properness:

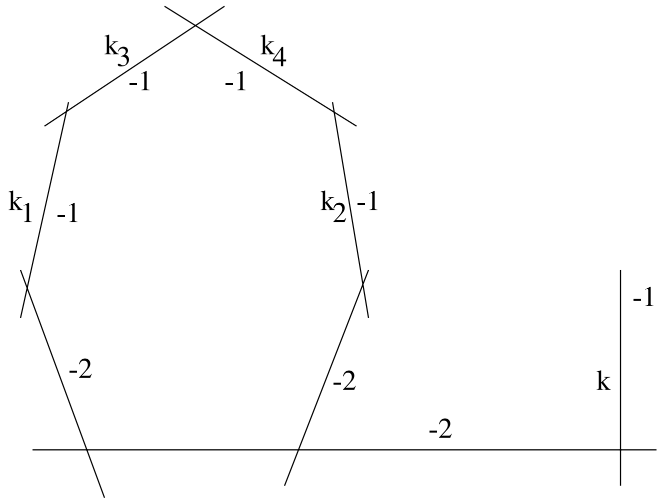
Let $\operatorname{Spec} R$ be the spectrum of a discrete valuation ring R with generic point η and closed point r. Let $\mathcal{X}$ be a smooth curve of genus 5 over $\operatorname{Spec} R$ such that the image of the morphism $\phi : \operatorname{Spec} R \longrightarrow \mathcal{M}_5$ obtained from $\mathcal{X}$ lies in U_0 and we are given a lift $\psi_\eta : \eta \longrightarrow \widetilde{U}_0$ of the restriction of ϕ to η. We need to extend ψ_η to all of $\operatorname{Spec} R$. The data of ψ_η is equivalent to the data of a birational map $f_\eta : \mathcal{X}_\eta \longrightarrow \mathbb{P}^2_\eta$ such that the configuration of double points of the image of $\mathcal{X}_\eta$ is as in Lemma 2.3,2. This map specializes to a map $f_r : \mathcal{X}_r \longrightarrow \mathbb{P}^2_r$ which is birational because $\mathcal{X}_r$ is not trigonal, nor bielliptic nor hyperelliptic. The configuration of double points of the image of $\mathcal{X}_r$ by f_r is a limit of the configuration of double points of $f_\eta(\mathcal{X}_\eta)$ and it cannot be more degenerate than this latter because the image of ϕ is contained in U_0. Thus the configuration of double points of $f_r(\mathcal{X}_r)$ is the same as that of $f_\eta(\mathcal{X}_\eta)$. Thus we have extended ψ_η to $\operatorname{Spec} R$.

U_1: The surface $S(U_1)$ is described in Lemma 2.3,4. The fibers of the morphism $\widetilde{U}_1 \longrightarrow U_1$ have dimension at least one. Since Q has a finite number of flex bitangents, the positive-dimensional fibers come from the automorphism group of $S(U_1)$. Hence, since the general fibers are one-dimensional, all the fibers are one-dimensional. The automorphism group of $S(U_1)$ fixes the double points of $S(U_1)$ since the double points are of different kinds. So the automorphism group of $S(U_1)$ preserves $l(U_1)$ (since $l(U_1)$ contains the two double points, see Lemma 2.3,4).

We claim that the only fixed points on $l(U_1)$ of any one-dimensional subgroup of the automorphism group of $S(U_1)$ are the two double points: If such a group has a third fixed point on $l(U_1)$, then it must fix every point on $l(U_1)$. The automorphism group is generated by an element f of PGL_3 which fixes p_1 and p_5 and exchanges p_3 and p_4 and the subgroup G of PGL_3 which fixes the points p_1, p_5, p_3, p_4 or, equivalently, the line $\langle p_3, p_4 \rangle$ and the point p_1. Therefore G is the only one-dimensional proper subgroup of the automorphism group. A conic C in $\mathbb{P}^2$ which is tangent to $\langle p_1, p_5 \rangle$ at p_5, contains p_1 and p_3 and is general for this property is not globally invariant under G since its second point of intersection with $\langle p_1, p_4 \rangle$ is not fixed by G. The conic C and its image by a general element of G have order of contact 2 at p_5, hence their strict transforms in $S(U_1)$ intersect $l(U_1)$ in two distinct points which are then not fixed under G (or the full automorphism group).

Fix a point $p \in l(U_1)$ distinct from the double points of $S(U_1)$. Since the stabilizer of p in the automorphism group of $S(U_1)$ is finite, for each curve X with $m_X \in U_1$, only a finite number of curves in $S(U_1)$ and isomorphic to X contain p. We let $\widehat{U}_1$ be the subvariety of $\widetilde{U}_1$ parametrizing nets of quadrics through p (note that none of the curves X with $m_X \in U_1$ contains any double point of $S(U_1)$ because otherwise X would not be smooth).

$V_i'(1 \leq i \leq 6)$: The surface $S(V_1) = S(V_i')$ is described in Lemma 2.3,5. The configuration of curves with negative self-intersection on the minimal desingularization of $S(V_1)$ is the following



With our previous notation (in Lemma 2.3 and the definition of V_1'), the line k_1 (resp. k_2) above is the strict transform of $\langle p_1' + p_1'' \rangle$ (resp. $\langle p_2' + p_2'' \rangle$), the line k_3 (resp. k_4) is the strict transform of $\langle u_1 + v_1 \rangle$ (resp. $\langle u_2 + v_2 \rangle$) and $k = l(V_1) = l(V_i')$ is the strict transform of $\langle s + t \rangle$.

Let m_1 and m_2 be the lines (in $\mathbb{P}^2$) through p_1 and p_2 with tangent directions given by p_3 and p_4 respectively. The data of an automorphism of $S(V_1)$ is equivalent to the data of an automorphism of $\mathbb{P}^2$ which fixes p_5 and preserves or exchanges the pairs (p_1, m_1) and (p_2, m_2). Equivalently, the automorphism of $\mathbb{P}^2$ must fix p_5 and the point $p_0 := m_1 \cap m_3$ and must either fix or exchange p_1 and p_2. It follows that any automorphism of $S(V_1)$ lifts to an automorphism of its minimal desingularization which leaves k invariant, fixes the point $k_3 \cap k_4$ (inverse image of p_0) and leaves invariant or exchanges (k_1, k_3) and (k_2, k_4). In particular, there is a one-dimensional subgroup (of index two) of the automorphism group of $S(V_1)$ which fixes $k_1 \cap k_3$ and $k_2 \cap k_4$. It can be seen as in the case of U_1 that the point of intersection with the -2 curve and $k_1 \cap k_3$ (resp. $k_2 \cap k_4$) are the only points on k_1 (resp. k_2) which are fixed under a one-dimensional subgroup of the automorphism group of $S(V_1)$. None of our smooth curves of genus 5 passes through the double point of $S(V_1)$ so their strict transforms do not intersect the (-2)-curves in the picture.

V_1': Choose a general point p on k_1 and let $\widehat{V}_1'$ be the subvariety of $\widetilde{V}_1$ parametrizing nets of quadrics through (the image of) p (in $S(V_1)$).

V_2': Let $\widehat{V}_2'$ be the subvariety of $\widetilde{V}_1$ parametrizing nets of quadrics through (the images of) p and $k_1 \cap k_3$ (in $S(V_1)$).

V_3': Choose a general point q on k_2. Let $\widehat{V}_3'$ be the subvariety of $\widetilde{V}_1$ parametrizing nets of quadrics through q and tangent to k_1 at $k_1 \cap k_3$.

V_4': Let $\widehat{V}_4'$ be the subvariety of $\widetilde{V}_1$ parametrizing nets of quadrics through q and $k_2 \cap k_4$ and tangent to k_1 at $k_1 \cap k_3$.

V_5': Choose a general conic C which corresponds to a line in $\mathbb{P}^2$ through p_1. Let $\widehat{V}_5'$ be the subvariety of $\widetilde{V}_1$ parametrizing nets of quadrics tangent to C and tangent to k_i at $k_i \cap k_{i+2}$ for $i = 1$ and $i = 2$.

V_6': Let $\widehat{V}_6'$ be the subvariety of $\widetilde{V}_1$ parametrizing nets of quadrics tangent to C, tangent to k and tangent to k_i at $k_i \cap k_{i+2}$ for $i = 1, 2$.

W_1: The surface $S(W_1)$ is described in Lemma 2.3,8: it has three nodes and contains two of the lines through the three nodes. By our choices, $l(W_1)$ is one of these two lines. Choose a point

p on $l(W_1)$ distinct from the nodes. As in the case of U_1, we can let $\widehat{W}_1$ be the subvariety of $\widetilde{W}_1$ parametrizing nets of quadrics through p.

U_2: The surface $S(U_2)$ is described in Lemma 2.3,6: it has one double point, contains 3 lines one of which ($l(U_2)$) intersects only one other line (at the double point), say $l'(= \langle p_1' + p_1'' \rangle)$. The line l' intersects the third line, say l'', outside the double point. Fix two general points p and q on $l(U_2)$ and l' respectively and let $\widehat{U}_2$ be the subvariety of $\widetilde{U}_2$ parametrizing nets of quadrics through p and q.

U_3: We have $S(U_3) = S(U_2)$. Let $\widehat{U}_2$ be the subvariety of $\widetilde{U}_3$ parametrizing nets of quadrics through p, q and $l' \cap l''$.

U_4: We have $S(U_4) = S(U_2)$. Choose a general point r on l''. Let $\widehat{U}_4$ be the subvariety of $\widetilde{U}_2$ parametrizing nets of quadrics through q and r and tangent to l' at $l' \cap l''$.

V_2: The surface $S(V_2)$ is described in Lemma 2.3,9: the two lines $\langle o_i, Sing(q_3) \rangle$ (see 2.3,9) do not intersect any lines in $S(V_2)$ (other than themselves) outside the double points of $S(V_2)$ and each contain two double points of $S(V_2)$. Choose general points p and q, one on each line. Let $\widehat{V}_2$ be the subvariety of $\widetilde{V}_2$ parametrizing nets of quadrics through p and q.

W_2, Y_2, Z_2: The surfaces $S(W_2), S(Y_2), S(Z_2)$ are described in Lemma 2.3 respectively in cases $11, 12, 14$. Each surface contains (at least) two lines which do not contain any point (other than a double point) fixed under a two-dimensional subgroup of the automorphism group of the surface. These cases are analogous to the case of V_2.

W_3: The surface $S(W_3)$ is described in Lemma 2.3,13. It contains one line ($l(W_3)$) and one pencil of conics. The irreducible conics in this pencil intersect $l(W_3)$ only at the double point of $S(W_3)$. Choose an irreducible conic C on $S(W_3)$ and a point r on C distinct from the double point of $S(W_4)$. Choose distinct points p and q (distinct from the double point) on $l(W_3)$. Let $\widehat{W}_3$ be the subvariety of $\widetilde{W}_3$ parametrizing nets of quadrics through p, q and r.

W_4: We have $S(W_4) = S(W_3)$. Let $\widehat{W}_4$ be the subvariety of $\widetilde{W}_4$ parametrizing nets of quadrics tangent to $l(W_3)$ at p and tangent to C at r.

V_3: The surface $S(V_3)$ is described in Lemma 2.3,15. It contains two lines which do not contain any point (other than a double point) fixed under a three-dimensional subgroup of the automorphism group of the surface. Choose two general points on the line $l(V_3)$ (equal to $\langle s + t \rangle$) and a general point on the second line l'. Let $\widehat{V}_3$ be the subvariety of $\widetilde{V}_3$ parametrizing nets of quadrics through these three points.

V_4: We have $S(V_3) = S(V_2)$. Choose a general point p on $l(V_3)$, two general point q and r on the second line l'. Let $\widehat{V}_4$ be the subvariety of $\widetilde{V}_3$ parametrizing nets of quadrics through q and r and tangent to $l(V_3)$ at p.

V_5: The surface $S(V_5) = S(V_2)$ has two pencils of conics whose smooth elements intersect the two lines in $S(V_3)$ only at the double points. Choose a general conic C from one of these pencils. Let $\widehat{V}_5$ be the subvariety of $\widetilde{V}_3$ parametrizing nets of quadrics tangent to C, tangent to $l(V_3)$ at p and tangent to l' at q. Q.E.D.

It follows from the above theorem that $[\overline{U}_i]_U, [\overline{V}_i]_U, [\overline{V}'_i]_U, [\overline{W}_i]_U, [\overline{Y}_i]_U, [\overline{Z}_2]_U$ generate $A_*(U)$. In [17] (page 299) Mumford defined a set of tautological classes in the Chow ring of $\overline{\mathcal{M}}_g$. We recall their definition:

Let $\overline{\mathcal{C}}_g$ be the coarse moduli space of one-pointed stable curves of genus g. Let ω be the relative dualizing sheaf of the natural morphism $\pi : \overline{\mathcal{C}}_g \longrightarrow \overline{\mathcal{M}}_g$ and put $K = c_1(\omega)$. Then

$$\kappa_l = \pi_* K^{l+1}, \ \lambda = c_1(\pi_* \omega), \ \lambda_l = c_l(\pi_* \omega) \ for \ l > 1.$$

Conjecturally (see [17] page 272), the restrictions of these classes to $\mathcal{M}_g$ generate the stable cohomology of $\mathcal{M}_g$, i.e., when g is big compared to k, then $H^{2k}(\mathcal{M}_g)$ is generated by these classes and $H^{2k+1}(\mathcal{M}_g) = 0$.

We denote by the same symbols the restrictions of the above classes to any subvariety of $\mathcal{M}_g$ (for $g = 3$ or 5). For a subvariety A of $\overline{\mathcal{M}}_g$, we denote by $\mathcal{C}_A$ the restriction of $\overline{\mathcal{C}}_g$ to A and by $\mathcal{C}_A^n$ its n-th fiber product over A. For each subset $I = \{i_1, \ldots, i_r\}$ of $\{1, \ldots, n\}$ we let $\Delta_I = \Delta_{i_1,\ldots,i_r}$ be the diagonal of $\mathcal{C}_A^n$ parametrizing elements $(x_1, \ldots, x_n)$ in the fibers of $\mathcal{C}_A^n \longrightarrow A$ such that $x_{i_1} = \ldots = x_{i_r}$. We also let K_i be the first Chern class of the pull-back of ω by the i-th projection $\mathcal{C}_A^n \longrightarrow \mathcal{C}_A$. We now show

Theorem 3.2. *The classes $[\overline{U}_i]_U, [\overline{V}_i]_U, [\overline{V}_i']_U, [\overline{W}_i]_U, [\overline{Y}_i]_U, [\overline{Z}_2]_U$ are polynomials in the tautological classes.*

Proof: We will characterize the points of each variety in terms of the existence of points (or divisors) with certain properties on the corresponding curves. Then we will use determinantal formulas to show that the classes are combinations of tautological classes. In order to be able to apply the determinantal formulas, we need U to be Cohen-Macaulay. This is certainly the case in characteristic zero but we do not know whether the same holds in positive characteristic (since the quotient of a smooth variety by a finite group may not be Cohen-Macaulay if the characteristic of the base field k divides the order of the group). Therefore we will need to replace U by a smooth finite cover and all the strata by their inverse images in this finite cover. By Lemma A page 332 in [9], if we show that the classes of the inverse images of the (closures of the) strata are combinations of the (inverse images of the) tautological classes, then it will follow that the classes of the (closures of the) strata themselves are combinations of the tautological classes. To avoid notational complications we will only use this finite cover implicitly, i.e., denote the finite cover of U and the inverse images of the strata by the same letters. This finite cover will only be used when we are applying determinantal formulas in Chern classes of bundles.

Consider the configuration of the double points of X^{st} given in Lemma 2.3,2. The configuration of points for any of our strata is a limit of this configuration in which some of the points have come together. We will define a finite cover $\mathcal{P}$ of U which, roughly speaking, parametrizes the points in configuration 2.3,2, sits inside $\mathcal{C}_U^{14}$ and whose intersections with various diagonals will map onto the closures of our strata.

Consider the codimension 14 subvariety of $\mathcal{C}_U^{14}$ parametrizing 14-tuples $(s, t, p_1', p_1'', \ldots, p_4', p_4'', u_1, v_1, u_2, v_2)$ of points on curves X ($m_X \in U$) such that

$$\left\{\begin{array}{l} h^0(2s+2t) \geq 2 \ , \ h^0(s+t+p_1'+p_1'') \geq 2 \ , \\ h^0(s+t+p_1'+p_1''+p_2'+p_2'') \geq 3 \ , \\ h^0(2s+t+p_1'+p_1''+p_2'+p_2'') \geq 4 \ , \\ h^0(s+t+p_1'+p_3'+u_1+v_1) \geq 3 \ , \\ h^0(s+t+p_3'+p_3'') \geq 2 \ , \\ h^0(s+t+p_3'+p_3''+p_4'+p_4'') \geq 3 \ , \\ h^0(2s+t+p_3'+p_3''+p_4'+p_4'') \geq 4 \ , \\ h^0(s+t+p_2'+p_4'+u_2+v_2) \geq 3. \end{array}\right. \tag{1}$$

The condition $h^0(2s+2t) \geq 2$ insures that $\overline{s} = \overline{t}$ so that we have the configuration of double points in Lemma 2.3,2 (or a degeneration of it) for the double points of X^{st}. The condition $h^0(s+t+p_1'+p_1'') \geq 2$ means that p_1' and p_1'' map to a double point p_1 of X^{st} and the condition $h^0(s+t+p_1'+p_1''+p_2'+p_2'') \geq 3$ means that p_1 and the images of p_2' and p_2'' in $|K_X - s - t|^*$ are collinear. Then $h^0(2s+t+p_1'+p_1''+p_2'+p_2'') \geq 4$ means that $\overline{s}, p_1$ and the images of p_2' and p_2'' are collinear which implies that p_2' and p_2'' map to the double point p_2 of X^{st} since the line $\langle \overline{s}, p_1 \rangle$ intersects X^{st} in $\overline{s}$, p_1 and p_2. Finally, the condition $h^0(s+t+p_1'+p_3'+u_1+v_1) \geq 2$ means that p_1, p_3 and the images of u and v are collinear.

The locus of divisors of the form $\sum_{i=1}^r n_i r_i$ with $0 \leq n_i \in \mathbb{Z}$ and $r_i \in X$, $m_X \in U$, for all i, such that $\sum_{i=1}^r n_i = n$ and $h^0(\sum_{i=1}^r n_i r_i) \geq k$ is a degeneracy locus for a map of vector bundles in the following way: Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}_U^{r+1}} \longrightarrow \mathcal{O}_{\mathcal{C}_U^{r+1}}\left(\sum_{i=1}^r n_i \Delta_{i,r+1}\right) \longrightarrow \frac{\mathcal{O}_{\mathcal{C}_U^{r+1}}\left(\sum_{i=1}^r n_i \Delta_{i,r+1}\right)}{\mathcal{O}_{\mathcal{C}_U^{r+1}}} \longrightarrow 0.$$

The sequence of higher direct images by the projection ρ to the first r factors is

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}_U^r} \longrightarrow \rho_*\left(\mathcal{O}_{\mathcal{C}_U^{r+1}}\left(\sum_{i=1}^r n_i \Delta_{i,r+1}\right)\right) \longrightarrow \rho_*\left(\frac{\mathcal{O}_{\mathcal{C}_U^{r+1}}\left(\sum_{i=1}^r n_i \Delta_{i,r+1}\right)}{\mathcal{O}_{\mathcal{C}_U^{r+1}}}\right)$$
$$\longrightarrow R^1\rho_*\mathcal{O}_{\mathcal{C}_U^{r+1}} \longrightarrow R^1\rho_*\left(\mathcal{O}_{\mathcal{C}_U^{r+1}}\left(\sum_{i=1}^r n_i \Delta_{i,r+1}\right)\right) \longrightarrow 0.$$

So we see that the locus where $h^0(\sum_{i=1}^r n_i r_i) \geq k$ is the locus where the rank of the map $\rho_*\left(\mathcal{O}_{\mathcal{C}_U^{r+1}}\left(\sum_{i=1}^r n_i \Delta_{i,r+1}\right.\right.$ $R^1\rho_*\mathcal{O}_{\mathcal{C}_U^{r+1}}$ is less than or equal to $n-k+1$.

Therefore our subvariety above is the degeneracy locus for the following flags and maps of vector bundles

$$\begin{array}{rcl}
A := \rho_*\left(\frac{\mathcal{O}_{\mathcal{C}_U^{15}}(2\Delta_{1,15}+2\Delta_{2,15})}{\mathcal{O}_{\mathcal{C}_U^{15}}}\right) & \longrightarrow & F := R^1\rho_*\mathcal{O}_{\mathcal{C}_U^{15}}\,, \\
A_1^i := \rho_*\left(\frac{\mathcal{O}_{\mathcal{C}_U^{15}}(\Delta_{1,15}+\Delta_{2,15}+\Delta_{i+3,15}+\Delta_{i+4,15})}{\mathcal{O}_{\mathcal{C}_U^{15}}}\right) & \subset & \\
\subset\ A_2^i := \rho_*\left(\frac{\mathcal{O}_{\mathcal{C}_U^{15}}(\Delta_{1,15}+\Delta_{2,15}+\Delta_{i+3,15}+\Delta_{i+4,15}+\Delta_{i+5,15}+\Delta_{i+6,15})}{\mathcal{O}_{\mathcal{C}_U^{15}}}\right) & \subset & \\
\subset\ A_3^i := \rho_*\left(\frac{\mathcal{O}_{\mathcal{C}_U^{15}}(2\Delta_{1,15}+\Delta_{2,15}+\Delta_{i+3,15}+\Delta_{i+4,15}+\Delta_{i+5,15}+\Delta_{i+6,15})}{\mathcal{O}_{\mathcal{C}_U^{15}}}\right) & \longrightarrow & R^1\rho_*\mathcal{O}_{\mathcal{C}_U^{15}}\,, \\
E^j := \rho_*\left(\frac{\mathcal{O}_{\mathcal{C}_U^{15}}(\Delta_{1,15}+\Delta_{2,15}\Delta_{3+j,15}+\Delta_{7+j,15}+\Delta_{11+j,15}+\Delta_{12+j,15})}{\mathcal{O}_{\mathcal{C}_U^{15}}}\right) & \longrightarrow & R^1\rho_*\mathcal{O}_{\mathcal{C}_U^{15}}\,,
\end{array}$$

for $i = 0, 4$ and $j = 0, 2$, where the ranks of $A \longrightarrow F$ and $A_1^i \longrightarrow F$ are at most 3 and the ranks of $A_2^i \longrightarrow F$, $A_3^i \longrightarrow F$ and $E^j \longrightarrow F$ are at most 4. By [14] (proof of Lemma 2.9), for X non-bielliptic, there are only a finite number of divisors $s+t$ such that $h^0(2s+2t) = 2$. For fixed $s+t$, there are only a finite number of divisors $p_i' + p_i''$ with $i = 1,2,3,4$ with the properties (1) since, as we saw, then $p_i := \pi_{st}(p_i') = \pi_{st}(p_i'')$ would be a double point of X^{st} for $i = 1,2,3,4$. Hence this locus maps onto U with finite fibers everywhere. So this locus has the right codimension as given in [10] pages 249-250 and 254. Therefore we can use the formulae on pages 249-250 and 254 of [10] to compute the class of this locus as a (Schur) polynomial in the Chern classes of the vector bundles A, A_k^i, E^j and F. Now, in the same way as in [17] pages 310-314, we see that the Chern classes of these vector bundles are polynomials in the classes of the diagonals and the first Chern classes K_i, $1 \leq i \leq 14$, of the pull-backs of ω by the projections $\mathcal{C}_U^{14} \longrightarrow \mathcal{C}_U$, for instance:

$$c(A) = (1-K_1)(1-2K_1)(1-K_2+2\Delta_{1,2})(1-2K_2+2\Delta_{1,2}).$$

However, the above locus contains the codimension 14 loci of 14-tuples of the form, for instance, $(s, t, p_1', p_1'', p_2', p_2'', p_3', p_3'', p_4', p_4'', p_1'', p_3'', p_2'', p_4'')$ with the same conditions as above on s, t and p_i', p_i''. These loci are of no interest to us. The class of the latter can be expressed as the product of $[\Delta_{4,11} \cap$

$\Delta_{8,12} \cap \Delta_{6,13} \cap \Delta_{10,14}]_{\mathcal{C}^{12}}$ by the class of the locus parametrizing 14-tuples $(s, t, p_1', p_1'', p_2', p_2'', p_3', p_3'', p_4', p_4'', u_1, v_1, u_2, v_2)$ with

$$h^0(2s+2t) \geq 2\ ,\ h^0(s+t+p_i'+p_i'') \geq 2\ ,$$
$$h^0(s+t+p_i'+p_i''+p_{i+1}'+p_{i+1}'') \geq 3$$
$$\text{and} \quad h^0(2s+t+p_i'+p_i''+p_{i+1}'+p_{i+1}'') \geq 4\,,$$

for $i = 1, 3$. So the classes of the unwanted loci are also expressible in terms of the classes of the diagonals and the K_i's. Therefore the class of the closure $\mathcal{P}$ of the subvariety parametrizing 14-tuples of *distinct* points with property (1) above is also a polynomial in the classes of the diagonals and the K_i's.

It is seen now, in the same way as in [13] page 55 that the class of the pushforward to U of the intersection of some of the diagonals with $\mathcal{P}''$ is a polynomial in the tautological classes. So all we need to do is to express the closures of all of our strata as such a push-forward.

$\overline{U}_1$: The moduli point of X is in U_1 when, for some divisor $s+t$ on X, the curve X^{st} has the configuration of points in Lemma 2.3,4. This is the limit of configuration 2 (in Lemma 2.3) where $\overline{s}$ and p_2 have come together. Therefore $\overline{U}_1$ is the push-forward to U of $\Delta_{1,5} \cap \mathcal{P}''$.

$\overline{V}_1$: The moduli point of X is in V_1 when, for some divisor $s+t$ on X, the curve X^{st} has the configuration of points in Lemma 2.3,5. This is the limit of configuration 2.3,2 in which p_1 and p_3 have come together as well as p_2 and p_4. We claim that the intersection $\Delta_{3,7} \cap \mathcal{P}$ pushes forward to $\overline{V}_1$. Indeed, the only specializations of configuration 2 in Lemma 2.3 in which p_1 (or p_2) and p_3 (or p_4) have come together are configurations 5, 6, 12 and 13: these are all specializations of configuration 5 so, in all cases, the moduli point of X is in $\overline{V}_1$.

$\overline{V}_2'$: The intersection $\Delta_{3,7,11} \cap \mathcal{P}$ pushes forward to $\overline{V}_2'$.

$\overline{V}_3'$: The intersection $\Delta_{3,4,7,11} \cap \mathcal{P}$ pushes forward to $\overline{V}_3'$.

$\overline{V}_4'$: The intersection $\Delta_{5,13} \cap \Delta_{3,4,7,11} \cap \mathcal{P}$ pushes forward to $\overline{V}_4'$.

$\overline{V}_5'$: The intersection $\Delta_{5,6,13} \cap \Delta_{3,4,7,11} \cap \mathcal{P}$ pushes forward to $\overline{V}_5'$.

$\overline{V}_6'$: The intersection $\Delta_{1,2} \cap \Delta_{5,6,13} \cap \Delta_{3,4,7,11} \cap \mathcal{P}$ pushes forward to $\overline{V}_6'$.

U_{null}: This is analogous to the case of $\overline{V}_1$ with configuration 2.3,5 replaced by 2.3,8. The intersection $\Delta_{3,5} \cap \mathcal{P}$ pushes forward to U_{null}.

$\overline{U}_2$: The moduli point of X is in $U_2 \cup U_3 \cup U_4$ if and only if X^{st} has the configuration of double points in Lemma 2.3,6. The intersection $\Delta_{1,5,9} \cap \mathcal{P}$ pushes forward to $\overline{U}_2$.

$\overline{U}_3$: The intersection $\Delta_{3,11} \cap \Delta_{1,5,9} \cap \mathcal{P}$ pushes forward to $\overline{U}_3$.

$\overline{U}_4$: The intersection $\Delta_{3,4,11} \cap \Delta_{1,5,9} \cap \mathcal{P}$ pushes forward to $\overline{U}_4$.

$\overline{V}_2$: The moduli point of X is in V_2 if and only if X^{st} has the configuration of double points in Lemma 2.3,9. The intersection $\Delta_{1,9} \cap \Delta_{3,5} \cap \mathcal{P}$ pushes forward to $\overline{U}_2$.

$\overline{V}_3$: The moduli point of X is in $V_3 \cup V_4 \cup V_5$ if and only if X^{st} has the configuration of double points in Lemma 2.3,15. The intersection $\Delta_{1,7,9} \cap \Delta_{3,5} \cap \mathcal{P}$ pushes forward to $\overline{V}_3$.

$\overline{V}_4$: The intersection $\Delta_{1,2,7,9} \cap \Delta_{3,5} \cap \mathcal{P}$ pushes forward to $\overline{V}_4$.

$\overline{V}_5$: The intersection $\Delta_{1,2,7,9} \cap \Delta_{3,4,5} \cap \mathcal{P}$ pushes forward to $\overline{V}_4$.

$\overline{W}_2$: The moduli point of X is in W_2 if and only if X^{st} has the configuration of double points in Lemma 2.3,11. The intersection $\Delta_{1,3,5} \cap \mathcal{P}$ pushes forward to $\overline{W}_2$.

$\overline{W}_3$: In this case all the points $\overline{s}, \overline{t}$ and $p_1, \ldots, p_5$ coincide. The moduli point of X is in $W_3 \cup W_4$ if and only if X^{st} has the configuration of double points in Lemma 2.3,13. The intersection $\Delta_{1,3,5,7} \cap \mathcal{P}$ pushes forward to $\overline{W}_3$.

$\overline{W}_4$: The moduli point of X is in W_4 if and only if X^{st} has the configuration of double points in Lemma 2.3,13 and $s = t$. The intersection $\Delta_{1,2,3,5,7} \cap \mathcal{P}$ pushes forward to $\overline{W}_4$.

$\overline{Y}_2$: The moduli point of X is in Y_2 if and only if X^{st} has the configuration of double points in Lemma 2.3,12. The intersection $\Delta_{3,5,7} \cap \mathcal{P}$ pushes forward to $\overline{Y}_2$.

$\overline{Z}_2$: This is the locus of curves with at least two vanishing theta-nulls. The moduli point of X is in Z_2 if and only if X^{st} has the configuration of double points in Lemma 2.3,14. The intersection $\Delta_{3,5} \cap \Delta_{7,9} \cap \mathcal{P}$ pushes forward to $\overline{V}_2$.

Q.E.D.

Corollary 3.3. $A_*(U) = 0$.

Proof: By Theorems 3.1 and 3.2, the Chow ring of U is generated by tautological classes. So, by [17] page 309 and [9] page 447, the ring of tautological classes is generated by $\lambda = 12\kappa_1$ and κ_2 (by continuity, the relations given by the two authors remain valid in positive characteristic). Since the class of the locus of trigonal curves is a positive multiple of λ by [13] page 55, we have $\lambda = \kappa_1 = 0$ in $A_*(U)$ (the proof is valid in characteristic $\neq 2$). The class of the locus of curves with a point p such that $h^0(3p) \geq 2$ can be computed from formula (7.7) in [17] to be $81\kappa_2 - 25\kappa_1\lambda + 48\lambda_2$. Since this locus is contained in the trigonal locus, we have $81\kappa_2 - 25\kappa_1\lambda + 48\lambda_2 = 0$ in $A_*(U)$. Since $\lambda_2 = \lambda^2/2$, we also have $\lambda_2 = 0$. Therefore $\kappa_2 = 0$. Q.E.D.

4. The Chow groups of B and V

We first show

Theorem 4.1. *The class of B in $V := U \cup B = \mathcal{M}_5 \setminus (T \cup H)$ is a combination of tautological classes.*

Proof: As in the proof of Theorem 3.2, we implicitly replace V by a smooth finite cover and we replace the subvarieties of V by their inverse images in the smooth cover.

To show that $[B]_V$ is a combination of tautological classes we construct, as in the proof of Theorem 3.2, the subvariety $\mathcal{P}'$ of $\mathcal{C}_V^7$ as the closure of the subvariety parametrizing 7-tuples $(s, p_1', p_1'', p_2', p_2'', p_3', p_3'')$ of *distinct* points on curves X (with $m_X \in V$) such that $h^0(4s) \geq 2$ and $h^0(2s + p_i' + p_i'') \geq 2$ for $i = 1, 2, 3$. As before, the class of $\mathcal{P}'$ can be expressed as a combination of the classes of the diagonals and the K_i's.

The image in V of the intersection $\mathcal{P}'' := \mathcal{P}' \cap \Delta_{2,3} \cap \Delta_{4,5} \cap \Delta_{6,7}$ contains B: Let $X \longrightarrow E$ be a bielliptic structure on a curve X of genus 5 and let $s_1, \ldots, s_8$ be its ramification points. Then, for all i, j between 1 and 8, we have $h^0(2s_i + 2s_j) = 2$ since $2s_i + 2s_j$ is the pullback of a divisor on E.

The intersection $\mathcal{P}''$ is proper:

- The map $\mathcal{P}'' \longrightarrow V$ is quasi-finite on its image: By [14] proof of Lemma 2.9, a curve X of genus 5 with moduli point in V has a finite number of divisors $s + t$ such that $h^0(2s + 2t) = 2$ unless it is bielliptic and then all such divisors which move in a one-dimensional family are of the form $s + \iota s$ where ι is a bielliptic involution on X. So to obtain $h^0(4s) \geq 2$ we must have $s = \iota s$. So X has a finite number of points s with $h^0(4s) = 2$ and a finite number of points t with $h^0(2s + 2t) \geq 2$.
- The image P' of $\mathcal{P}'' \cap C_U^7$ in U has dimension 8: the variety P' parametrizes curves which admit an embedding in $\mathbb{P}^2$ with configuration of double points as in Lemma 2.3,2 such that $\overline{s} = \overline{t}$ is a cusp as well as three other p_i's. Therefore P' has codimension 4 or dimension 8.
- B has dimension 8

Therefore, the class of $\mathcal{P}''$ is a combination of the K_i's and the diagonals and the class of the image of $\mathcal{P}''$ in V is a combination of tautological classes. Therefore, since, by the above, B is a component of the image of $\mathcal{P}''$ in V, we obtain that a linear combination with positive coefficients of $[B]_V$ and $[P']_V$ is a combination of tautological classes. By Corollary 3.3, the only possibly nonzero codimension 4 class in $A_*(V)$ is $[B]_V$. We deduce that $[P']_V$ is a nonnegative multiple of $[B]_V$. Therefore $[B]_V$ is a combination of tautological classes. Q.E.D.

We now proceed to find generators for $A_*(B)$. Let B_i be the subvariety of B parametrizing curves X with i distinct bielliptic structures, i.e., Q contains i lines. We have $B_1 = B$, $B_4 = B_5$, $\dim(B_2) = 5$, $\dim(B_3) = 3$ and $\dim(B_5) = 2$. Put $B_i^0 = B_i \setminus B_{i+1}$. For X such that $m_X \in B_1^0$, let l be the line in Π which is contained in Q. Let $u_1, \dots, u_4$ be the nodes of Q that are on the intersection of l with the quartic component R of Q and, for each $i \in \{1, \dots, 4\}$, let m_i be the line through u_i which is tangent to the branch of Q which belongs to R. We define the following strata in B_1^0.

$B_{1,0}^0$: The open subvariety of B_1^0 parametrizing curves such that, for all i, the line m_i has contact of order 3 with Q at u_i and is elsewhere transverse to Q.

$B_{1,1}^0$: The locally closed subvariety of B_1^0 parametrizing curves such that for some i, the line m_i has contact of order 4 with Q at u_i and for all j the order of contact of m_j with Q at u_j is at most 4.

${B_{1,1}^0}'$: The locally closed subvariety of B_1^0 parametrizing curves such that for some i, the line m_i is tangent to Q at a smooth point and, for all j, the order of contact of m_j with Q at u_j is at most 4 and m_j does not contain another node of Q.

$B_{1,2}^0$: The closed subvariety of B_1^0 parametrizing curves such that for some i, the line m_i has contact of order 5 with Q at u_i.

${B_{1,2}^0}'$: The closed subvariety of B_1^0 parametrizing curves such that for some i, the line m_i contains another node of Q.

For X such that $m_X \in B_2^0$ let l_1 and l_2 be the two lines in Q, let C be the cubic in Q and let u_1, u_2, u_3 (resp. u_4, u_5, u_6) be the points of intersection of C with l_1 (resp. l_2). For m_X general in B_2^0, there are four lines m_{ij}, $1 \le j \le 4$ through u_i ($1 \le i \le 6$) which are tangent to C at another point. For m_X not necessarily general in B_2^0, some of these lines may come together or their point of tangency may coincide with u_i which is then a flex of C. We define the following strata in B_2^0.

$B_{2,0}^0$: The open subvariety of B_2^0 such that for all i, j, the line m_{ij} has contact of order 2 with Q at u_i and its point of tangency with C is not a node of Q.

$B_{2,1}^0$: The closed subvariety of B_2^0 parametrizing curves such that for some i, j, the line m_{ij} has contact of order 4 with Q at u_i.

${B_{2,1}^0}''$: The closed subvariety of B_2^0 such that Q is the union of two lines and a nodal cubic.

${B_{2,1}^0}'$: The closed subvariety of $B_2^0 \setminus {B_{2,1}^0}''$ such that, for some i, j, the point of tangency of m_{ij} with C is a node of Q (i.e., is on l_1 or l_2).

Note that for any stratum $B_{i,j}^0$, ${B_{i,j}^0}'$, ${B_{2,1}^0}''$ above, the codimension of $B_{i,j}^0$ (resp. ${B_{i,j}^0}'$, resp. ${B_{2,1}^0}''$) in B_i is equal to j.

Theorem 4.2. *The Chow groups of the varieties $B_{i,j}^0$, ${B_{i,j}^0}'$, ${B_{2,1}^0}''$, B_3^0 and B_5 are trivial.*

Proof: The first two parts of this proof proceed along the same lines as the proof of Theorem 3.1. We will therefore omit the details that are similar to those in that proof.

In this preliminary part of the proof, we will let P be any of the varieties $B_{1,j}^0$, ${B_{1,j}^0}'$. Let $S(P)$ be the quartic Del Pezzo surface (defined up to projective equivalence) such that for any X with $m_X \in P$, the curve X embeds in $S(P)$ (i.e., $S(P) = S(m_i)$ for all X with moduli point in P). Fix a quadric q_0 of rank 3 and a projective embedding of $S(P)$ in q_0 such that q_0 corresponds to u_i in m_i. By [3] page 365, all the quadrics of rank 4 parametrized by l have the same vertex (which is not on X). By continuity, this vertex is contained in $Sing(q_0)$ and it is not on $S(P)$ because X is the intersection of $S(P)$ with the quadrics parametrized by l. Fix a general point $p_0 \in Sing(q_0)$.

Since the embedding of X in $\mathbb{P}^4$ is defined up to projective equivalence, we can suppose that p_0 is the common vertex of the quadrics parametrized by l. So, if we let $\mu : \mathbb{P}^4 \longrightarrow \mathbb{P}^3$ be the projection from p_0, for any X such that $m_X \in P$, the curve X is the complete intersection of $S(P)$ with $\mu^* q$ for some quadric q in $\mathbb{P}^3$ such that $\mu^* q \neq q_0$. Let $U(P)$ be the open subvariety of $L := \frac{\mu^*|\mathcal{O}_{\mathbb{P}^3}(2)|}{\{q_0\}}$ which parametrizes pencils of quadrics $\langle \mu^* q, q_0 \rangle$ such that $\mu^* q \cap S(P)$ is a smooth canonical curve with moduli point in P. Below we describe (for each P) a linear subspace $L(P)$ of L whose intersection with $U(P)$ has codimension 1 complement in $L(P)$ and maps onto P in a finite way.

$B^0_{1,0}$: The surface $S(B^0_{1,0})$ is described in Lemma 2.3,10. The space $L(B^0_{1,0})$ is equal to L (The automorphism group of $S(B^0_{1,0})$ is one-dimensional, it acts on $\mathbb{P}^4$ and leaves $Sing(q_0)$ globally invariant. The stabilizer of p_0 is finite since p_0 is general.)

$B^0_{1,1}$: The surface $S(B^0_{1,1})$ is described in Lemma 2.3,12. It contains two lines which intersect at its double point. Choose a general point p on one of these lines. Let $L(B^0_{1,1})$ be the linear subspace of L parametrizing pencils of quadrics through p.

$B^0_{1,1}{}'$: The surface $S(B^0_{1,1}{}')$ is described in Lemma 2.3,11. The space $L(B^0_{1,1}{}')$ is defined in a way analogous to the previous case.

$B^0_{1,2}$: The surface $S(B^0_{1,2})$ is described in Lemma 2.3,13. It contains only one line. We claim that this line intersects X in two distinct points which are fixed points for the bielliptic involution of X: Let $X \longrightarrow E$ be a morphism of degree 2 onto an elliptic curve and $s_1, \dots, s_8$ its ramification points. For $i \neq j$, the pencil l_{ij} of quadrics in $\mathbb{P}^4$ containing X and $\langle s_i + s_j \rangle$ is not contained in Q: Otherwise, by [3] page 365, all the quadrics parametrized by l_{ij} are singular at some point, say x_0. The base locus $S(l_{ij})$ of l_{ij} is then a cone with vertex x_0 over a quartic elliptic curve in $\mathbb{P}^3$. The curve X maps onto this curve by a morphism of degree 2. So, since X has only one bielliptic structure, this curve is isomorphic to E. Therefore all the lines in $S(l_{ij})$ (in particular $\langle s_i + s_j \rangle$) contain p_0. So s_i and s_j project to the same point on E: contradiction.

Now, $h^0(2s_i + 2s_j) = 2$ implies that $S(l_{ij})$ contains less than 16 lines and therefore is not smooth by Lemma 2.1. Hence, by Lemma 2.3, the line l_{ij} is bitangent to Q. Now suppose for a moment that Q is the union of a line L and a smooth plane quartic R. We claim that l_{ij} is bitangent to R. Indeed, l_{ij} is either bitangent to R or contains a node of Q which corresponds to a quadric q of rank 3. If l_{ij} contains a node of Q, the configuration of double points of $X^{s_i s_j}$ is as in Lemma 2.3 case 8 or 9 or 11 or 13. So the vanishing theta-null cut by the ruling of q is $|s_i + s_j + p_1' + p_1''|$. The quadric q is the pull-back by $\mu : \mathbb{P}^4 \longrightarrow \mathbb{P}^3$ of a quadric in $\mathbb{P}^3$ since it belongs to l. So the vanishing theta-null cut by its ruling is the pull-back of a g^1_2 on E. It follows that we have $s_i + s_j + p_1' + p_1'' = 2s_i + 2s_j$, i.e., the configuration of double points of $X^{s_i s_j}$ is 2.3 case 11 or 13 and l_{ij} is bitangent to R.

Since R has 28 bitangents and there are 28 divisors $s_i + s_j$, we see that *these are the only divisors of degree 2 on X which are not pullbacks from E and verify $h^0(2s_i + 2s_j) = 2$*. So, for R smooth, the line in $S(B^0_{1,2})$ intersects X in s_i and s_j for some $i \neq j$. For R not necessarily smooth, m_i is a limit of bitangents of smooth quartics and by continuity the line in $S(B^0_{1,2})$ intersects X in two points which are fixed points for the bielliptic involution of X. These are distinct since such points cannot come together when X varies in a family unless X becomes singular.

Now choose two general points on this line. Let $L(B^0_{1,2})$ be the linear subspace of L parametrizing pencils of quadrics through these points.

$B^0_{1,2}{}'$: The surface $S(B^0_{1,2}{}')$ is described in Lemma 2.3,15. It contains two lines which intersect at one of its double points. Choose a general point on each line. Let $L(B^0_{1,2}{}')$ be the linear subspace of L parametrizing pencils of quadrics through these points.

Now let P denote, momentarily, one of the varieties $B^0_{2,i}$ or ${B^0_{2,1}}'$. Again, we have a quartic Del Pezzo surface $S(P)$ such that every curve with moduli point in P embeds in $S(P)$ and $S(P)$ is projectively isomorphic to the base locus of one of the lines m_{ij}. The surfaces $S(B^0_{2,0})$, $S(B^0_{2,1})$, $S({B^0_{2,1}}')$ are described respectively in 2.3 cases 8, 12 and 15. For each P, the surface $S(P)$ contains (exactly) two lines k_1 and k_2 which intersect (each other and) any other line in $S(P)$ only at the double points of $S(P)$ (these lines are $\langle s+t\rangle$ and $\langle p_1'+p_1''\rangle$ with the notation of 2.3). When we let the quartic R in the case of $B^0_{1,2}$ degenerate to the union $C \cup l_2$, some of the bitangents to R degenerate to the lines m_{ij} with $1 \leq i \leq 3$. Therefore, by continuity, each of the lines k_1 and k_2 in $S(P)$ intersects X in two distinct points which are fixed points of the bielliptic involution of X corresponding to l_1. In particular, the tangent lines to X at these four points all meet in one point which is $\mathrm{Sing}(q_5)$ since q_5 is on l_1 with the notation of 2.3. This point is neither on $\mathrm{Sing}(q_0)$ nor on $S(P)$. It is, however, on the singular locus of the quadric of rank 3 which is the intersection of l_1 and l_2. It is also in the intersection of the two planes which are tangent to $S(P)$ along k_1 and k_2 respectively. We therefore choose a general point x_0 on the intersection of these two planes (this intersection is a line since the two planes are contained in the hyperplane tangent at $k_1 \cap k_2$ to a quadric containing $S(P)$ which is smooth at $k_1 \cap k_2$). We let L be the linear system parametrizing quadrics of rank 3 with singular locus $\langle p_0, x_0\rangle$. Let $U(P)$ be the open subset of L parametrizing quadrics such that their intersection with $S(P)$ is a smooth curve with moduli point in P.

$B^0_{2,0}$: The open subset $U(B^0_{2,0})$ maps onto $B^0_{2,0}$ in a finite way (the stabilizer of x_0 in the automorphism group of $S(B^0_{2,0})$ (which has dimension 1) is finite).

$B^0_{2,1}$, ${B^0_{2,1}}'$: Fix a general point p on k_1. Let $L(B^0_{2,1})$ (resp. $L({B^0_{2,1}}')$) be the sublinear system of L parametrizing quadrics through p. Then $U(B^0_{2,1}) \cap L$ (resp. $U({B^0_{2,1}}') \cap L$) maps onto $B^0_{2,1}$ (resp. ${B^0_{2,1}}'$) in a finite way.

We prove the triviality of the Chow rings of ${B^0_{2,1}}''$, B^0_3 and B_5 by a different method. By [3] pages 316-317, 321, 324, 362, the (ramified) double cover $\widetilde{Q} \longrightarrow Q$, where $\widetilde{Q}$ parametrizes the rulings of the quadrics parametrized by Q, is ramified at all the nodes of Q and the branches of $\widetilde{Q}$ at each node are not exchanged by the covering involution; furthermore, the Prym variety of this double cover is the jacobian of X and the data of the double cover is equivalent to the data of X. Since all the components of Q are rational for $m_X \in {B^0_{2,1}}'' \cup B^0_3 \cup B_5$, the double cover $\widetilde{Q} \longrightarrow Q$ is determined by Q and so the data of X is equivalent to the data of the nodal plane quintic Q. It follows that

1. ${B^0_{2,1}}''$ is isomorphic to the moduli space of nodal plane quintics which are unions of two lines and a nodal irreducible cubic. The Chow ring of this space is trivial: Fix two lines l_1 and l_2, a general point r_i on l_i and a general point x in the plane. Then the space of quintics which are the union of l_1, l_2 and an irreducible cubic with a node at x and passing through r_1, r_2 maps in a finite way onto ${B^0_{2,1}}''$.
2. B^0_3 is isomorphic to the moduli space of nodal plane quintics which are unions of three lines and an irreducible conic. Fix three general lines l_1, l_2, l_3 and two general points x_1 and x_2 on l_1 and l_2 respectively. The space of quintics with only nodes which are the union of l_1, l_2, l_3 and an irreducible conic through x_1 and x_2 maps in a finite way onto B^0_3.
3. B_5 is isomorphic to the moduli space of nodal plane quintics which are unions of five lines. By duality, this is isomorphic to the moduli space of 5 distinct points in $\mathbb{P}^2$ of which no three are collinear. Therefore the space of five *ordered* distinct points in the plane of which no three are collinear maps in a finite way onto B_5. By [6] page 114, the former is isomorphic to a smooth anticanonical Del Pezzo surface of degree 5 minus all of its lines: this has trivial Chow ring. Q.E.D.

Instead of computing the classes of the closures of the above subvarieties of B, we deduce from the above theorem that $A_*(B)$ is generated by pullbacks of cycles from the moduli space $\mathcal{Q}$ of plane quintics which contain a line and have only nodes as singularities.

Let $\tilde{\mathcal{Q}} \subset (\mathbb{P}^2)^* \times \mathcal{Q}$ be the space parametrizing pairs (l, Q) such that the line l is contained in the plane quintic Q. The morphism $\tilde{\mathcal{Q}} \longrightarrow \overline{\mathcal{M}}_3$ sending (l, Q) to the plane quartic curve $Q \setminus l$ allows us to identify $\tilde{\mathcal{Q}}$ with an open subset of the projectivization of the Hodge bundle over $\overline{\mathcal{M}}_3$. The image $\widetilde{\mathcal{M}}_3$ of $\tilde{\mathcal{Q}} \longrightarrow \overline{\mathcal{M}}_3$ is the subvariety of $\overline{\mathcal{M}}_3$ parametrizing stable curves C which *embed* into $\mathbb{P}^2 = |\omega_C|^*$ as plane quartics, its fiber at a given plane quartic C is the set of lines in $|\omega_C|^*$ which are transverse to the image of C, i.e., the complement in $|\omega_C|$ of the full dual variety of the image of C.

So, if we denote by ξ the first Chern class of the tautological line bundle on $\tilde{\mathcal{Q}}$ (as an open subset of a projective bundle on $\widetilde{\mathcal{M}}_3$), the Chow ring (well-defined because $\tilde{\mathcal{Q}}$ is an open subset of a projective bundle over $\widetilde{\mathcal{M}}_3$) of $\tilde{\mathcal{Q}}$ is generated by the Chow ring of $\widetilde{\mathcal{M}}_3$ and ξ. Now, by the above, the class of the complement of $\tilde{\mathcal{Q}}$ in the projective Hodge bundle is the sum of 12ξ and the pull-back of a class on $\widetilde{\mathcal{M}}_3$. Hence (since our Chow rings are with rational coefficients), the Chow ring of $\tilde{\mathcal{Q}}$ is isomorphic to $A_*(\widetilde{\mathcal{M}}_3)$. The Chow ring of $\overline{\mathcal{M}}_3$ is computed in [9], it is:

$$\mathbb{Q}[\lambda, \delta_0, \delta_1, \kappa_2]/I,$$

where we also denote by λ and κ_2 Mumford's tautological classes for $\overline{\mathcal{M}}_3$, δ_0 is the class of the boundary component Δ_0 of $\overline{\mathcal{M}}_3$ whose general elements are irreducible curves with one node, δ_1 is the class of the boundary component Δ_1 of $\overline{\mathcal{M}}_3$ whose general elements are unions of smooth curves of genus 2 and smooth elliptic curves meeting at 1 point, and I is an ideal. This result has been proved in characteristic zero. However, the proofs of the triviality of the Chow rings of the strata for $\overline{\mathcal{M}}_3$ are valid in characteristic $\neq 2$ and 3 and the relations (and computations of classes of subvarieties as combinations of tautological classes) proved in [9] remain valid in positive characteristic by continuity. Since the elements of Δ_1 cannot be embedded as canonical plane quartics, we deduce that the restriction of δ_1 to $\widetilde{\mathcal{M}}_3$ is 0. By [13] section 6, the Q-class (see [17] section 3) of the hyperelliptic locus in $\overline{\mathcal{M}}_3$ is equal to $9\lambda - \delta_0 - 3\delta_1$, therefore, the (usual) class of the hyperelliptic locus in $\widetilde{\mathcal{M}}_3$ is $2(9\lambda - \delta_0)$. Since hyperelliptic curves cannot be embedded as canonical plane quartics either, it follows that $9\lambda - \delta_0$ is zero in $\widetilde{\mathcal{M}}_3$. Next we deduce from the formulas in [9] pages 343, 366 and 368 (valid by continuity in positive characteristic) that the restriction of κ_2 to $\widetilde{\mathcal{M}}_3$ is a multiple of δ_0^2. So $A_*(\widetilde{\mathcal{M}}_3)$, and hence $A_*(\tilde{\mathcal{Q}})$, is generated by δ_0 (δ_0 is nonzero because it is independent of λ and δ_1 in $\overline{\mathcal{M}}_3$ and the general elements of Δ_0 *can* be embedded as plane quartics; however, it follows from relation 1) on page 389 of [9] that $\delta_0^3 = 0$ in $A_*(\widetilde{\mathcal{M}}_3)$).

Since the morphism $\tilde{\mathcal{Q}} \longrightarrow \mathcal{Q}$ is finite (and an isomorphism on an open set which meets the inverse image of Δ_0 in $\tilde{\mathcal{Q}}$), we obtain that $A_*(\mathcal{Q})$ is also generated by δ_0 (and $\delta_0 \neq 0$).

Let B' be the locus of bielliptic curves of genus 5 with at least 5 vanishing theta-nulls. By the above and Corollary 3.3, the classes $[B]_V$, $[B']_V$ and $[B']_V^2$ are the only possibly nonzero classes in the Chow ring of $V = U \cup B$. We have

Proposition 4.3. *The class of B' in V is a combination of tautological classes.*

Proof: Let $X \longrightarrow E$ and $s_1, \ldots, s_8$ be as in the proof of Theorem 4.1 and suppose that X has only one bielliptic structure. Suppose that the quartic R in Q has a node q_0 and Q is general for this property. Then there are 6 lines $n_1, \ldots, n_6$ through the node of R which are elsewhere tangent to R. As in the proof of Theorem 4.2, for the cases of $B^0_{1,2}$ and of the strata in B^0_2, for each $i \in \{1, \ldots, 6\}$, there are two lines $\langle s_{k(i)} + s_{h(i)} \rangle$ and $\langle s_{m(i)} + s_{n(i)} \rangle$ which are contained in the base locus $S(n_i)$ of n_i so that $|s_{k(i)} + s_{h(i)} + s_{m(i)} + s_{n(i)}|$ is the vanishing theta-null on X cut by the ruling of q_0. Now construct the subvariety $\mathcal{P}'''$ of $\mathcal{C}^7$ which is the closure of the subvariety parametrizing 4-tuples (s_1, s_2, s_3, s_4) of distinct points such that $h^0(2s_1 + 2s_i) \geq 2$ for $i = 2, 3, 4$, $h^0(4s_1 + 2s_2) \geq 3$ and $h^0(s_1 + s_2 + s_3 + s_4) \geq 2$. Then, as in the proof of Theorem 4.1, the pushforward of $\mathcal{P}'''$ to V is a combination of tautological classes. The rest of the proof is analogous to the proof of Theorem 4.1. Q.E.D.

5. The Chow groups of T and W

The class of $\overline{T}$ in $\mathcal{M}_5$ was computed in [13] (Theorem 3 page 53, the proof is valid in characteristic $\neq 2$ since the computation is as in the proof of Theorem 3.2 above: the variety $\overline{T}$ is the image in $W := T \cup U \cup B$ of the locus in $\mathcal{C}^2_W$ defined by the condition $h^0(2s + t) \geq 2$) and is a positive multiple of λ. Below we write a stratification for T and show that the classes in W of the closures of the strata are combinations of tautological classes. Recall (see Section 1 above) that $\overline{T} \cap B = \emptyset$ and that a non-hyperelliptic trigonal curve has a unique g^1_3.

By, e.g. [20] page 175, every smooth canonical trigonal curve embeds in a smooth rational normal scroll. In $\mathbb{P}^4$ there is only one such scroll D which has degree 3 and is the projection of the Veronese surface in $\mathbb{P}^5$ from a point on it. The point from which one projects blows up to a smooth rational curve E of self-intersection -1 on D. The scroll D is isomorphic to the projective bundle associated to the vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1$ and E is the only section of D over $\mathbb{P}^1$ with self-intersection -1. The curve E and the fibers of D over $\mathbb{P}^1$ are lines in $\mathbb{P}^4$ and D is embedded in $\mathbb{P}^4$ by the line bundle $\mathcal{O}_D(E + 2F)$ where F is a fiber. Every smooth trigonal canonical curve X of genus 5 is the residual intersection of D with a cubic hypersurface containing some fiber of D over $\mathbb{P}^1$. So X is a member of the linear system $|\mathcal{O}_D(3(E + 2F) - F)| = |\mathcal{O}_D(3E + 5F)|$. The automorphism group of D has dimension 6 and acts on this linear system. The fibers of D cut on X the divisors of its g^1_3 and $|K_X| = |\mathcal{O}_X(1)| = |(E + 2F).X|$. It follows that $|E.X| = |K_X - 2g^1_3|$ and, in particular, $|E.X|$ has degree 2. Choose three general fibers F_0, F_1 and F_2 with general points p_i on F_i and let e_i be the point of intersection of F_i with E. We define the following strata in T:

- T_0: The image in T of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose g^1_3 has no double ramification points, no ramification points on E and has ramification points at p_0, p_1 and p_2.
- T_1: The image in T of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose g^1_3 has ramifications point at e_0, p_1 and p_2, has no other ramification point on E and no double ramification points.
- T'_1: The image in T of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose g^1_3 has no ramification points on E, has exactly one double ramification point which is p_0 and has (simple) ramification points at p_1 and p_2.
- T_2: The image in T of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose g^1_3 has no double ramification points and has ramification points at e_0, e_1 and p_2.
- T'_2: The image in T of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose g^1_3 has exactly one double ramification point which is p_2, has ramification points at e_0 and p_1 and no other ramification point on E.
- T''_2: The image in T of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose g^1_3 has exactly one double ramification point which is at e_0, has no other ramification point on E and has ramification points at p_1 and p_2.
- T'''_2: The image in T of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose g^1_3 has no ramification points on E, has a (simple or double) ramification point at p_0 and double ramification points at p_1 and p_2.

T_3: The image in T of the locally closed subset of $|\mathcal{O}_D(3E+5F)|$ parametrizing smooth curves whose g^1_3 has a double ramification point at p_2 and ramification points at e_0 and e_1.

T'_3: The image in T of the locally closed subset of $|\mathcal{O}_D(3E+5F)|$ parametrizing smooth curves whose g^1_3 has exactly one double ramification point which is at e_0 and has (simple) ramification points at e_1 and p_2.

T''_3: The image in T of the locally closed subset of $|\mathcal{O}_D(3E+5F)|$ parametrizing smooth curves whose g^1_3 has a ramification point at e_0, no other ramification point on E and has double ramification points at p_1 and p_2.

T'''_3: The image in T of the locally closed subset of $|\mathcal{O}_D(3E+5F)|$ parametrizing smooth curves whose g^1_3 has double ramification points at e_0 and p_2, a ramification point at p_1 and no other ramification point on E.

T_4: The image in T of the locally closed subset of $|\mathcal{O}_D(3E+5F)|$ parametrizing smooth curves whose g^1_3 has double ramification points at e_0 and e_1 and a ramification point at p_2.

We have

Theorem 5.1. *The Chow groups of T_i, T'_i, T''_i, T'''_i are trivial. The classes of the closures $\overline{T}_i, \overline{T}'_i, \overline{T}''_i, \overline{T}'''_i$ of the above strata in W are combinations of tautological classes.*

Proof: Every locally closed subvariety above which maps onto one of our strata is an open dense subset with codimension 1 complement of a linear subsystem of $|\mathcal{O}_D(3E+5F)|$. Therefore it has trivial Chow ring. The proof of the first part is now analogous to the proof of Theorem 3.1. For the second part we proceed as in the proof of Theorem 3.2 (as in that proof, to apply determinantal formulas, we implicitly replace W by a smooth finite cover and we replace all the subvarieties of W by their inverse images).

For X trigonal, put $|K_X - 2g^1_3| = \{p+q\}$.

$\overline{T}_1$: The moduli point m_X is in $\overline{T}_1$ if and only if one of the points p or q, say p, is a ramification point of g^1_3. Consider the subvariety of $\mathcal{C}^4_W$ defined by the conditions

$$h^0(2t+s) \geq 2 \textit{ and } h^0(2t+s+p+q) \geq 3$$

This subvariety can be expressed, as in the proof of Theorem 3.2 as the degeneracy locus of a map of vector bundles with respect to some flag and it has the right codimension so we can compute its class using determinantal formulas. Since T_1 is the pushforward to W of the intersection of this subvariety with the diagonal $\Delta_{1,3}$, we deduce, as in Theorem 3.2, that $[\overline{T}_1]_W$ is a combination of tautological classes.

$\overline{T}'_1$: This is the pushforward to W of the subvariety of $\mathcal{C}_W$ defined by the condition $h^0(3t) \geq 2$ which is a degeneracy locus of a map of vector bundles and has the right codimension. So the class of $\overline{T}'_1$ is a combination of tautological classes.

$\overline{T}_2$: The moduli point m_X is in $\overline{T}_2$ if and only if p and q are both ramification points of g^1_3. Consider the subvariety P of $\mathcal{C}^6_W$ defined by the conditions

$$h^0(2t+s) \geq 2,\ h^0(2t+s+p+q) \geq 3$$

$$h^0(2t+s+p+q+2r) \geq 4,\ h^0(2t+s+p+q+2r+u) \geq 5$$

As before P is a degeneracy locus of the right dimension. Its intersection with $\Delta_{1,3} \cap \Delta_{4,5}$ pushes forward to $\overline{T}_2$ (we cannot have $p = q$ and p is a ramification point of the g^1_3, since then X would be singular (tangent to E and a fiber at the same point)) so, as before, the class of $\overline{T}_2$ is a combination of tautological classes.

$\overline{T}'_2$: The intersection of the variety P above with $\Delta_{1,2} \cap \Delta_{4,5}$ pushes forward to $\overline{T}'_2$ and has the correct codimension. We conclude as before.

$\overline{T}''_2$: The intersection of P with $\Delta_{1,2,3}$ pushes forward to $\overline{T}''_2$ and has the correct dimension.

$\overline{T}'''_2$: Consider the subvariety of $\mathcal{C}^6_W$ defined by the conditions

$$h^0(2t+s) \geq 2,\ h^0(2t+s+p+q) \geq 3$$

The intersection R of this subvariety with $\Delta_{1,5} \cap \Delta_{2,6}$ is contained in P and its class is a combination of the diagonals and the K_i's (see the proof of Theorem 3.2). Therefore the class of the closure P' of $P \setminus R$ is also a combination of diagonals and K_i's and its pushforward to W is a combination of tautological classes. It is immediately seen that $\overline{T}'''_2$ is the pushforward of $P' \cap \Delta_{1,2} \cap \Delta_{5,6}$ to W.

$\overline{T}_3$: We remark that $\overline{T}_3 \cup \overline{T}'_3 = \overline{T}_2 \cap \overline{T}'_1$. Therefore a linear combination with positive coefficients of the classes of $\overline{T}_3$ and $\overline{T}'_3$ is a combination of tautological classes. So the fact that the class of $\overline{T}_3$ is a combination of tautological classes follows from the same fact for $\overline{T}'_3$ which we prove below.

$\overline{T}'_3$: The variety $\overline{T}'_3$ is the pushforward to W of the intersection $P \cap \Delta_{1,2,3} \cap \Delta_{4,5}$.

$\overline{T}''_3$: As in the case of $\overline{T}_3$, we remark that $\overline{T}''_3 \cup \overline{T}'''_3 = \overline{T}'_2 \cap \overline{T}_1$.

$\overline{T}'''_3$: The variety $\overline{T}'''_3$ is the pushforward to W of the intersection $P' \cap \Delta_{1,2,3} \cap \Delta_{5,6}$.

$\overline{T}_4$: The variety $\overline{T}_4$ is the pushforward to W of the intersection $P \cap \Delta_{1,2,3} \cap \Delta_{4,5,6}$.

Q.E.D.

References

1. E. Arbarello and M. Cornalba, *The Picard groups of the moduli spaces of curves*, Topology **26** (1987), 153–171.
2. E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris, *Geometry of algebraic curves*, vol. 1, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985.
3. A. Beauville, *Variétés de Prym et jacobiennes intermédiaires*, Annales Sc. de l'École Norm. Sup. **10** (1977), 309–391, 4ème série.
4. F.R. Cossec and I.V. Dolgachev, *Enriques surfaces* i, Progress in Mathematics, vol. 76, Birkhäuser, Boston, 1989.
5. S. Diaz, *A bound on the dimension of complete subvarieties of M_g*, Duke Math. Journal **51** (1984), 405–408.
6. I. Dolgachev and D. Ortland, *Point sets in projective spaces and theta functions*, Astérisque **165** (1988).
7. D. Edidin, *The codimension-two homology of the moduli space of stable curves is algebraic*, Duke Math. Journal **67** (1992), 241–272.
8. C. Faber, *Some results on the codimension-two Chow group of the moduli space of curves*, Algebraic Curves and Projective Geometry, vol. LNM 1389, Springer-Verlag, 1988, pp. 66–75.
9. ______, *Chow rings of moduli spaces of curves, I and II*, Annals of Mathematics **132** (1990), 331–449.
10. W. Fulton, *Intersection theory*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.
11. J. Harer, *The 4-th cohomology group of the moduli space of curves*, To appear.
12. ______, *The second homology group of the mapping class group of an orientable surface*, Inventiones Math. **72** (1983), 221–239.
13. J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Inventiones Math. **67** (1982), 23–86.
14. E. Izadi, *The geometric structure of $\mathcal{A}_4$, the structure of the Prym map, double solids and Γ_{00}-divisors*, Journal für die Reine und Angewandte Mathematik **462** (1995), 93–158.
15. E. Looijenga, *Smooth Deligne-Mumford compactifications by means of Prym-level structures*, Journal of Algebraic Geometry **3** (1994), 283–293.
16. D. Mumford, *Prym varieties I*, Contributions to Analysis (L.V. Ahlfors, I. Kra, B. Maskit, and L. Niremberg, eds.), Academic Press, 1974, pp. 325–350.
17. ______, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and Geometry, volume 2, Papers dedicated to I.R. Shafarevich on the occasion of his sixtieth birthday (M. Artin and J. Tate, eds.), Progress in Mathematics, vol. 36, Birkhäuser, 1983, pp. 271–328.
18. M. Nagata, *On rational surfaces I*, Mem. Coll. Sci. Univ. Tokyo, Series A, Math. **32** (1960), 351–370.

19. M. Pikaart and A.J. de Jong, *Moduli of curves with non-abelian level structure*, The Moduli Spaces of Curves, Proceedings of the 1994 Texel conference (R. Dijkgraaf, C. Faber, and G. van der Geer, eds.), Progress in Mathematics, vol. 129, Birkhäuser, 1995, pp. 483–509.
20. B. Saint-Donat, *On Petri's analysis of the linear system of quadrics through a canonical curve*, Math. Ann. **206** (1973), 157–175.

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