SUBVARIETIES OF ABELIAN VARIETIES

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Let (A, Θ) be a principally polarized abelian variety (*ppav*) of dimension g over the field $\mathbb{C}$ of complex numbers. This means that Θ is an ample divisor on A, well-determined up to translation, with $h^0(A, \Theta) := \dim H^0(A, \Theta) = 1$. Let $[\Theta] \in H^2(A, \mathbb{Z})$ be the cohomology class of the theta divisor Θ. Then the cohomology class $[\Theta]^{g-e}$ is divisible by $(g-e)!$. The class $\frac{[\Theta]^{g-e}}{(g-e)!}$ is not divisible and it is called the dimension e minimal cohomology class in (A, Θ). This class is positive in the sense that some multiple of it can be represented by an algebraic subvariety (for instance $[\Theta]^{g-e}$ is the class of a complete intersection of $g-e$ general translates of Θ) and, furthermore, any subvariety whose class is a multiple of $\frac{[\Theta]^{g-e}}{(g-e)!}$ is nondegenerate, i.e., generates A as a group. We are interested in the representability of multiples of the minimal classes by algebraic subvarieties of A. We begin by discussing two special cases.

1. Jacobians

Let C be a smooth, complete, irreducible curve of genus g over the complex numbers. The jacobian $JC = Pic^0 C$ of C is the connected component of its Picard group parametrizing degree 0 invertible sheaves. For any nonnegative integer e, the choice of an invertible sheaf $\mathcal{L}$ of degree e on C gives a morphism

$$\begin{array}{rcl} \phi_{\mathcal{L}} : C^{(e)} & \longrightarrow & JC \\ D_e & \longmapsto & \mathcal{O}_C(D_e) \otimes \mathcal{L}^{-1} \end{array}$$

where $C^{(e)}$ is the e-th symmetric power of C. For $e \geq g$, such a morphism is surjective. When $e = g-1$, the image of $C^{(g-1)}$ by $\phi_{\mathcal{L}}$ is a theta divisor on JC which we will denote by Θ_C (always well-determined up

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Date: March 15 2001.

1991 *Mathematics Subject Classification.* Primary 14K12, 14C25; Secondary 14B10, 14H40.

This material is based upon work partially supported by the National Science Foundation under Grant No. DMS-0071795. Any opinions, findings and conclusions or recomendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation (NSF)..

to translation). For any e between 1 and g the image of $\phi_{\mathcal{L}}$ has class $\frac{[\Theta]^{g-e}}{(g-e)!}$. If $e = 1$, the map $\phi_{\mathcal{L}}$ is an embedding and its image is called an Abel curve. By a theorem of Matsusaka [14], the dimension 1 minimal class is represented by an algebraic curve C in (A, Θ) if and only if (A, Θ) is the polarized jacobian (JC, Θ_C) of C. The higher-dimensional analogue of this theorem has the following counterexample: by a result of Clemens and Griffiths, the intermediate jacobian of a smooth cubic threefold in $\mathbb{P}^4$ is not the jacobian of a curve but it contains a surface (the image of the Fano variety of lines in the cubic threefold) whose cohomology class is the dimension 2 minimal class [5]. Debarre has shown that for any e strictly between 1 and $g-1$ jacobians form an irreducible component of the family of ppav in which the dimension e minimal class is represented by an algebraic subvariety [8].

As the above suggests, not every ppav is a jacobian. In fact, the moduli space of ppav of dimension g has dimension $\frac{g(g+1)}{2}$ whereas the moduli space of curves of genus $g \geq 2$ has dimension $3g-3$. As soon as $g \geq 4$, we have $\frac{g(g+1)}{2} > 3g-3$ and so not all ppav of dimension g are jacobians. The question then becomes how else can one parametrize ppav? The first step of a generalization of the notion of jacobian is the construction of a Prym variety which we describe below.

2. Pryms

Suppose that C is a smooth, complete and irreducible curve of genus $g+1$ with an étale double cover $\pi : \widetilde{C} \to C$. Then $\widetilde{C}$ has genus $\tilde{g} := 2g+1$. Let $\sigma : \widetilde{C} \to \widetilde{C}$ be the involution of the cover π. The involution σ acts on the jacobian $J\widetilde{C}$ and the Prym variety P of π is an abelian variety of dimension g defined by

$$P := im(\sigma - 1) \subset J\widetilde{C}.$$

The principal polarization of $J\widetilde{C}$ induces twice a principal polarization on P which we will denote Ξ.

A priori, there are two ways of obtaining subvarieties of a Prym variety: by projection and intersection. Since P is a quotient of $J\widetilde{C}$ via $\sigma - 1$, we can take the images of subvarieties of $J\widetilde{C}$ in P. For $e = 1$, we can take the images in P of Abel curves in $J\widetilde{C}$. We obtain in this way embeddings of $\widetilde{C}$ in P whose images are called Prym-embedded curves. The class of a Prym-embedded curve is $2\frac{[\Xi]^{g-1}}{(g-1)!}$. Welters classfied all curves of class twice the minimal class in a ppav [17]. The list is short but it is not limited to Prym-embedded curves. Therefore the analogue of Matsusaka's theorem is false for curves representing twice the minimal class. For $e > 1$, using Pontrjagin product, it is easy to see that the projection in P of the image of $\phi_{\mathcal{L}}$ has class $2^e \frac{[\Xi]^{g-e}}{(g-e)!}$ (provided it has dimension e).

Secondly, since the Prym variety is also a subvariety of $J\widetilde{C}$, we can intersect the images of the symmetric powers of $\widetilde{C}$ with it. Since $\Theta_{\widetilde{C}}$ induces twice Ξ on P, for $\mathcal{L}$ of degree e, the intersection of the image of $\phi_{\mathcal{L}}$ with P has class $2^{\tilde{g}-e} \frac{[\Xi]^{\tilde{g}-e}}{(\tilde{g}-e)!}$, provided that the intersection is proper. There is at least one case in which one can do better: there are a finite number of theta divisors in $J\widetilde{C}$ whose intersection with P is 2Ξ as a divisor (see for instance [15]). In particular, we have a nice way of parametrizing the theta divisor of the ppav (P, Ξ).

It can easily happen that the intersection $\phi_{\mathcal{L}}(\widetilde{C}^{(e)}) \cap P$ is not proper. In such a case the cohomolgy class of the resulting subvariety needs to be determined by other means. Such subvarieties appear in the work of Recillas [16], Donagi [9], Clemens-Griffiths [5] and Beauville [3]. Following Beauville, we shall call them special subvarieties. A different way of defining a special subvariety of a Prym variety which allows one to compute its cohomology class is as follows [16], [9], [5] and [3]. Let g^r_e be a *complete* linear system of dimension r and degree e on C. Let L be the corresponding invertible sheaf on C and let $\mathcal{L}$ be an invertible sheaf on $\widetilde{C}$ whose Norm is $Nm(\mathcal{L}) = L$ (i.e., if $\mathcal{L} \cong \mathcal{O}_{\widetilde{C}}(D)$, then $L \cong \mathcal{O}_C(\pi_* D)$). Consider g^r_e as a subvariety of $C^{(e)}$, isomorphic to $\mathbb{P}^r$. Assuming that g^r_e contains reduced divisors, the inverse image of g^r_e in $\widetilde{C}^{(e)}$ is reduced. It splits as the union of two connected components whose images in $J\widetilde{C}$ by $\phi_{\mathcal{L}}$ are contained in P and the translate P' of P such that $P \cup P'$ is the kernel of the Norm $Nm : J\widetilde{C} \to JC$. Therefore, after translating one of these subvarieties, we obtain two subvarieties of P. They are isomorphic if e is odd but not if e even. They both have cohomology class $2^{e-2r-1} \frac{[\Xi]^{g-r}}{(g-r)!}$ at least when $1 \leq e \leq 2g+1$ and $e > 2r$ [3].

To see that these special subvarieties are indeed the nonproper intersections that we mentioned above, one needs to note that the fibers of the map $J\widetilde{C} \to JC$ are translates of $P \cup P'$. The special subvarieties are intersections of these fibers with images of maps $\phi_{\mathcal{L}}$. Note that when the g^r_e is nonspecial, i.e., $h^1(g^r_e) = 0$, the special subvariety is in fact a proper intersection and Beauville's cohomology class is equal (as it should be) to the cohomology class of the proper intersection above. As we shall see below, looking at special subvarieties as such nonproper intersections allows us to define them for arbitrary ppav.

For $g \leq 5$, all ppav are Prym varieties (in the generalized sense of Beauville [2]). For $g \geq 5$ Prym varieties of dimension g depend on the same number of moduli as curves of genus $g+1$, meaning $3g$ moduli. Therefore, for $g \geq 6$, a general ppav is not a Prym variety. So we need to find a different way to parametrize a ppav.

3. Prym-Tjurin varieties

Again, one would want to use a construction involving curves. Higher degree coverings $\widetilde{C} \to C$ do not yield general ppav because the dimension of the Prym variety (defined as the connected component of $\mathcal{O}_{\widetilde{C}}$ of the kernel of the Norm (or pushforward on divisors) $J\widetilde{C} \to JC$) is too high and therefore the families of ppav that one would obtain are too small, their dimensions being the dimensions of the moduli spaces for the bottom curves C. Looking back at a Prym variety, we note that it was defined as a special type of abelian subvariety of a jacobian. An abelian subvariety A of a jacobian (JX, Θ_X) such that Θ_X induces m-times a principal polarization Θ on A is called a Prym-Tjurin variety. Welters has proved that all ppav are Prym-Tjurin varieties [17].

To say that Θ_X induces m-times Θ on A is equivalent to saying that the class of the image of an Abel curve in A is m-times the minimal class for curves [17]. Here we are taking the image of an Abel curve by the composition

$$JX \xrightarrow{\cong} \hat{JX} \longrightarrow \hat{A} \xrightarrow{\cong} A$$

where $\hat{}$ denotes the dual abelian variety, the first and the last map are induced by the polarizations Θ_X and Θ respectively, and the middle map is the transpose of the embedding of A in JX.

Therefore, finding a structure of Prym-Tjurin variety on a ppav is equivalent to finding a reduced and irreducible curve $\overline{X}$ in A representing m-times the minimal class and such that A embeds in the jacobian of the normalization X of $\overline{X}$. Given such a structure, we can find subvarieties of A as in the case of Prym varieties: by projection and intersection. Since the class of the image of an Abel curve is m-times the minimal class, Pontrjagin product shows that the projection in P of the image of $\phi_{\mathcal{L}}$ has class $m^e \frac{[\Theta]^{g-e}}{(g-e)!}$ (provided it has dimension e). Since Θ_X induces m-times Θ on A, for $\mathcal{L}$ of degree e, the intersection of the image of $\phi_{\mathcal{L}}$ with P has class $m^{g_X - e}\frac{[\Theta]^{g_X-e}}{(g_X-e)!}$, provided that the intersection is proper (g_X is the genus of X). However, unlike Prym varieties, it is not clear whether one can find translates of Θ_X whose intersection with A is m-times a theta divisor. Kanev has shown that this is possible under a restrictive hypothesis which we explain below.

Any abelian subvariety of JX is the image of an endomorphism of JX (which is not unique). The datum of an endomorphism of JX is equivalent to the datum of a correspondence, i.e., a divisor in $X \times X$, up to addition and subtraction of fibers of the two projections. This is best seen as follows. Start with a divisor $D \subset X \times X$. To D one can associate an endomorphism of JX in the following way

$$\begin{array}{rrcl} \psi_D : & JX & \longrightarrow & JX \\ & \mathcal{O}_X(E) & \longmapsto & \mathcal{O}_X(p_{2*}((p_1^* E) \cdot D)) \end{array}$$

where p_1 and p_2 are the two projections $X \times X \to X$. If D is linearly equivalent to a sum of fibers of p_1 and p_2, then ψ_D is the zero endomorphism. If we exchange the roles of p_1 and p_2 in the above definition then ψ_D is replaced by its image under the Rosati involution. The correspondence D is said to be symmetric if there are (not necessarily effective) divisors a and b on X such that $D - D^t$ is linearly equivalent to $p_1^*(a) + p_2^*(b)$, where D^t is the transpose of D, i.e., the image of D under the involution exchanging the two factors of $X \times X$. So D is symmetric if and only if ψ_D is fixed by the Rosati involution. We shall assume that this is the case. This is not restrictive since any abelian subvariety of JX is always the image of an endomorphism which is fixed by the Rosati involution (see e.g. [17]).

Kanev [13] has shown that if the endomorphism can be represented by a symmetric fixed-point-free correspondence D (i.e., the support of D does not intersect the diagonal of $X \times X$), then one can find theta divisors Θ_X such that $\Theta_X|_A = m\Theta$ as divisors. Furthermore, fixing an invertible sheaf $\mathcal{L}_0 \in Pic^{g_X - 1}X$, an invertible sheaf $\mathcal{L} \in P \subset JX$ is on Θ if and only if $h^0(\mathcal{L} \otimes \mathcal{L}_0) \geq m$ and $\mathcal{L} \notin \Theta$ if and only if $h^0(\mathcal{L} \otimes \mathcal{L}_0) = 0$. This gives a nice parametrization of Θ and even allows one to analyze the singularities of Θ. It is not known however, whether every ppav is a Prym-Tjurin variety for a (symmetric) fixed-point-free correspondence. In addition, two correspondences could induce the same endomorphism of JX while one is fixed-point-free and the other is not. In general it is difficult to determine whether a given endomorphism can be induced by a fixed-point-free correspondence.

As we noted above, we can generalize the notion of special subvarieties to Prym-Tjurin varieties by defining them to be non-proper intersections of A with images of symmetric powers of X. It would be interesting to compute the cohomology classes of these special subvarieties and see whether the analogue of Beauville's formula holds, meaning, the cohomology class of a special subvariety of dimension r is $m^{e-2r-1}\frac{[\Theta]^{g-r}}{(g-r)!}$.

Welters showed that every principally polarized abelian variety is a Prym-Tjurin variety [17]. Birkenhacke and Lange showed that every principally polarized abelian variety is a Prym-Tjurin variety for an integer $m \leq 3^g(g-1)!$ (see [4] page 374 Corollary 2.4)[1].

4. Deforming curves

The question is to find the smallest integer m for which $m\frac{[\Theta]^{g-1}}{(g-1)!}$ can be represented by an algebraic curve. This naturally defines a stratification of the moduli space $\mathcal{A}_g$ of ppav. Using results of Kanev, Debarre [6] shows that if (A, Θ) is the Prym-Tjurin variety for a symmetric fixed-point-free correspondence, then either $Sing(\Theta)$ is empty or its dimension is at least $g - 2m - 2$. Since the theta divisor of a general ppav is smooth, this suggests that, for a general ppav A, the smallest integer m for which there is a curve of class $m\frac{[\Theta]^{g-1}}{(g-1)!}$ in A which in addition gives A a structure of Prym-Tjurin variety should be at least $\frac{g-1}{2}$. It is unlikely however that this bound is effective. Debarre has proved in [7] that the smallest integer m for which $m\frac{[\Theta]^{g-1}}{(g-1)!}$ is the class of an algebraic curve is at least $\sqrt{\frac{g}{8}} - \frac{1}{4}$ if (A, Θ) is general.

The difficulty is to produce curves in ppav in nontrivial ways. One approach that we have considered is to deform curves in jacobians of curves out of the jacobian locus. More precisely, let C be a curve of genus g with a g^1_d (a pencil of degree d). Define

$$X_e(g^1_d) := \{D_e : \exists D \in C^{(d-e)} \text{ such that } D_e + D \in g^1_d\} \subset C^{(e)}$$

(for the precise scheme-theoretical definition see [11] when $e = 2$ and [12] for $e > 2$). If $d \geq e + 1$, the restriction of a given morphism $\phi_{\mathcal{L}}$ to $X_e(g^1_d)$ is nonconstant and so we can map $X_e(g^1_d)$ to JC. The cohomology class of the image of $X_e(g^1_d)$ in JC is m-times the minimal class with $m = \binom{d-2}{e-1}$. Given a one-parameter infinitesimal deformation of the jacobian of C out of the jacobian locus $\mathcal{J}_g$ we ask when the curve $X_e(g^1_d)$ deforms with it. Infinitesimal deformations of JC are parametrized by $H^1(T_{JC})$ where T_{JC} is the tangent sheaf of JC. The principal polarization Θ_C provides an isomorphism between $H^1(T_{JC})$ and the second tensor power $H^1(\mathcal{O}_C)^{\otimes 2}$. Under this isomorphism the globally unobstructed deformations of the pair (JC, Θ_C) are identified with the symmetric square $S^2H^1(\mathcal{O}_C)$. Therefore any quadric in the

[1]Their proof uses 3-theta divisors. Using the fact that a general 2-theta divisor is smooth, the exact same proof would give $m \leq 2^g(g-1)!$. For abelian varieties with a smooth theta divisor, the same proof would give $m \leq (g-1)!$. One needs the Lefschetz hyperplane theorem which also works for mildly singular theta divisors, see e.g. [10] Chapter 2.

canonical space of C defines a linear form on the space of these infinitesimal deformations. When we say that an infinitesimal deformation $\eta \in S^2H^1(\mathcal{O}_C)$ is in the annihilator of a quadric, we mean that it is in the kernel of the corresponding linear form. We prove the following in [11]

Theorem *Suppose C nonhyperelliptic and $d \geq 4$. If the curve $X_2(g^1_d)$ deforms out of $\mathcal{J}_g$ then*

(1) *either $d = 4$*
(2) *or $d = 5$, $h^0(g^1_5) = 3$ and C has genus 5 or genus 4 and only one g^1_3.*

In the case $g = 5$ if $X_2(g^1_5)$ deforms in a direction $\eta \in S^2H^1(\mathcal{C}_C)$ out of $\mathcal{J}_5$, then η is in the annihilator of the quadric $\cup_{D\in g^1_5}\langle D\rangle$.

Here $\langle D\rangle$ denotes the span of the divisor D in the canonical space of C. For $d = 3$ the image of $X_2(g^1_3)$ in JC is an Abel curve and so by the result of [14], the curve $X_2(g^1_3)$ cannot deform out of $\mathcal{J}_g$. For $d = 4$, it follows from the theory of Prym varieties that $X_2(g^1_4)$ deforms out of $\mathcal{J}_g$ (into the locus of Prym varieties): in fact $X_2(g^1_4)$ is a Prym-embedded curve [16]. For $d = 5$, $h^0(g^1_d) = 3$ and $g = 4$ (with only one g^1_3) or $g = 5$ we believe that $X_2(g^1_5)$ deforms out of $\mathcal{J}_g$ but we do not have a proof of this. An interesting question is what are these deformations of (JC, Θ_C) into which $X_2(g^1_5)$ deforms. Can one describe them in a concrete geometric way.

For $e > 2$, the analogous result would be the following. The curve $X_e(g^1_d)$ deforms out of $\mathcal{J}_g$ if and only if

- either $e = h^0(g^1_d)$ and $d = 2e$
- or $e = h^0(g^1_d) - 1$ and $d = 2e + 1$.

We expect this to be true most of the time. There could, however, be special pairs (C, g^1_d) for which the curve $X_e(g^1_d)$ deforms out of the jacobian locus but g^1_d does not verify the above conditions. For instance, so far my calculations [12] seem to indicate that if there is a divisor $D \in X$ with $h^0(D) \geq 2$, then X might deform in directions η whose images in the projectivization $\mathbb{P}(S^2H^1(\mathcal{O}_C))$ are in the span of the image of $\cup_{D'\in|D|}\langle D'\rangle$. Finally, note that a standard Brill-Noether calculation shows that for general curves of genus ≥ 7, the smallest d for which they can have g^1_d's satisfying $d = 2h^0(g^1_d)$ is $d = 2g - 4$. In such a case the class of $X_e(g^1_d)$ is m-times the minimal class with $m = \binom{2g-6}{g-3}$ which is then what we would find for a general ppav. We address the case $e > 2$ in [12].

5. The genus

The cohomology class is one discrete invariant that one can associate to a curve in a ppav. Another discrete invariant is the genus of the curve. We refer the reader to the nice paper by Bardelli, Ciliberto and Verra [1] for a discussion of this.

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