Math. Nachr. **230** (2001), 73 – 91

Some Properties of Second Order Theta Functions on Prym Varieties

By E. Izadi of Athens and C. Pauly of Nice

(Received September 21, 1999; accepted June 6, 2000)

Abstract. Let $P \cup P'$ be the two component Prym variety associated to an étale double cover $\widetilde{C} \to C$ of a non–hyperelliptic curve of genus $g \geq 6$ and let $|2\Xi_0|$ and $|2\Xi_0'|$ be the linear systems of second order theta divisors on P and P' respectively. The component P' contains canonically the Prym curve $\widetilde{C}$. We show that the base locus of the subseries of divisors containing $\widetilde{C} \subset P'$ is exactly the curve $\widetilde{C}$. We also prove canonical isomorphisms between some subseries of $|2\Xi_0|$ and $|2\Xi_0'|$ and some subseries of second order theta divisors on the Jacobian of C.

1. Introduction

Let C be a curve of genus $g \geq 5$ with an étale double cover $\pi : \widetilde{C} \to C$. Let $Nm : \mathrm{Pic}(\widetilde{C}) \to \mathrm{Pic}(C)$ be the norm map. Consider the Prym varieties

$$Nm^{-1}(\mathcal{O}) \;=\; P \cup P'$$

which are characterized by the facts that $\mathcal{O} \in P$, $\mathcal{O} \notin P'$. Let $\sigma : \widetilde{C} \to \widetilde{C}$ be the involution of the cover $\pi : \widetilde{C} \to C$. The curve $\widetilde{C}$ admits a natural embedding in P' given by the morphism

$$\begin{array}{rccl} i \;:\; & \widetilde{C} & \longrightarrow & P'\,, \\ & \tilde{p} & \longmapsto & \mathcal{O}_{\widetilde{C}}(\tilde{p} - \sigma\tilde{p})\,. \end{array}$$

A symmetric Riemann theta divisor $\widetilde{\Theta}_0$ on the Jacobian $J\widetilde{C}$ of $\widetilde{C}$ induces twice a symmetric principal polarization Ξ_0 on P (resp. Ξ_0' on P'). Let $\Gamma_{\widetilde{C}}$ be the space of sections of $\mathcal{O}_{P'}(2\Xi_0')$ vanishing on the image of i. In his work on the Schottky problem, Donagi proved in [Do1] (Lemma 4.8 page 597) that the base locus $\mathrm{Bs}(\mathbb{P}\Gamma_{\widetilde{C}})$ of $\mathbb{P}\Gamma_{\widetilde{C}}$ is $i(\widetilde{C})$ for a Wirtinger cover $\pi : \widetilde{C} \to C$. Since he proves that for a Wirtinger cover the equality between $\mathrm{Bs}(\mathbb{P}\Gamma_{\widetilde{C}})$ and $i(\widetilde{C})$ is scheme–theoretical outside the double points

1991 *Mathematics Subject Classification.*
Keywords and phrases.

of $i(\widetilde{C})$, it follows from his proof that, for a general double cover, the base locus is the union of $i(\widetilde{C})$ and possibly a finite set of points. We prove (Sections 3 and 6)

Theorem 1.1. *If $g \geq 6$ and C is non–hyperelliptic or if $g = 5$ and C is nonbielliptic, the scheme–theoretical base locus in P' of the linear system $\mathbb{P}\Gamma_{\widetilde{C}}$ is $i(\widetilde{C})$.*

The proof of Theorem 1.1 has two steps. First we show that $\mathrm{Bs}(\mathbb{P}\Gamma_{\widetilde{C}})$ equals $i(\widetilde{C})$ set–theoretically (Section 3). In order to prove the scheme–theoretic equality, we introduce and study divisors $D := \Delta(E)$ in the linear systems $|2\Xi_0|$ and $|2\Xi'_0|$ associated to certain semi–stable rank 2 vector bundles E over the curve C (Proposition 4.1). We calculate the tangent spaces to the divisors $\Delta(E)$ along the curve $i(\widetilde{C})$, for $\Delta(E) \in |2\Xi'_0|$, and show that at any given point of $i(\widetilde{C})$ their intersection is equal to the tangent space to $i(\widetilde{C})$.

Let Θ_0 be a symmetric theta divisor on the Jacobian JC and let α be the square–trivial invertible sheaf associated to the double cover $\widetilde{C} \to C$. Translation by α induces an involution T_α on JC, which lifts canonically to a linear involution acting on $H^0(JC, \Theta_0 + T^*_\alpha\Theta_0)$. Mumford constructs in [M2] (see also [vGP] Proposition 1) canonical isomorphisms

$$(1.1)\qquad \begin{array}{rcl} \mu_+ : \; H^0(P, 2\Xi_0) & \xrightarrow{\sim} & H^0(JC, \Theta_0 + T^*_\alpha\Theta_0)_+\,, \\ \mu_- : \; H^0(P', 2\Xi'_0) & \xrightarrow{\sim} & H^0(JC, \Theta_0 + T^*_\alpha\Theta_0)_- \end{array}$$

where the subscript $\pm$ denotes the $\pm$eigenspaces of the involution. We are interested in some naturally defined subspaces of these vector spaces.

In connection with the Schottky problem, van Geemen and van der Geer [vGvdG] introduced the subspace

$$\Gamma_{00} \;=\; \{s \in H^0(A, 2\Theta) \mid \mathrm{mult}_0(s) \geq 4\}$$

for any abelian variety A with symmetric principal polarization Θ. It was conjectured by van Geemen, van der Geer and Donagi ([vGvdG] and [Do2] page 110) that if (A, Θ) is a Jacobian, then the base locus $\mathrm{Bs}(\mathbb{P}\Gamma_{00})$ of $\mathbb{P}\Gamma_{00}$ is the surface $C - C = \{\mathcal{O}_C(p-q) \mid p, q \in C\} \subset JC$ as a set and, if (A, Θ) is not in the closure of the locus of Jacobians, then $\mathrm{Bs}(\mathbb{P}\Gamma_{00}) = \{\mathcal{O}\}$. For Jacobians, the conjecture was proved by Welters [W1]. For non–Jacobians, the conjecture was proved in dimension 4 by the first author [I1]. Some evidence was also given for non–Jacobian Pryms by the first author in [I2].

Consider the subspaces $\Gamma^{(2)}_{\widetilde{C}}$ of $H^0(P', 2\Xi'_0)$ of elements vanishing with multiplicity greater than or equal to 2 along $i(\widetilde{C})$ and the subspace

$$\Gamma^\alpha_{C-C} \;:=\; \{s \in H^0(JC, \Theta_0 + T^*_\alpha\Theta_0) \mid C - C \subset Z(s)\}$$

where $Z(s)$ denotes the zero divisor of the section s. This space splits into $\pm$eigenspaces $\Gamma^{\alpha\pm}_{C-C}$ under the involution induced by T_α.

The infinitesimal study of the above mentioned divisors $\Delta(E)$ at the origin $\mathcal{O} \in P$ and along the curve $i(\widetilde{C})$ allows us to prove the following result (Section 5).

Theorem 1.2. *Assume C non–hyperelliptic of genus $g \geq 5$. Via the canonical isomorphisms* (1.1)*, we have equalities among the following subspaces*

1. $\Gamma^{\alpha+}_{C-C} = \Gamma_{00}$, *i. e., for all $s \in H^0(P, 2\Xi_0)$,*

$$\mathrm{mult}_0(s) \;\geq\; 4 \quad \Longleftrightarrow \quad C - C \subset Z(\mu_+(s))\,.$$

2. $\Gamma^{\alpha-}_{C-C} = \Gamma^{(2)}_{\widetilde{C}}$, *i. e., for all $s \in H^0\big(P', 2\Xi'_0\big)$,*

$$\big(\textit{for all } \tilde{p} \in \widetilde{C}\,,\ \mathrm{mult}_{i(\tilde{p})}(s) \geq 2\big) \quad \Longleftrightarrow \quad C - C \subset Z(\mu_-(s))\,.$$

One can view statement 1 as an analogue for Prym varieties of the equivalence (see e. g. [F] page 489, [W1] Prop 4.8 or [vGvdG])

$$\text{for all } \ s \in H^0(JC, 2\Theta_0)\,, \quad \mathrm{mult}_0(s) \;\geq\; 4 \quad \Longleftrightarrow \quad C - C \subset Z(s)\,.$$

Alternatively, one can derive equality 1 from an analytic identity between Prym and Jacobian theta functions (formula (41) [F]). Equality 2, however, seems to be new.

2. Preliminaries and notation

In this section we introduce the notation and recall some well–known facts on Prym varieties (see e. g. [M2], [vGP], [B2], [B4], [BNR]). Throughout the paper we will suppose the genus of C to be at least 5. Let ω be the dualizing sheaf of C and consider the two additional Prym varieties

$$Nm^{-1}(\omega) \;=\; P_{even} \cup P_{odd}$$

which are characterized by the fact that $\dim H^0\big(\widetilde{C}, \lambda\big)$ is even (resp. odd) for $\lambda \in P_{even}$ (resp. P_{odd}). The variety P_{even} carries the naturally defined reduced Riemann theta divisor

$$\Xi \;:=\; \big\{\lambda \in P_{even} \mid h^0(\lambda) > 0\big\}$$

a translate of which is Ξ_0. Let $\mathcal{SU}_C(2,\alpha)$ and $\mathcal{SU}_C(2,\omega\alpha)$ be the moduli spaces of semi–stable vector bundles of rank 2 with determinant α and $\omega\alpha$ respectively. Taking direct image gives morphisms

$$\varphi \;:\; P \cup P' \;\longrightarrow\; \mathcal{SU}_C(2,\alpha)\,, \quad \varphi \;:\; P_{even} \cup P_{odd} \;\longrightarrow\; \mathcal{SU}_C(2,\omega\alpha)\,.$$

Let $\mathcal{L}_\alpha$ (resp. $\mathcal{L}_{\omega\alpha}$) be the generator of the Picard group of $\mathcal{SU}_C(2,\alpha)$ $\big($resp. $\mathcal{SU}_C(2,\omega\alpha)\big)$. It is known that

$$\big(\varphi|_P\big)^*\mathcal{L}_\alpha \;=\; \mathcal{O}(2\Xi_0)\,, \quad \big(\varphi|_{P_{even}}\big)^*\mathcal{L}_{\omega\alpha} \;=\; \mathcal{O}(2\Xi)\,.$$

We denote by $\mathcal{O}\big(2\Xi'_0\big)$ (resp. $\mathcal{O}(2\Xi')$) the pull–back of the line bundle $\mathcal{L}_\alpha$ (resp. $\mathcal{L}_{\omega\alpha}$) to the Prym P' (resp. P_{odd}), i. e., $\big(\varphi|_{P'}\big)^*\mathcal{L}_\alpha = \mathcal{O}\big(2\Xi'_0\big)$ and $\big(\varphi|_{P_{odd}}\big)^*\mathcal{L}_{\omega\alpha} = \mathcal{O}(2\Xi')$. We consider the following morphisms

$$\begin{aligned} \psi \;&:\; JC \;\longrightarrow\; \mathcal{SU}_C(2,\alpha)\,, && \xi \;\longmapsto\; \xi \oplus \alpha\xi^{-1}\,, \\ \psi \;&:\; \mathrm{Pic}^{g-1}(C) \;\longrightarrow\; \mathcal{SU}_C(2,\omega\alpha)\,, && \xi \;\longmapsto\; \xi \oplus \omega\alpha\xi^{-1}\,. \end{aligned}$$

One computes the pull–backs

$$\psi^*\mathcal{L}_\alpha \;=\; \Theta_0 + T_\alpha^*\Theta_0\,, \quad \psi^*\mathcal{L}_{\omega\alpha} \;=\; \Theta + T_\alpha^*\Theta\,,$$

where

$$\Theta \;:=\; \left\{L \in \operatorname{Pic}^{g-1}(C) \mid h^0(L) > 0\right\}$$

and Θ_0 is a symmetric theta divisor in the Jacobian JC, i.e., a translate of Θ by a theta–characteristic. By abuse of notation, we will also write $\mathcal{L}_\alpha$ and $\mathcal{L}_{\omega\alpha}$ for $\psi^*\mathcal{L}_\alpha$ and $\psi^*\mathcal{L}_{\omega\alpha}$ respectively. Note that ψ induces linear isomorphisms at the level of global sections:

$$\begin{aligned} \psi^* \;&:\; H^0(\mathcal{S}U_C(2,\alpha),\mathcal{L}_\alpha) \;\cong\; H^0(JC,\mathcal{L}_\alpha)\,, \\ \psi^* \;&:\; H^0(\mathcal{S}U_C(2,\omega\alpha),\mathcal{L}_{\omega\alpha}) \;\cong\; H^0\big(\operatorname{Pic}^{g-1}(C),\mathcal{L}_{\omega\alpha}\big)\,. \end{aligned} \tag{2.1}$$

There is a well–defined morphism

$$D \;:\; \mathcal{S}U_C(2,\alpha) \;\longrightarrow\; |\mathcal{L}_{\omega\alpha}| \quad (\text{resp. } D : \mathcal{S}U_C(2,\omega\alpha) \longrightarrow |\mathcal{L}_\alpha|)$$

where the support of $D(E)$ (reduced for E general) is

$$D(E) \;=\; \left\{\xi \in JC \;\big(\text{resp. } \operatorname{Pic}^{g-1}(C)\big) \mid h^0(C, E\otimes\xi) > 0\right\}.$$

The two involutions of the Jacobian JC given by

$$T_\alpha \;:\; \xi \;\longmapsto\; \xi\otimes\alpha\,, \quad (-1) \;:\; \xi \;\longmapsto\; \xi^{-1}$$

induce (up to ± 1) linear involutions T_α^* and $(-1)^*$ on the spaces of global sections $H^0(JC,\mathcal{L}_\alpha)$ and $H^0\big(\operatorname{Pic}^{g-1}(C),\mathcal{L}_{\omega\alpha}\big)$.

Lemma 2.1. *The projective linear involutions T_α^* and $(-1)^*$ acting on $\mathbb{P}H^0(JC,\mathcal{L}_\alpha)$ are equal.*

Proof. We observe that the composite map $T_\alpha \circ (-1) : \xi \mapsto \alpha\xi^{-1}$ satisfies $\psi \circ (T_\alpha \circ (-1)) = \psi$. Since ψ^* is a linear isomorphism (2.1), we have $(T_\alpha \circ (-1))^* = \pm id_{H^0}$. Therefore $T_\alpha^* = \pm(-1)^*$. □

Thus the two spaces decompose into ±eigenspaces. Note that in order to distinguish the two eigenspaces, we need a lift of the 2–torsion point α into the Mumford group. We will take the following convention: the +eigenspace (resp. −eigenspace) contains the Prym varieties P and P_{even} (resp. P' and P_{odd}), i.e., we have canonical (up to multiplication by a nonzero scalar) isomorphisms:

$$H^0(JC,\mathcal{L}_\alpha)_+ \;=\; H^0(P, 2\Xi_0)\,, \quad H^0(JC,\mathcal{L}_\alpha)_- \;=\; H^0\big(P', 2\Xi_0'\big)\,, \tag{2.2}$$

$$\begin{aligned} H^0\big(\operatorname{Pic}^{g-1}(C),\mathcal{L}_{\omega\alpha}\big)_+ \;&=\; H^0(P_{even}, 2\Xi)\,, \\ H^0\big(\operatorname{Pic}^{g-1}(C),\mathcal{L}_{\omega\alpha}\big)_- \;&=\; H^0(P_{odd}, 2\Xi')\,. \end{aligned} \tag{2.3}$$

Since the surface $C-C$ is invariant under the involution $(-1) : \xi \mapsto \xi^{-1}$, the subspace Γ^α_{C-C} is invariant under $(-1)^*$ and decomposes into a direct sum of ±eigenspaces for $(-1)^* = T_\alpha^*$:

$$\Gamma^\alpha_{C-C} \;=\; \Gamma^{\alpha+}_{C-C} \oplus \Gamma^{\alpha-}_{C-C}\,.$$

2.1. Prym–Wirtinger duality

For the details see [B4] Lemma 2.3. There exists an integral Cartier divisor on the product $\mathcal{S}U_C(2,\alpha) \times \mathcal{S}U_C(2,\omega\alpha)$ whose support is given by

$$\left\{(E,F) \in \mathcal{S}U_C(2,\alpha) \times \mathcal{S}U_C(2,\omega\alpha) \mid h^0(C, E \otimes F) > 0\right\}.$$

Its associated section can be viewed as an element of the tensor product

$$H^0(\mathcal{S}U_C(2,\alpha), \mathcal{L}_\alpha) \otimes H^0(\mathcal{S}U_C(2,\omega\alpha), \mathcal{L}_{\omega\alpha})$$

and it can be shown that the corresponding linear map

$$H^0(\mathcal{S}U_C(2,\alpha), \mathcal{L}_\alpha)^* \longrightarrow H^0(\mathcal{S}U_C(2,\omega\alpha), \mathcal{L}_{\omega\alpha}) \tag{2.4}$$

is an isomorphism and is equivariant for the linear involutions induced by the map $E \mapsto E \otimes \alpha$. Hence using the identifications (2.2) and (2.3) we obtain canonical isomorphisms,

$$H^0(P, 2\Xi_0)^* \xrightarrow{\sim} H^0(P_{even}, 2\Xi), \quad H^0\big(P', 2\Xi_0'\big)^* \xrightarrow{\sim} H^0(P_{odd}, 2\Xi'). \tag{2.5}$$

3. The base locus of $\mathbb{P}\Gamma_{\widetilde{C}}$

In this section we compute the set–theoretical base locus of the subseries $\mathbb{P}\Gamma_{\widetilde{C}}$ on the Prym variety P'. Our strategy is to show (Lemma 3.3) that points in the base locus $Bs\big(\mathbb{P}\Gamma_{\widetilde{C}}\big)$ determine reducible divisors when pulled back to some parameter space S covering P_{odd} and then show that the set of such reducible divisors is the canonical curve (Lemma 3.5).

Suppose C non–hyperelliptic. We denote by $\widetilde{C}_m$ the m–th symmetric power of $\widetilde{C}$ and let S be the subvariety of $\widetilde{C}_{2g-2}$ defined as

$$S \;=\; \left\{D \in \widetilde{C}_{2g-2} \mid Nm(D) \in |\omega| \text{ and } h^0(D) \equiv 1 \bmod 2\right\}.$$

Then, by [B1] Corollaire page 365, the variety S is normal and irreducible of dimension $g-1$. The variety S comes equipped with two natural surjective morphisms

$$Nm \;:\; S \longrightarrow |\omega|, \quad u \;:\; S \longrightarrow P_{odd}$$

where u associates to an effective divisor D its line bundle $\mathcal{O}_{\widetilde{C}}(D)$. Note that u is birational and Nm is finite of degree 2^{2g-3}. Also denote by u the extended morphism $u : \widetilde{C}_{2g-2} \to \operatorname{Pic}^{2g-2}\big(\widetilde{C}\big)$ and consider the commutative diagram

$$\begin{array}{ccc} S & \hookrightarrow & \widetilde{C}_{2g-2} \\ \big\downarrow u & & \big\downarrow u \\ P_{odd} & \hookrightarrow & \operatorname{Pic}^{2g-2}\big(\widetilde{C}\big). \end{array} \tag{3.1}$$

Consider the Brill–Noether locus in P_{odd} which is defined set–theoretically by

$$\Xi_3 \;:=\; \left\{\lambda \in P_{odd} \mid h^0(\lambda) \geq 3\right\}.$$

The scheme structure on Ξ_3 is defined by taking the scheme–theoretical intersection [W2]

$$\Xi_3 \;:=\; W^2_{2g-2}\big(\widetilde{C}\big) \cap P_{odd}$$

where $W^2_{2g-2}\big(\widetilde{C}\big) \subset \operatorname{Pic}^{2g-2}\widetilde{C}$ is the Brill–Noether locus of line bundles having at least 3 sections (see [ACGH]).

Lemma 3.1. *The subscheme $\Xi_3 \subset P_{odd}$ is not empty and is of pure codimension* 3.

Proof. Theorem 9 [DCP] asserts that Ξ_3 is not empty and every irreducible component has dimension at least $g-4$. Suppose that there is an irreducible component I of dimension greater than or equal to $g-3$. Then its inverse image $u^{-1}(I)$ has dimension greater than or equal to $g-1$, hence, since S is irreducible, $u^{-1}(I) = S$ and $\Xi_3 = P_{odd}$. The last equality cannot happen, since otherwise, using translation by an element of the form $\mathcal{O}_{\widetilde{C}}(\tilde{p} - \sigma\tilde{p})$, we would have $\Xi = P_{even}$. □

Observe that u is equivariant for the action of σ on S and P_{odd}. Denote by $Z = u^{-1}(\Xi_3)$ the inverse image of the subscheme Ξ_3. By the previous lemma Z is of pure codimension 1 in S. We will see in a moment that there is a Cartier divisor $\mathcal{D}$ on S whose support is the support of Z. Let $\omega_{\widetilde{C}}$ be the dualizing sheaf of $\widetilde{C}$. Consider the following divisors in $\widetilde{C}_{2g-2}$

$$\begin{aligned} U_{\tilde{p}} \;&:=\; \big\{D \in \widetilde{C}_{2g-2} \mid \exists D' \in \widetilde{C}_{2g-3} \text{ with } D = D' + \tilde{p}\big\} \;=\; \tilde{p} + \widetilde{C}_{2g-3}\,,\\ V_{\tilde{p}} \;&:=\; \big\{D \in \widetilde{C}_{2g-2} \mid h^0\big(\omega_{\widetilde{C}}(-D-\tilde{p})\big) \geq 1\big\} \end{aligned}$$

and let $\overline{U}_{\tilde{p}}$ and $\overline{V}_{\tilde{p}}$ be their intersections with S. The divisor $\overline{U}_{\tilde{p}}$ is reduced: to see this, it suffices to show that at a general point $D = \tilde{p} + D' \in S \cap U_{\tilde{p}}$, the tangent space $T_D S$ to the variety S is not contained in the tangent space $T_D U_{\tilde{p}}$ to $U_{\tilde{p}}$. If $h^0(D) = 1$ $\big($which is the case for a general $D = \tilde{p} + D' \in S \cap U_{\tilde{p}}\big)$, then the differential of u at D, $du_D \;:\; T_D\widetilde{C}_{2g-2} \xrightarrow{\sim} T_{u(D)}\widetilde{\Theta} \subset H^0\big(\widetilde{C}, \omega_{\widetilde{C}}\big)^*$, is an isomorphism. Via this differential, the space $T_D\widetilde{C}_{2g-2}$ can be identified with the linear span $\langle D\rangle \subset H^0\big(\widetilde{C}, \omega_{\widetilde{C}}\big)^*$ of the divisor D. Under the isomorphism du_D the tangent spaces $T_D S$ and $T_D U_{\tilde{p}}$ map respectively to the tangent space to P_{odd}, i. e. $H^0(C, \omega\alpha)^*$, and the linear span $\langle D'\rangle$. If $H^0(C, \omega\alpha)^* \subset \langle D'\rangle$, then, projecting from $H^0(C, \omega\alpha)^*$ (or, equivalently, taking Nm), we obtain that $\langle NmD'\rangle$ is a hyperplane in the hyperplane $\langle NmD\rangle \subset H^0(C, \omega)^*$. This is impossible because, by Serre Duality and Riemann–Roch, it implies that $h^0(\pi(\tilde{p})) = 2$.

We denote by $\mathcal{O}_S(1)$ the pull–back by the norm map of the hyperplane line bundle on $|\omega|$. Then it is easily seen that, for any $\tilde{p} \in \widetilde{C}$,

$$Nm^*(|\omega(-p)|) \;=\; \overline{U}_{\tilde{p}} + \overline{U}_{\sigma\tilde{p}} \in |\mathcal{O}_S(1)|\,. \tag{3.2}$$

Let $\widetilde{\Theta}_\lambda$ denote the translate of $\widetilde{\Theta}$ by λ. Then, for any points $\tilde{p}, \tilde{q} \in \widetilde{C}$, we have an equality among divisors on $\widetilde{C}_{2g-2}$ (see [W1] page 6)

$$u^*\big(\widetilde{\Theta}_{\tilde{p}-\tilde{q}}\big) \;=\; U_{\tilde{p}} + V_{\tilde{q}}\,. \tag{3.3}$$

The analogue on the even Prym variety of the following lemma was previously proved by R. Smith and R. Varley. In the case of genus 3 it is in their paper [SV1] (Prop. 1 page 358) and for higher genus it will be published in their upcoming paper [SV2].

Lemma 3.2. *There exists an effective Cartier divisor $\mathcal{D}$ on S whose support is equal to*

$$\operatorname{supp} Z \;=\; \left\{ D \in S \mid h^0\big(\mathcal{O}_{\widetilde{C}}(D)\big) \geq 3 \right\}.$$

Moreover, we have the following equality among effective Cartier divisors

$$u^*\big(\Xi_{\tilde{p}-\sigma\tilde{p}} + \Xi_{\sigma\tilde{p}-\tilde{p}}\big) \;=\; \overline{U}_{\tilde{p}} + \overline{U}_{\sigma\tilde{p}} + \mathcal{D} \quad \textit{for all} \quad \tilde{p} \in \widetilde{C}\,. \tag{3.4}$$

In particular, $u^*\mathcal{O}_{P_{odd}}(2\Xi') = \mathcal{O}_S(1) \otimes \mathcal{O}_S(\mathcal{D})$.

Proof. We are going to define $\mathcal{D}$ as the residual divisor of the restricted divisor $\overline{V}_{\tilde{q}}$, for a given point $\tilde{q} \in \widetilde{C}$ and then show that it does not depend on the choice of $\tilde{q}$. We first observe that we have an equality of sets

$$\overline{V}_{\tilde{q}} \;=\; \overline{U}_{\sigma\tilde{q}} \cup Z$$

which can be seen as follows: for $D \in \widetilde{C}_{2g-2}$ such that $h^0(D) = h^0\big(\omega_{\widetilde{C}}(-D)\big) = 1$ the assumption $D \in \overline{V}_{\tilde{q}}$ and the formula $D + \sigma D = \pi^*(Nm(D))$ imply that $\tilde{q} \in \operatorname{supp} \sigma D \iff D \in \overline{U}_{\sigma\tilde{q}}$. If $h^0(D) = h^0\big(\omega_{\widetilde{C}}(-D)\big) \geq 2$, then $D \in \operatorname{supp} Z$. Again a calculation involving Zariski tangent spaces shows that $\overline{V}_{\tilde{q}}$ is reduced generically on $\overline{U}_{\sigma\tilde{p}}$. Hence we can define $\mathcal{D}$ by $\overline{V}_{\tilde{q}} = \overline{U}_{\sigma\tilde{q}} + \mathcal{D}$. Now we substitute this expression into (3.3), which we restrict to S

$$u^*\big(\widetilde{\Theta}_{\tilde{p}-\tilde{q}}\big)\big|_S \;=\; \overline{U}_{\tilde{p}} + \overline{U}_{\sigma\tilde{q}} + \mathcal{D}\,.$$

Now we fix $\tilde{q}$ and we take the limit when $\tilde{p} \to \tilde{q}$. Since $\mathcal{O}_{P_{odd}}\big(\widetilde{\Theta}\big) = \mathcal{O}_{P_{odd}}(2\Xi')$, we see that $\overline{U}_{\tilde{p}} + \overline{U}_{\sigma\tilde{p}} + \mathcal{D} \in |u^*\mathcal{O}_{P_{odd}}(2\Xi')|$. So by (3.2) we obtain the line bundle equality claimed in the lemma and we see that the scheme–structure on $\mathcal{D}$ does not depend on the point $\tilde{q}$. To prove (3.4), we compute using (3.3)

$$u^*\big(\widetilde{\Theta}_{\tilde{p}-\sigma\tilde{p}} + \widetilde{\Theta}_{\sigma\tilde{p}-\tilde{p}}\big) \;=\; \overline{U}_{\tilde{p}} + \overline{V}_{\sigma\tilde{p}} + \overline{U}_{\sigma\tilde{p}} + \overline{V}_{\tilde{p}}\,.$$

Now we restrict to S and use the commutativity of diagram (3.1) and the divisorial equality $\widetilde{\Theta}_{\tilde{p}-\sigma\tilde{p}} \cap P_{odd} = 2\Xi_{\tilde{p}-\sigma\tilde{p}}$ to obtain

$$u^*\big(2\Xi_{\tilde{p}-\sigma\tilde{p}} + 2\Xi_{\sigma\tilde{p}-\tilde{p}}\big) \;=\; 2\overline{U}_{\tilde{p}} + 2\overline{U}_{\sigma\tilde{p}} + 2\mathcal{D}\,.$$

Since $\overline{U}_{\tilde{p}} + \overline{U}_{\sigma\tilde{p}} + \mathcal{D} \in |u^*\mathcal{O}_{P_{odd}}(2\Xi')|$ we can divide this equality by 2 and we are done. □

Let μ be a point of $Bs\big(\mathbb{P}\Gamma_{\widetilde{C}}\big)$. By Lemma 5.2 the linear map $i^{*t} : |\omega|^* \longrightarrow \big|2\Xi_0'\big|^*$ is injective and, since $|\omega|^*$ is the span of the image of $\widetilde{C}$ in $\big|2\Xi_0'\big|^*$, the space $\mathbb{P}\Gamma_{\widetilde{C}}$ is the annihilator of $|\omega|^* \subset \big|2\Xi_0'\big|^*$. So $Bs\big(\mathbb{P}\Gamma_{\widetilde{C}}\big) = |\omega|^* \cap Kum(P')$ and μ corresponds to a hyperplane $H_\mu \in |\omega|^*$. Since $\mu \in Kum(P')$, the image of μ by Wirtinger duality is the divisor $\Xi_\mu + \Xi_{\mu^{-1}} \in |2\Xi'|$.

Lemma 3.3. *With the previous notation, we have an equality*

$$\text{for all} \quad \mu \in Bs\big(\mathbb{P}\Gamma_{\widetilde{C}}\big) \quad Nm^*\big(H_\mu\big) + \mathcal{D} \;=\; u^*\big(\Xi_\mu + \Xi_{\mu^{-1}}\big)\,. \tag{3.5}$$

Proof. The equality follows from the commutativity of the right–hand square of the diagram

$$\begin{array}{ccccccccc}
\widetilde{C} & \xrightarrow{\pi} & C & \xrightarrow{\varphi_{can}} & |\omega|^* & \xrightarrow{Nm^*} & |\mathcal{O}_S(1)| & \xrightarrow{+\mathcal{D}} & |\mathcal{O}_S(1)\otimes\mathcal{O}_S(\mathcal{D})| \\
\downarrow i & & & & \downarrow i^{*t} & & & \nearrow u^* & \\
P' & & \longrightarrow & & |2\Xi_0'|^* & \cong & |2\Xi'|. & &
\end{array}$$

The commutativity of the right–hand square follows from that of the outside square because $\varphi_{can}(C)$ generates $|\omega|^*$. In other words we need to check the assertion of the lemma only for hyperplanes of the form $|\omega(-p)|$ for $p \in C$. This follows immediately from (3.2) and (3.4). □

Corollary 3.4. *For every $\mu \in Bs\big(\mathbb{P}\Gamma_{\widetilde{C}}\big)$, the hyperplane $Nm^*(H_\mu)$ is reducible.*

Proof. By the above Lemma we have

$$u^*\big(\Xi_\mu + \Xi_{\mu^{-1}}\big) - \mathcal{D} \;=\; Nm^*\big(H_\mu\big)\,.$$

If $Nm^*\big(H_\mu\big)$ is irreducible, then the support of one of the divisors $u^*\big(\Xi_\mu\big)$ or $u^*\big(\Xi_{\mu^{-1}}\big)$, say $u^*\big(\Xi_\mu\big)$, is contained in the support of $\mathcal{D}$. This is impossible because $u^*\big(\Xi_\mu\big)$ is the inverse image of a divisor in P_{odd} and $\operatorname{supp}\mathcal{D}$ is the inverse image of the codimension 3 support of Ξ_3. □

The set–theoretical assertion of Theorem 1.1 now follows from the following lemma.

Lemma 3.5. *If C is not bielliptic, we have a set–theoretical equality*

$$\big\{H \in |\omega|^* \;:\; Nm^*(H) \text{ is reducible}\big\} \;=\; \varphi_{can}(C)\,.$$

If C is bielliptic, the LHS is contained in the union of $\varphi_{can}(C)$ and the finite set of points $t \in |\omega|^*$ such that the projection from t induces a morphism of degree 2 from C onto an elliptic curve.

Proof. Suppose that $Nm^*(H)$ is reducible. Then a local computation shows that the hyperplane H is everywhere tangent to the branch locus of Nm. It is immediately seen that the branch locus B of Nm is the dual hypersurface of the canonical curve. The components of the singular locus $\operatorname{Sing}(B)$ of B are of two different types which can be described as follows

Type 1 whose points are hyperplanes tangent to $\varphi_{can}(C)$ in more than one point.

Type 2 whose points are hyperplanes osculating to $\varphi_{can}(C)$.

To prove that $\mu \in \varphi_{can}(C)$, we need to prove that there is a point on $H \cap B$ which is smooth on B because the dual variety of B is the closure of the set of hyperplanes tangent to B at a smooth point and this is equal to $\varphi_{can}(C)$. In other words we need to show that $H \cap B$ is not contained in $\operatorname{Sing}(B)$. Since $H \cap B$ has pure codimension 2, it suffices to show that no codimension 2 component of $\operatorname{Sing}(B)$ is contained in a hyperplane.

Suppose a codimension 2 component B_i of type i ($i = 1$ or 2) is contained in a hyperplane H in $|\omega|$ and let $t \in |\omega|^*$ be the corresponding point. Then the set of hyperplanes in $|\omega|^*$ through t and doubly tangent (resp. osculating) to $\varphi_{can}(C)$ has dimension $g-3$. We have

Lemma 3.6. *For any $t \in \varphi_{can}(C)$ the restriction ρ to $\varphi_{can}(C)$ of the projection from t is birational onto its image. If $t \in |\omega|^* \setminus \varphi_{can}(C)$, then ρ is either birational onto its image or of degree two onto an elliptic curve.*

Proof. First note that the degree of the image C_t of C by the projection is at least $g-2$ because C_t is a non–degenerate curve in a projective space of dimension $g-2$. If $t \in \varphi_{can}(C)$, then the degree of ρ is equal to $2g-3$. The degree r of the restriction of ρ to C_t satisfies $r \cdot \deg(C_t) = 2g-3$. Therefore $\frac{2g-3}{r} \geq g-2$. Or $r \leq 2 + \frac{1}{g-2}$ which implies $r \leq 2$. However, r cannot be equal to 2 because $2g-3$ is odd. If $t \notin \varphi_{can}(C)$, then the same argument gives again $r \leq 2$ because $g \geq 5$. Hence, if ρ is not generically injective, then $r = 2$ and $\deg(C_t) = g-1$. Therefore C_t is either smooth rational or an elliptic curve. Since C is not hyperelliptic, we have that C_t is an elliptic curve. □

First suppose that $C \to C_t$ is birational. If $i = 1$, projecting from t, we see that the set of hyperplanes in $|\omega|^*/t$ doubly tangent to C_t has dimension $(g-3)$ that is equal to the dimension of the dual variety of C_t which is impossible. If $i = 2$, then the set of hyperplanes in $|\omega|^*/t$ osculating C_t has dimension $g-3$ which is also impossible.

If $C \to C_t$ is of degree 2, then indeed every hyperplane tangent to C_t pulls back to a hyperplane twice tangent (or osculating if the point of tangency is a branch point of $C \to C_t$) to $\varphi_{can}(C)$ and we have a codimension 2 family of type B_1 contained in the hyperplane H corresponding to t. Then $Nm^*(H)$ could be reducible. □

The previous lemma proves Theorem 1.1 set–theoretically for a non bielliptic curve. In the bielliptic case, we have to work a little more. By Lemma 3.5 a hyperplane $H \notin \varphi_{can}(C)$, such that $Nm^*(H)$ might be reducible, corresponds to a point $e \in |\omega|^*$ such that the projection from e induces a morphism γ of degree 2 from C to an elliptic curve E. In other words, e is the common point of all chords $\langle \gamma^* q \rangle$ ($q \in E$). In that case there exists a 1–dimensional family (parametrized by E) of trisecants, namely the chords $\langle \gamma^* q \rangle$, to the Kummer variety $Kum(P')$. By [De] the Prym variety is a Jacobian and by [S] (see also [B3] page 610) the double cover $\pi : \widetilde{C} \to C$ is of the following two types

1. C is trigonal;
2. C is a smooth plane quintic and $h^0(\mathcal{O}_C(1) \otimes \alpha) = 0$.

Lemma 3.7. *No double cover of a bielliptic curve C of genus $g \geq 6$ is of the above two types.*

Proof. For a bielliptic curve C, the Brill–Noether locus $W^1_{g-1}(C)$ has two irreducible components, which are fixed by the reflection in ω ([W1] Corollary 3.10). For a smooth plane quintic this Brill–Noether locus is irreducible, ruling out 1. For a trigonal curve this Brill–Noether locus has two irreducible components, which are interchanged by reflection in ω, ruling out 2. $\square$

Remark 3.8. If $g = 5$ and C is bielliptic, we do not know whether the common point of all the chords for a given bielliptic structure lies on $Kum(P')$ (see also [B3] Remark (1) page 611). We expect it not to be on $Kum(P')$.

4. Rank 2 bundles and 2Ξ–divisors

Consider the induced action of the involution σ on the moduli space $\mathcal{SU}_{\widetilde{C}}(2,\mathcal{O})$ given by $\widetilde{E} \mapsto \sigma^*\widetilde{E}$. Since the covering π is unramified, the fixed point set for the σ–action

$$Fix_\sigma \mathcal{SU}_{\widetilde{C}}(2,\mathcal{O}) \;=\; \left\{ \big[\widetilde{E}\big] \in \mathcal{SU}_{\widetilde{C}}(2,\mathcal{O}) \mid \exists \theta :\ \sigma^*\widetilde{E} \xrightarrow{\sim} \widetilde{E} \right\}$$

has two connected components which are the isomorphic images of $\mathcal{S}U_C(2,\mathcal{O})$ and $\mathcal{S}U_C(2,\alpha)$ by π^*. Similarly, since $\sigma^*\omega_{\widetilde{C}} \xrightarrow{\sim} \omega_{\widetilde{C}}$, the involution σ acts on $\mathcal{SU}_{\widetilde{C}}\big(2,\omega_{\widetilde{C}}\big)$ and

$$Fix_\sigma \mathcal{SU}_{\widetilde{C}}\big(2,\omega_{\widetilde{C}}\big) \;=\; \pi^*\mathcal{S}U_C(2,\omega) \cup \pi^*\mathcal{S}U_C(2,\omega\alpha)\,.$$

Proposition 4.1. *Consider a bundle $E \in \mathcal{S}U_C(2,\omega\alpha)$ such that $E \notin \varphi(P_{odd})$ and put $\widetilde{E} = \pi^*E$. Then there is a divisor $\Delta(E) \in |2\Xi_0|$ with the following properties.*

1. *If $D\big(\widetilde{E}\big)$ does not contain P, then*

$$D\big(\widetilde{E}\big) \;=\; 2\Delta(E)\,.$$

For E general, P is not contained in $D\big(\widetilde{E}\big)$ and $\Delta(E)$ is reduced.

2. *Let pr_+ be the projection $|\mathcal{L}_\alpha| \to |2\Xi_0|$ with center $\big|2\Xi_0'\big|$ (see (2.2)). Then we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{S}U_C(2,\omega\alpha) & \xrightarrow{D} & |\mathcal{L}_\alpha| \\ & \searrow^{\Delta} & \big\downarrow pr_+ \\ & & |2\Xi_0| \;=\; |\mathcal{L}_\alpha|_+ \end{array}$$

Remark 4.2. Similarly, when $E \in \mathcal{S}U_C(2,\omega\alpha)$ such that $E \notin \varphi(P_{even})$, we obtain divisors $\Delta(E) \in \big|2\Xi_0'\big|$ as described in Proposition 4.1 by projecting on the $-$eigenspace $pr_- : |\mathcal{L}_\alpha| \longrightarrow |\mathcal{L}_\alpha|_- = \big|2\Xi_0'\big|$.

Proof. 1. Given a bundle $F \in Fix_\sigma \mathcal{SU}_{\widetilde{C}}\big(2,\omega_{\widetilde{C}}\big)$ and a line bundle $\xi \in J\widetilde{C}$ which is anti–invariant under σ, i. e., $\sigma^*\xi \xrightarrow{\sim} \xi^{-1}$, we have a natural non–degenerate quadratic form with values in the canonical bundle $\omega_{\widetilde{C}}$

$$\begin{array}{cccc} q : & F\otimes\xi & \longrightarrow & \omega_{\widetilde{C}}\,, \\ & s & \longmapsto & s\wedge\sigma^*s \end{array}$$

where s is a local section of $F\otimes\xi$. Note that we have canonical isomorphisms

$$\sigma^*(F\otimes\xi) \;=\; F\otimes\xi^{-1} \;=\; \mathrm{Hom}\big(F\otimes\xi,\omega_{\widetilde{C}}\big)\,.$$

Therefore we are in a position to apply the Atiyah–Mumford lemma [M1] to the family of bundles (here F is fixed, with $\sigma^*F \xrightarrow{\sim} F$)

$$\{F \otimes \xi\}_{\xi \in P}$$

which states that the parity of $h^0\big(\widetilde{C}, F \otimes \xi\big)$ is constant when ξ varies in P.

From now on, we suppose $F = \widetilde{E} = \pi^* E$, with $E \in \mathcal{S}U_C(2, \omega\alpha)$, then

$$h^0\big(\widetilde{C}, \widetilde{E}\big) \ = \ 2h^0(C, E) \ \equiv \ 0 \bmod 2\,.$$

For the first equality we use the fact that $H^0\big(\widetilde{C}, \widetilde{E}\big) = H^0(C, E) \oplus H^0(C, E\alpha)$ and, by Riemann–Roch and Serre duality, $h^0(C, E) = h^1(C, E) = h^0(C, \omega \otimes E^*) = h^0(C, E\alpha)$.

First suppose that $E \in \mathcal{S}U_C(2, \omega\alpha)$ is general. Then the divisor $D\big(\widetilde{E}\big)$ does not contain the Prym variety P (e. g. because, for general E, $h^0(E) = 0 \iff h^0\big(\widetilde{E}\big) = 0 \iff \mathcal{O} \notin D\big(\widetilde{E}\big)$), so the restriction of the divisor $D\big(\widetilde{E}\big) \in \big|2\Theta_{\widetilde{C}}\big|$ to P is a divisor in the linear system $|4\Xi_0|$. Moreover, for $\xi \in D\big(\widetilde{E}\big) \cap P$

$$\mathrm{mult}_\xi D\big(\widetilde{E}\big) \ \geq \ h^0\big(\widetilde{C}, \widetilde{E} \otimes \xi\big) \ \geq \ 2$$

because $h^0\big(\widetilde{C}, \widetilde{E} \otimes \xi\big) \equiv h^0\big(\widetilde{C}, \widetilde{E}\big) \equiv 0 \bmod 2$. Hence any point $\xi \in D\big(\widetilde{E}\big) \cap P$ is a singular point of $D\big(\widetilde{E}\big)$, which implies that $D\big(\widetilde{E}\big) \cap P$ is an everywhere non–reduced divisor. We have

Lemma 4.3. *Suppose that $D\big(\widetilde{E}\big) \cap P$ is a divisor in P. Then there is a divisor $\Delta(E) \in |2\Xi_0|$ such that $D\big(\widetilde{E}\big) \cap P = 2\Delta(E)$.*

Proof. A local equation of $\Delta(E)$ is given by the pfaffian of a skew–symmetric perfect complex of length one $L \to L^*$ representing the perfect complex $R\,pr_{1*}\big(\mathcal{P} \otimes pr_2^* \widetilde{E}\big)$ where $\mathcal{P}$ is the Poincaré line bundle over the product $P \times \widetilde{C}$ and pr_1, pr_2 are the projections on the two factors. The construction of the complex $L \to L^*$ is given in the proof of Proposition 7.9 [LS]. □

If E is of the form $E = \pi_* L$ for some $L \in P_{even}$, we have $\Delta(E) = T_L^* \Xi + T_{\omega L^{-1}}^* \Xi$. It follows from this equality that $\Delta(E)$ is reduced for general E.

So far we have defined a rational map $\Delta : \mathcal{S}U_C(2, \omega\alpha) \longrightarrow |2\Xi_0|$. It will follow from part 2 of the proposition that Δ can be defined away form $\varphi(P_{odd})$.

2. First we consider the composite (rational) map

$$\mathrm{Pic}^{\,g-1}(C) \ \xrightarrow{\psi} \ \mathcal{S}U_C(2, \omega\alpha) \ \xrightarrow{\Delta} \ |2\Xi_0|\,.$$

A straight–forward computation shows that for all $\xi \in \mathrm{Pic}^{\,g-1}(C)$ such that $\pi^*\xi \notin P_{odd}$ the divisor $\Delta(\psi(\xi)) = \Delta\big(\xi \oplus \omega\alpha\xi^{-1}\big)$ equals the translated divisor $T_{\pi^*\xi}\widetilde{\Theta}$ restricted to P. Hence, by [M2], the map $\Delta \circ \psi$ is given by the full linear system $|\mathcal{L}_{\omega\alpha}|_+$ of invariant elements of $|\mathcal{L}_{\omega\alpha}|$. By Prym–Wirtinger duality (2.4) and (2.5) $|\mathcal{L}_{\omega\alpha}|_+^* \cong |\mathcal{L}_\alpha|_+ \cong |2\Xi_0|$ and we obtain the commutative diagram in the proposition. Geometrically, Δ is obtained by restricting the projection with center the $-$eigenspace $|\mathcal{L}_\alpha|_-$ to the embedded moduli space $\mathcal{S}U_C(2, \omega\alpha) \subset |\mathcal{L}_\alpha|$. Since by [NR] $|\mathcal{L}_\alpha|_- \cap \mathcal{S}U_C(2, \omega\alpha) = \varphi(P_{odd})$ we see that Δ is well–defined for $E \notin \varphi(P_{odd})$ even if $D(\widetilde{E}\,) \supset P$. □

Remark 4.4. We observe that we obtain by the same construction a rational map

$$\Delta \;:\; \mathcal{S}U_C(2, \omega) \;\longrightarrow\; |2\Xi_0|\,.$$

The images under Δ of the two moduli spaces $\mathcal{S}U_C(2, \omega)$ and $\mathcal{S}U_C(2, \omega\alpha)$ coincide, which is easily deduced from the following formula. Let β be a 4–torsion point such that $\beta^{\otimes 2} = \alpha$ and $\pi^*\beta \in P[2]$. Then, for any $E \in \mathcal{S}U_C(2, \omega)$, we have $E \otimes \beta \in \mathcal{S}U_C(2, \omega\alpha)$ and

$$T^*_{\pi^*\beta}\Delta(E) \;=\; \Delta(E \otimes \beta)\,.$$

Similar statements hold for $\mathcal{S}U_C(2, \alpha)$.

5. Proof of Theorem 1.2

5.1. Proof of $\Gamma^{\alpha+}_{C-C} = \Gamma_{00}$

The strategy is to show that the two linear maps

$$\phi_1 \;:\; H^0(P, 2\Xi_0)_0 \;\longrightarrow\; \mathrm{Sym}^2 T^*_0 P \;=\; \mathrm{Sym}^2 H^0(\omega\alpha)$$

and

$$\phi_2 \;:\; H^0(JC, \mathcal{L}_\alpha)_{+0} \;\longrightarrow\; H^0(C \times C, \delta^*\mathcal{L}_\alpha - 2\Delta)_+ \;=\; \mathrm{Sym}^2 H^0(\omega\alpha)$$

differ by multiplication by a scalar under the isomorphism (2.2) $H^0(JC, \mathcal{L}_\alpha)_{+0} \cong H^0(P, 2\Xi_0)_0$. Here the subscript 0 denotes the subspace (on P or JC) consisting of global sections vanishing at the origin. The map ϕ_1 sends $s \in H^0(P, 2\Xi_0)_0$ to the quadratic term of its Taylor expansion at the origin $\mathcal{O} \in P$ and ϕ_2 is the pull–back of invariant sections of $\mathcal{L}_\alpha$ under the difference map

$$\begin{array}{rccc} \delta \;: & C \times C & \longrightarrow & JC\,, \\ & (p, q) & \longmapsto & \mathcal{O}_C(p - q)\,. \end{array}$$

By restricting to the fibers of the two projections $p_i : C \times C \to C$ and using the See–saw Theorem, we compute that $\delta^*\mathcal{L}_\alpha = p_1^*(\omega\alpha) \otimes p_2^*(\omega\alpha)(2\Delta_C)$ where $\Delta_C \subset C \times C$ is the diagonal. Since $\phi_2^{-1}(0) = \Delta_C$ and the sections of $\mathcal{L}_\alpha$ are symmetric, we see that im $\phi_2 \subset \mathrm{Sym}^2 H^0(\omega\alpha) \subset H^0(\omega\alpha)^{\otimes 2} = H^0(p_1^*(\omega\alpha) \otimes p_2^*(\omega\alpha)) \subset H^0(p_1^*(\omega\alpha) \otimes p_2^*(\omega\alpha)(2\Delta_C))$. So if ϕ_1 and ϕ_2 are proportional, we will have

$$\Gamma_{00} \;=\; \ker\phi_1 \;=\; \ker\phi_2 \;=\; \Gamma^{\alpha+}_{C-C}\,.$$

To show that $\phi_1 = \lambda\phi_2$ for some $\lambda \in \mathbb{C}^*$, we compute $\phi_1(s_E)$ and $\phi_2(s_E)$ for special sections, namely those with divisor of zeros $Z(s_E) = \Delta(E)$ for some vector bundle $E \in \mathcal{S}U_C(2, \omega\alpha)$ with $h^0(E) = h^0(E \otimes \alpha) = 2$. Recall that by Riemann–Roch and Serre duality we have $h^0(E) = h^0(E \otimes \alpha)$ for $E \in \mathcal{S}U_C(2, \omega\alpha)$. Now to compute $\phi_1(s_E)$, we need to determine the tangent cone to $\Delta(E)$ at $\mathcal{O} \in P$. As before we put $\widetilde{E} = \pi^*E$. By [L] Prop. V.2, this tangent cone is the intersection of the anti–invariant part $H^0(\omega\alpha) = H^0\big(\omega_{\widetilde{C}}\big)_-$ of $H^0\big(\omega_{\widetilde{C}}\big) = T_0^* J\widetilde{C}$ with the affine cone over the projective cone over the Grassmannian $Gr\big(2, H^0\big(\widetilde{E}\,\big)^*\big) \subset \mathbb{P}\Lambda^2 H^0\big(\widetilde{E}\big)^*$ under the linear map

$$\mu^* \;:\; H^0\big(\omega_{\widetilde{C}}\big)^* \;\longrightarrow\; \Lambda^2 H^0\big(\widetilde{E}\,\big)^* \tag{5.1}$$

which is the dual of the map $\mu : \Lambda^2 H^0\big(\widetilde{E}\,\big) \to H^0\big(\omega_{\widetilde{C}}\big)$ obtained from exterior product by the isomorphism $\wedge^2 \widetilde{E} \cong \omega_{\widetilde{C}}$. Note that the σ–invariant part $\big[\Lambda^2 H^0\big(\widetilde{E}\,\big)^*\big]_+$ is canonically isomorphic to the 2–dimensional subspace $\Lambda^2 H^0(E)^* \oplus \Lambda^2 H^0(E\alpha)^* \subset \Lambda^2 H^0\big(\widetilde{E}\,\big)^*$ because $H^0\big(\widetilde{E}\,\big)_+ = H^0(E)$ and $H^0\big(\widetilde{E}\,\big)_- = H^0(E\alpha)$. Since $\wedge^2 E \cong \wedge^2(E \otimes \alpha) \cong \omega\alpha$, the restriction of μ to $\wedge^2 H^0(E)$ $\big($resp. $\wedge^2 H^0(E \otimes \alpha)\big)$ which is obtained from exterior product by the isomorphism $\wedge^2 E \cong \omega\alpha$ $\big($resp. $\wedge^2(E \otimes \alpha) \cong \omega\alpha\big)$ maps into $H^0(\omega\alpha)$. Therefore the linear map μ^* (5.1) maps σ–anti–invariant sections into σ–invariant sections, i. e.,

$$\mu_+^* \;:\; H^0(\omega\alpha)^* \;\longrightarrow\; \Lambda^2 H^0(E)^* \oplus \Lambda^2 H^0(E\alpha)^* \,. \tag{5.2}$$

Since the intersection $\mathbb{P}(\Lambda^2 H^0(E)^* \oplus \Lambda^2 H^0(E\alpha)^*) \cap Gr\big(2, H^0\big(\widetilde{E}\,\big)^*\big)$ consists of the two points $\mathbb{P}\Lambda^2 H^0(E)^*$ and $\mathbb{P}\Lambda^2 H^0(E\alpha)^*$, it follows that the intersection of $H^0(\omega\alpha) \subset H^0\big(\omega_{\widetilde{C}}\big)$ with the cone over $Gr\big(2, H^0\big(\widetilde{E}\,\big)^*\big)$ is the union of the two lines $\wedge^2 H^0(E)$ and $\wedge^2 H^0(E \otimes \alpha)$. Therefore the tangent cone of $\Delta(E)$ at the origin is the union of the two hyperplanes in $|\omega\alpha|^*$ which are the zeros of $a, b \in H^0(\omega\alpha)$ such that

$$a\,\mathbb{C} \;=\; \mathrm{im}\,\big(\Lambda^2 H^0(E) \longrightarrow H^0(\omega\alpha)\big)\,, \quad b\,\mathbb{C} \;=\; \mathrm{im}\,\big(\Lambda^2 H^0(E\alpha) \longrightarrow H^0(\omega\alpha)\big)\,. \tag{5.3}$$

In other words, up to multiplication by a nonzero scalar,

$$\phi_1(s_E) \;=\; a \otimes b + b \otimes a \in \mathrm{Sym}^2 H^0(\omega\alpha)\,.$$

We now compute $\phi_2(s_E)$. First we note that the pull–back map induced by δ is equivariant for the involution $(-1) : \xi \mapsto \xi^{-1}$ acting on JC and the involution $(p, q) \mapsto (q, p)$ acting on $C \times C$. Since $\Delta(E) = pr_+(D(E))$ by Proposition 4.1, this implies that

$$\phi_2(s_E) \;=\; \phi_2(pr_+(s_E)) \;=\; pr_+(\delta^*(s_E))\,. \tag{5.4}$$

On the RHS pr_+ denotes the projection $H^0(\omega\alpha) \otimes H^0(\omega\alpha) \longrightarrow \mathrm{Sym}^2 H^0(\omega\alpha)$. Therefore we compute

$$\delta^*(D(E)) \;=\; \big\{(p, q) \in C \times C \mid h^0(E(p - q)) > 0\big\}$$

and take its symmetric part. It follows from [vGI] Lemma 3.2 that

$$\delta^*(D(E)) \;=\; C \times Z_a + Z_b \times C + 2\Delta_C \tag{5.5}$$

where Z_a (resp. Z_b) is the divisor of zeros of a (resp. b). Hence it follows from (5.4) and (5.5) that $\phi_2(pr_+(s_E)) = a \otimes b + b \otimes a$ up to multiplication by a nonzero scalar. We can now conclude that $\phi_1 = \lambda \phi_2$ for some $\lambda \in \mathbb{C}^*$ because, by the following lemma (Prop. 3.7 [vGI]), we have enough bundles $E \in \mathcal{S}U_C(2, \omega\alpha)$ with $h^0(E) = 2$ to generate linearly the image $\mathrm{Sym}^2 H^0(\omega\alpha)$ of ϕ_1 and ϕ_2.

Lemma 5.1. (Prop. 3.7 [vGI].) *For general sections $a, b \in H^0(\omega\alpha)$, we can find a semi–stable bundle $E \in \mathcal{S}U_C(2, \omega\alpha)$ with $h^0(E) = 2$ such that* (5.5) *holds.*

5.2. Proof of $\Gamma^{\alpha -}_{C-C} = \Gamma^{(2)}_{\widetilde{C}}$

First note that any anti–invariant section of $\mathcal{L}_\alpha$ vanishes at $\mathcal{O} \in JC$. Denote by

$$\tau \;:\; H^0(JC, \mathcal{L}_\alpha)_- \;\longrightarrow\; T_0^* JC \;=\; H^0(\omega)$$

the map which sends an element s of $H^0(JC, \mathcal{L}_\alpha)_-$ to the linear term of its Taylor expansion at the origin (Gauss map). Recall the natural embedding of the curve $\widetilde{C}$ into the Prym variety P'

$$i \;:\; \widetilde{C} \;\longrightarrow\; P', \quad \tilde{p} \;\longmapsto\; \mathcal{O}_{\widetilde{C}}(\tilde{p} - \sigma\tilde{p}) . \tag{5.6}$$

Then $i^*\mathcal{O}\big(2\Xi_0'\big) \cong \omega_{\widetilde{C}}$ and since all $2\Xi_0'$–divisors are symmetric and i is equivariant for the involution, i induces a linear map

$$i^* \;:\; H^0\big(P', 2\Xi_0'\big) \;\longrightarrow\; H^0(C, \omega) \;=\; H^0\big(\widetilde{C}, \omega_{\widetilde{C}}\big)_+ . \tag{5.7}$$

Lemma 5.2. *The linear maps τ and i^* are proportional via the isomorphism* (2.2) *and are surjective.*

Proof. It will be enough to show that the canonical divisors $i^*(\Delta(\pi_*\lambda))$ and $\tau(D(\pi_*\lambda))$ are equal for a general element $\lambda \in P_{odd}$. In both cases the divisor coincides with the divisor $Nm(\delta)$, where δ is the unique effective divisor in the linear system $|\lambda|$. The computations are straightforward and left to the reader. □

Therefore we can conclude that

$$H^0(JC, \mathcal{L}_\alpha)^{(3)}_{0-} \;=\; \ker\tau \;=\; \ker i^* \;=\; \Gamma_{\widetilde{C}} ,$$

where $H^0(JC, \mathcal{L}_\alpha)^{(3)}_{0-}$ denotes the subspace of $H^0(JC, \mathcal{L}_\alpha)_-$ of elements with multiplicity greater than or equal to 2 (hence greater than or equal to 3 by anti–symmetry) at the origin. We now proceed as in the proof of part 1 of Theorem 1.2. We consider the two linear maps

$$\phi_1 \;:\; \Gamma_{\widetilde{C}} \;\longrightarrow\; \Lambda^2 H^0(\omega\alpha) ,$$
$$\phi_2 \;:\; H^0(JC, \mathcal{L}_\alpha)^{(2)}_{0-} \;\longrightarrow\; H^0(C \times C, \delta^*\mathcal{L}_\alpha(-2\Delta))_- \;=\; \Lambda^2 H^0(\omega\alpha)$$

which are defined as follows. As in part 1, ϕ_2 is the map given by pull–back under the difference map δ. To define ϕ_1, let $N_{\widetilde{C}/P'}$ denote the normal bundle of $i\big(\widetilde{C}\big)$ in P'. Then ϕ_1 is obtained by restricting a section $s \in \Gamma_{\widetilde{C}}$ to the first infinitesimal neighborhood of $\widetilde{C}$. In other words

$$\Gamma^{(2)}_{\widetilde{C}} \;=\; \ker \left\{ \phi_1 : \Gamma_{\widetilde{C}} \longrightarrow H^0\Big(\widetilde{C}, N^*_{\widetilde{C}/P'} \otimes i^*\mathcal{O}\big(2\Xi_0'\big)\Big)_- \;=\; H^0\Big(\widetilde{C}, N^*_{\widetilde{C}/P'} \otimes \omega_{\widetilde{C}}\Big)_- \right\} .$$

The vector bundle $N^*_{\widetilde{C}/P'}$ fits into the exact sequence

$$0 \;\longrightarrow\; N^*_{\widetilde{C}/P'} \;\longrightarrow\; H^0(\omega\alpha) \otimes \mathcal{O}_{\widetilde{C}} \;\longrightarrow\; \omega_{\widetilde{C}} \;\longrightarrow\; 0 \tag{5.8}$$

where the right–hand map is the embedding $H^0(\omega\alpha)\otimes\mathcal{O}_{\widetilde{C}} \hookrightarrow H^0(\omega_{\widetilde{C}})\otimes\mathcal{O}_{\widetilde{C}}$ followed by evaluation $H^0(\omega_{\widetilde{C}})\otimes\mathcal{O}_{\widetilde{C}} \to \omega_{\widetilde{C}}$. Therefore this map is the pull–back of evaluation $H^0(\omega\alpha)\otimes\mathcal{O} \xrightarrow{ev} \omega\alpha$. Let M be the kernel of the latter, i. e., we have the exact sequence

$$0 \longrightarrow M \longrightarrow H^0(\omega\alpha)\otimes\mathcal{O} \xrightarrow{ev} \omega\alpha \longrightarrow 0, \tag{5.9}$$

whose pull–back by π is (5.8).

We twist (5.9) by $\omega\alpha$ and take cohomology

$$0 \longrightarrow H^0(C, M\otimes\omega\alpha) \longrightarrow H^0(\omega\alpha)\otimes H^0(\omega\alpha) \xrightarrow{m} H^0(\omega^2) \longrightarrow \dots$$

where m is the multiplication map. We deduce that

$$H^0\left(\widetilde{C}, N^*_{\widetilde{C}/P'}\otimes\omega_{\widetilde{C}}\right)_- = H^0(C, M\otimes\omega\alpha) = \ker m = \Lambda^2 H^0(\omega\alpha)\oplus I_C^{Pr}(2)$$

where $I_C^{Pr}(2)$ is the space of quadrics through the Prym–canonical curve. It remains to show that $\operatorname{im}\phi_1 = \Lambda^2 H^0(\omega\alpha)$. This will follow from the next two lemmas. First we will compute, as in part 1, the image under ϕ_1 of some special sections $s_E \in \Gamma_{\widetilde{C}}$, namely s_E such that $Z(s_E) = \Delta(E)$ with E a general bundle in $\mathcal{SU}_C(2,\omega\alpha)$ with $h^0(E) = 2$, i. e., we determine the tangent spaces to $\Delta(E)$ along the curve $i(\widetilde{C})$. This is done in the following lemma.

Lemma 5.3. *Let a, b be the sections defined by* (5.3). *Then we have*

$$\phi_1(s_E) = a\wedge b \in \Lambda^2 H^0(\omega\alpha)$$

up to multiplication by a nonzero scalar.

Proof. First we need to show that for a general semi–stable bundle E with $h^0(E) = 2$ the divisor $\Delta(E)$ is smooth at a general point $i(\tilde{p}) \in \Delta(E)$. For this decompose a general Prym–canonical divisor into two effective divisors of degree $g-1$, i. e., $D+D' \in |\omega\alpha|$. Put $L = \mathcal{O}(D)$. Then $h^0(D) = 1 = h^0(\omega(-D)) = h^0(\omega\alpha(-D)) = h^0(\alpha(D))$. If $E = L\oplus\omega\alpha L^{-1}$, then $\widetilde{E} = \pi^*E = \pi^*L\oplus\omega_{\widetilde{C}}\pi^*L^{-1}$, $D(\widetilde{E}) = \widetilde{\Theta}_{\pi^*L} + \widetilde{\Theta}_{\omega_{\widetilde{C}}\pi^*L^{-1}}$ and $\Delta(E) = \widetilde{\Theta}_{\pi^*L}\Big|_{P'} + \widetilde{\Theta}_{\omega_{\widetilde{C}}\pi^*L^{-1}}\Big|_{P'}$. At a general point $i(\tilde{p}) \in \widetilde{\Theta}_{\pi^*L}$, we see immediately that the tangent space to $\widetilde{\Theta}_{\pi^*L}$ does not contain the tangent space to P', i. e., $\Delta(E)$ is smooth at $i(\tilde{p})$. Next we compute the tangent space to the divisor $\Delta(E)$ at a smooth point $i(\tilde{p}) \in \Delta(E)$. The smoothness of $\Delta(E)$ at $i(\tilde{p})$ implies that $h^0(\widetilde{C}, \widetilde{E}(\tilde{p}-\sigma\tilde{p})) = 2$. We choose a basis $\{u, v\}$ of the 2–dimensional vector space $H^0(\widetilde{C}, \widetilde{E}(\tilde{p}-\sigma\tilde{p}))$. Then by [L] Prop. V.2 and the same reasoning as in the proof of part 1 of Theorem 1.2, we see that the projectivized tangent space $\mathbb{T}_{i(\tilde{p})}\Delta(E)$ to $\Delta(E)$ at $i(\tilde{p})$, which is a hyperplane in $\mathbb{P}T_{i(\tilde{p})}P' \cong |\omega\alpha|^*$ is the zero locus of the section in $\gamma(\tilde{p}) \in H^0(\omega\alpha)$, which is the image of $u\wedge\sigma^*v := u\otimes\sigma^*v - v\otimes\sigma^*u$ under the exterior product map

$$\begin{aligned} & H^0(\widetilde{E}(\tilde{p}-\sigma\tilde{p}))\otimes\sigma^* H^0(\widetilde{E}(\tilde{p}-\sigma\tilde{p})) \\ = \quad & H^0(\widetilde{E}(\tilde{p}-\sigma\tilde{p}))\otimes H^0(\widetilde{E}(\sigma\tilde{p}-\tilde{p})) \xrightarrow{\mu} H^0(\omega_{\widetilde{C}}) \end{aligned}$$

Since $\det E = \omega\alpha$, we see that $\gamma(\tilde{p}) = \mu(u\wedge\sigma^*v) \in H^0(\omega\alpha) \subset H^0(\omega_{\widetilde{C}})$. We will now describe the map $\gamma : \widetilde{C} \to |\omega\alpha| :\ \tilde{p} \mapsto \gamma(\tilde{p})$. Note that, since $h^0(\widetilde{E}) = 4$, we have

$h^0\big(\widetilde{E}(-\sigma\tilde{p})\big) = 2$ for $\tilde{p}$ general. Hence $\{u, v\}$ is also a basis for $H^0\big(\widetilde{E}(-\sigma\tilde{p})\big)$. Consider the inclusion

$$H^0\big(\widetilde{E}(-\sigma\tilde{p})\big) \subset H^0\big(\widetilde{E}\big) \;=\; H^0(E) \oplus H^0(E\alpha)$$

and decompose $u{=}u_+ + u_-$, $v{=}v_+ + v_-$ with $u_+, v_+{\in}H^0(E){=}H^0\big(\widetilde{E}\big)_+$ and $u_-, v_-{\in}$ $H^0(E\alpha) = H^0\big(\widetilde{E}\big)_-$. Then the element $\gamma(\tilde{p})$ is the image of $(u_+ \wedge v_+, -u_- \wedge v_-) \in \Lambda^2 H^0(E) \oplus \Lambda^2 H^0(E\alpha)$ under the exterior product map $\Lambda^2 H^0(E) \oplus \Lambda^2 H^0(E\alpha) \to H^0(\omega\alpha)$, i.e., $\gamma(\tilde{p}) \in \mathbb{P}(\mathbb{C}\, a \oplus \mathbb{C}\, b) \subset |\omega\alpha|$. Since $\widetilde{C} \subset \Delta(E)$, we have $\varphi_{\alpha can}(p) \in \mathbb{T}_{i(\tilde{p})}(\Delta(E))$. So for general $\tilde{p}$, $\gamma(\tilde{p})$ is the unique divisor of the pencil $\mathbb{P}(\mathbb{C}\, a \oplus \mathbb{C}\, b)$ containing $\tilde{p}$. Hence we can conclude that the section $\phi_1(s_E) \in H^0(M \otimes \omega\alpha)$ considered as a tensor in $H^0(\omega\alpha) \otimes H^0(\omega\alpha)$ is $a \wedge b$. $\square$

Since, a priori, we do not know that $\mathbb{P}\Gamma_{\widetilde{C}}$ is spanned by divisors of the form $\Delta(E)$, we need to establish a symmetry property for any divisor $D \in \mathbb{P}\Gamma_{\widetilde{C}}$. This is done as follows.

Let $\tilde{s}, \tilde{t} \in \widetilde{C}$ be two points of $\widetilde{C}$ with respective images $s, t \in C$ and let D be an element of $\mathbb{P}\Gamma_{\widetilde{C}}$. Assume that $i(\tilde{s}), i\big(\tilde{t}\big) \in D$ are smooth points of D and let $\mathbb{T}_s D$ and $\mathbb{T}_t D$ denote the projectivized tangent spaces to the divisor D at the points $i(\tilde{s})$ and $i\big(\tilde{t}\big)$. Since we can identify the projectivized tangent space to the Prym variety P' at any point with the Prym–canonical space $|\omega\alpha|^*$, we may view $\mathbb{T}_s D$ and $\mathbb{T}_t D$ as hyperplanes in $|\omega\alpha|^*$. Note that $\mathbb{T}_s D$ only depends on $s \in C$ and not on the lift $\tilde{s} \in \widetilde{C}$. Then we have

Lemma 5.4. *With the preceding notation, we have an equivalence*

$$\varphi_{\alpha can}(s) \in \mathbb{T}_t D \iff \varphi_{\alpha can}(t) \in \mathbb{T}_s D\,.$$

P r o o f. Consider the invertible sheaf $x = \mathcal{O}_{\widetilde{C}}\big(\tilde{s} - \sigma\tilde{s} + \tilde{t} - \sigma\tilde{t}\big) \in P$ and the corresponding embedding

$$i_x \;:\; \widetilde{C} \;\longrightarrow\; P', \quad \tilde{p} \;\longmapsto\; \mathcal{O}_{\widetilde{C}}(\tilde{p} - \sigma\tilde{p}) \otimes x\,.$$

The curve $i_x\big(\widetilde{C}\big)$ is the curve $i\big(\widetilde{C}\big)$ translated by x. A straight–forward computation shows that $i_x^{-1}\big(\mathcal{O}_{P'}\big(2\Xi_0'\big)\big) = \omega_{\widetilde{C}} x^{-2}$ and by a result of Beauville (see [IvS] page 569) the induced linear map on global sections $H^0\big(P', 2\Xi_0'\big) \to H^0\big(\omega_{\widetilde{C}} x^{-2}\big)$ is surjective. We observe that

$$i_x\big(\sigma\tilde{t}\big) \;=\; i(\tilde{s})\,, \quad i_x(\sigma\tilde{s}) \;=\; i\big(\tilde{t}\big)\,,$$

and that the projectivized tangent line to the curve $i_x\big(\widetilde{C}\big)$ at the point $i_x\big(\sigma\tilde{t}\big)$ (resp. $i_x(\sigma\tilde{s})$) is the point $\varphi_{\alpha can}(t)$ (resp. $\varphi_{\alpha can}(s)$) in $|\omega\alpha|^* \cong \mathbb{P}T_{i(\tilde{s})}P'$ $\big($resp. $\cong \mathbb{P}T_{i(\tilde{t})}P'\big)$. Let $\mathbb{T}_{\tilde{s}}$ $\big($resp. $\mathbb{T}_{\tilde{t}}\big)$ denote the embedded tangent line in $\big|2\Xi_0'\big|^*$ to the curve $i_x\big(\widetilde{C}\big)$ at the point $i_x\big(\sigma\tilde{t}\big)$ (resp. $i_x(\sigma\tilde{s})$), so that $\mathbb{T}_{\tilde{s}}$ $\big($resp. $\mathbb{T}_{\tilde{t}}\big)$ passes through the point $i(\tilde{s})$ $\big($resp. $i\big(\tilde{t}\big)\big)$ with tangent direction $\varphi_{\alpha can}(t)$ (resp. $\varphi_{\alpha can}(s)$). Then the lemma will follow if we show that these two tangent lines intersect in a point $I\big(\tilde{s}, \tilde{t}\big)$, i.e.

$$\mathbb{T}_{\tilde{s}} \cap \mathbb{T}_{\tilde{t}} \;=\; I\big(\tilde{s}, \tilde{t}\big) \in \big|2\Xi_0'\big|^*\,. \tag{5.10}$$

This property follows from a dimension count: since C is non–hyperelliptic, we have $x^{-2} \neq \mathcal{O}_{\widetilde{C}}$, so $h^0\big(\omega_{\widetilde{C}}x^{-2}\big) = 2g-2$. Since $h^0\big(\omega_{\widetilde{C}}x^{-2}\big(-2\sigma\tilde{s}-2\sigma\tilde{t}\big)\big) = h^0\big(\omega_{\widetilde{C}}\big(-2\tilde{s}-2\tilde{t}\big)\big) \geq 2g-5$, the tangent lines $\mathbb{T}_{\tilde{t}}$ and $\mathbb{T}_{\tilde{s}}$ are contained in a projective 2–plane, hence intersect. To obtain the equivalence stated in the lemma, let H_D denote the hyperplane in $\big|2\Xi_0'\big|^*$ corresponding to the divisor $D \in \mathbb{P}\Gamma_{\widetilde{C}}$. Assume e. g. that $\varphi_{\alpha can}(s) \in \mathbb{T}_t D$. This means that H_D contains $\mathbb{T}_{\tilde{t}}$. Since $i(\tilde{s}) \in H_D$, it follows from (5.10) that H_D also contains $\mathbb{T}_{\tilde{s}}$, so $\varphi_{\alpha can}(t) \in \mathbb{T}_s D$. □

At this stage we can conclude: by Lemma 5.4 we know that for all $s \in \Gamma_{\widetilde{C}}$, $\phi_1(s) \in H^0(\omega\alpha) \otimes H^0(\omega\alpha)$ lies either in the symmetric or skew–symmetric eigenspace, i. e. $\operatorname{im}\phi_1 \subset I_C^{Pr}(2) \subset \operatorname{Sym}^2 H^0(\omega\alpha)$ or $\operatorname{im}\phi_1 \subset \Lambda^2 H^0(\omega\alpha)$. Lemma 5.3 asserts that $\operatorname{im}\phi_1 \subset \Lambda^2 H^0(\omega\alpha)$.

As in (5.4), we have that $\phi_2(pr_-(s_E)) = pr_-(\delta^*(s_E))$, where pr_- denotes the projection $H^0(\omega\alpha) \otimes H^0(\omega\alpha) \to \Lambda^2 H^0(\omega\alpha)$ and s_E is as above. Hence we see that $\phi_2(pr_-(s_E)) = a \wedge b$. By Lemma 5.3 the projectivizations of ϕ_1 and ϕ_2 coincide on all divisors of the form $\Delta(E)$ whose images generate $\mathbb{P}\wedge^2 H^0(\omega\alpha)$. Hence $\phi_1 = \phi_2$ up to a nonzero scalar and ϕ_1 and ϕ_2 are surjective.

Remark 5.5. An alternative way of proving that $\operatorname{im}\phi_1 \subset \Lambda^2 H^0(\omega\alpha)$ would be to twice take the derivative of the quadrisecant identity for Prym varieties [F] Prop. 6 $\big($fix two points and consider the other two as canonical coordinates on the universal cover of $\widetilde{C}$.$\big)$

6. The scheme–theoretical base locus of $\mathbb{P}\Gamma_{\widetilde{C}}$

From Section 3 we know that the sets $Bs\big(\mathbb{P}\Gamma_{\widetilde{C}}\big)$ and $i\big(\widetilde{C}\big)$ are equal. To prove the scheme–theoretical equality, it will be enough to show that, for all $\tilde{p} \in \widetilde{C}$, the projectivized tangent spaces at $i(\tilde{p})$ to divisors $D \in \mathbb{P}\Gamma_{\widetilde{C}}$ cut out the projectivized tangent space at $i(\tilde{p})$ to $i\big(\widetilde{C}\big)$, which is $\varphi_{\alpha can}(p) \in |\omega\alpha|^* = \mathbb{P}T_{i(\tilde{p})}P'$, i. e.,

$$\bigcap_{D \in \mathbb{P}\Gamma_{\widetilde{C}}} \mathbb{T}_{i(\tilde{p})} D \;=\; \varphi_{\alpha can}(p)\,. \tag{6.1}$$

If we take $D = \Delta(E)$ $\big($we consider here the divisor $\Delta(E) \in \big|2\Xi_0'\big|$ defined in Remark 4.2$\big)$ for some semi–stable vector bundle E with $h^0(E) = 2$, then the hyperplane $\mathbb{T}_{i(\tilde{p})}(\Delta(E)) \subset |\omega\alpha|^*$ corresponds to the unique section of the pencil $\mathbb{P}(\mathbb{C}\,a \oplus \mathbb{C}\,b)$ vanishing at p (proof of Lemma 5.3). Since for general $a, b \in |\omega\alpha|$ we can find a vector bundle E (Lemma 5.1) such that equality in Lemma 5.3 holds, we can conclude (6.1).

Acknowledgements

We would like to thank R. Smith *and* R. Varley *for helpful discussions. This work was carried out during visits of the first author of the University of Nice and of the second author of the University of Georgia. We wish to thank these universities for their hospitality. The first author was partially supported by a grant from the National Security Agency.*

References

[ACGH] Arbarello, E., Cornalba, M., Griffiths, P. A., and Harris, J.: Geometry of Algebraic Curves, Vol. 1, Springer–Verlag, New York, Berlin, Heidelberg, Tokyo, 1985

[B1] Beauville, A.: Sous–Variétés Spéciales des Variétés de Prym, Comp. Math. **45**, Vol. 3 (1982), 357 – 383

[B2] Beauville, A.: Fibrés de Rang 2 sur une Courbe, Fibré Déterminant et Fonctions Thêta, Bull. Soc. Math. France **116** (1988), 431 – 448

[B3] Beauville, A.: Prym Varieties: A Survey, Proceedings of Symposia in Pure Mathematics Vol. **49**, Part 1 (1989), 607 – 620

[B4] Beauville, A.: Fibrés de Rang 2 sur une Courbe, Fibré Déterminant et Fonctions Thêta II, Bull. Soc. Math. France **119** (1991), 259 – 291

[BNR] Beauville, A., Narasimhan, M. S., and Ramanan, S.: Spectral Curves and the Generalised Theta Divisor, J. Reine Angew. Math. **398** (1989), 169 – 179

[De] Debarre, O.: The Trisecant Conjecture for Pryms, Proceedings of Symposia in Pure Mathematics Vol. **49**, Part 1 (1989), 621 – 626

[Do1] Donagi, R.: Big Schottky, Invent. Math. **89** (1987), 569 – 599

[Do2] Donagi, R.: The Schottky Problem, Theory of Moduli, Lecture Notes in Math. Vol. **1337**, pp. 84 – 137, Springer–Verlag, 1988

[DCP] De Concini, C., and Pragacz, P.: On the Class of Brill–Noether Loci for Prym Varieties, Math. Ann. **302** (1995), 687 – 697

[F] Fay, J.: Schottky Relations on $\frac{1}{2}(C - C)$, Theta Functions, Proc. Sympos. Pure Math. **49**, Part 1 (1989), 485 – 502, Bowdoin, 1987

[vGvdG] van Geemen, B., and van der Geer, G.: Kummer Varieties and the Moduli Spaces of Abelian Varieties, Am. J. Math. **108** (1986), 615 – 642

[vGI] van Geemen, B., and Izadi, E.: The Tangent Space to the Moduli Space of Vector Bundles on a Curve and the Singular Locus of the Theta Divisor of the Jacobian, J. Algebraic Geometry **10** (2001), 133 – 177

[vGP] van Geemen, B., and Previato, E.: Prym Varieties and the Verlinde Formula, Math. Ann. **294** (1992), 741 – 754

[I1] Izadi, E.: The Geometric Structure of $\mathcal{A}_4$, the Structure of the Prym Map, Double Solids and Γ_{00}–Divisors, Journal für die Reine und Angewandte Mathematik **462** (1995), 93 – 158

[I2] Izadi, E.: Second Order Theta Divisors on Pryms, Bull. Soc. Math. France **127** (1999), 1 – 23

[IvS] Izadi, E., and van Straten, D.: The Intermediate Jacobians of the Theta Divisors of Four–Dimensional Principally Polarized Abelian Varieties, J. Alg. Geom. **4** (1995), 557 – 590

[L] Laszlo, Y.: Un Théorème de Riemann pour les Diviseurs Thêta sur les Espaces de Modules de Fibrés Stables, Duke Math. J. **64** (1991), 333 – 347

[LS] Laszlo, Y., and Sorger, C.: The Line Bundle on the Moduli of Parabolic G–Bundles over Curves and Their Sections, Ann. Sci. Ec. Norm. Sup. **30** (1997), 499 – 525

[M1] Mumford, D.: Theta Characteristics of an Algebraic Curve, Ann. Sci. Ec. Norm. Sup. **4** (1971), 181 – 192

[M2] Mumford, D.: Prym Varieties I. In: Contributions to Analysis, Academic Press, New York, 1974

[NR] Narasimhan, M. S., and Ramanan, S.: Generalized Prym Varieties as Fixed Points, J. Ind. Math. Soc. **39** (1975), 1 – 19

[S] Shokurov, V.: Distinguishing Prymians from Jacobians, Invent. Math. **65** (1981), 209 – 219

[SV1] Smith, R., and Varley, R.: On the Geometry of Two–Dimensional Prym Varieties, Pac. J. of Math. **188** (1999), 353 – 368

[SV2] Smith, R., and Varley, R.: The Prym Torelli Problem: An Update and a Reformulation as a Question in Birational Geometry, preprint

[W1] Welters, G. E.: The Surface $C-C$ on Jacobi Varieties and Second Order Theta Functions, Acta. Math. **157** (1986), 1 – 22

[W2] Welters, G. E.: A Theorem of Giesecker–Petri Type for Prym Varieties, Ann. Sci. Ec. Norm. Sup. **18** (1985), 671 – 683

Department of Mathematics
Boyd Graduate Studies Research Center
University of Georgia
Athens, GA 30602–7403
USA
E – mail:
izadi@math.uga.edu

Laboratoire J.–A. Dieudonné
Université de Nice Sophia Antipolis
Parc Valrose
06108 Nice Cedex 02
France
E – mail:
pauly@math.unice.fr