J. reine angew. Math. **462** (1995), 93—158

Journal für die reine und angewandte Mathematik

The geometric structure of $\mathscr{A}_4$, the structure of the Prym map, double solids and Γ_{00}-divisors

By *E. Izadi* at Cambridge

Introduction

Let $\mathscr{A}_4$ be the moduli space of principally polarized abelian varieties (ppav) of dimension 4 over the field $\mathbb{C}$ of complex numbers. Let $\mathscr{P}_5$ be the moduli space of "Prym-curves" $(\tilde{X}, X)$ of genus 5 (i.e., X is a stable curve of genus 5 and $\tilde{X}$ is an allowable double cover of X in the sense of [B1]). The Prym map $P: \mathscr{P}_5 \to \mathscr{A}_4$ is surjective and proper [B1]. The moduli space $\mathscr{A}_4$ has dimension 10 and $\mathscr{P}_5$ has dimension 12. Hence the *generic* fibers of P have dimension 2.

To a *generic* ppav A, Donagi associated a cubic threefold T with an even point μ of order 2 in its intermediate jacobian (see [Do2] and [Do4]). Let F denote the variety of lines in T. Donagi observed that the double cover of F obtained from μ by restriction is isomorphic to $P^{-1}(A)$. He denoted by λ the involution on $P^{-1}(A)$ with quotient F. Donagi's construction is as follows: Start with a generic Prym-curve $(\tilde{X}, X)$. One has a canonical double cover $\tilde{Q} \to Q$ (where Q is a smooth plane quintic) with $P(\tilde{Q}, Q) = JX$. The point of order 2 in JX associated to $\tilde{X} \to X$, gives, by pullback, three nonzero points of order 2 in JQ. If we denote $\lambda(\tilde{X}, X)$ by $(\tilde{X}_\lambda, X_\lambda)$, the Prym varieties of Q associated to the three points of order 2 are JT, JX and JX_λ. The Torelli theorem for smooth cubic threefolds and smooth curves permits us then to recover T, X and X_λ from their jacobians. The three abelian varieties JT, JX and JX_λ each come with a point of order 2: JT comes with μ, while the points of order 2 in JX and JX_λ have A as Prym varieties. Donagi has a beautiful configuration of points of order 2 which we reproduce in 6.31, Figures 1–4.

We construct the cubic threefold T and the involution λ in a different way which permits us to define them for every element of $\mathscr{A}_4 \backslash (\bar{\mathscr{J}}_{\text{hyp}} \cup \mathscr{A}_{4\text{dec}})$. Here $\mathscr{J}_{\text{hyp}}$ is the locus of jacobians of hyperelliptic curves in $\mathscr{A}_4$, $\bar{\mathscr{J}}_{\text{hyp}}$ is its closure and $\mathscr{A}_{4\text{dec}}$ is the locus of decomposable ppav's.

We start with a study of the fibers of the Prym map. For a decomposable ppav A it is easily seen that the dimension of $P^{-1}(A)$ is greater than 2. For the jacobian JC of a smooth non-hyperelliptic curve C, $P^{-1}(JC)$ is two-dimensional [B1]. We do not know the dimension of the fibers of P at hyperelliptic jacobians. Let $\mathcal{J}_4$ be the locus of jacobians of smooth curves and $\bar{\mathcal{J}}_4$ its closure in $\mathcal{A}_4$. We show that the fibers of P are always two-dimensional outside of $\bar{\mathcal{J}}_4$ and that they always contain smooth Prym-curves. For any $A \in \mathcal{A}_4 \backslash \bar{\mathcal{J}}_4$, we show that there is a dense open subset of $P^{-1}(A)$ whose elements are either smooth or what we call of type (1, 1) (see 3.3). We show

Theorem 1. *There is an involution $\lambda : (\tilde{X}, X) \mapsto (\tilde{X}_\lambda, X_\lambda)$ defined on*

$$\mathcal{P}_5 \backslash P^{-1}(\mathcal{J}_{\mathrm{hyp}} \cup \mathcal{A}_{4\,\mathrm{dec}}).$$

The curve $\tilde{X}_\lambda$ parametrizes the Prym-embeddings of $\tilde{X}$ in any symmetric theta divisor Θ of $A = P(\tilde{X}, X) = P(\tilde{X}_\lambda, X_\lambda)$.

For a smooth non-hyperelliptic curve C, it is shown in [B1], 170–175, 183, and [DS] that the fiber of P at JC has two components each isomorphic to the quotient of the second symmetric product of C by the automorphism group of C. We show that the involution λ exchanges the two components of $P^{-1}(JC)$.

We then reveal a beautiful connection between Prym-curves, Γ_{00}-divisors and double solids. By a Γ_{00}-divisor we mean an element of the linear system $|2\Theta|$ which has multiplicity greater than or equal to 4 at 0. We let $|2\Theta|_{00}$ be the linear system containing the Γ_{00}-divisors. This linear system was first studied in connection with the Schottky problem: the problem of characterizing jacobians among all ppav's [GG]. For an indecomposable ppav of dimension 4, it has (projective) dimension 4.

Suppose that A is not in $\bar{\mathcal{J}}_{\mathrm{hyp}} \cup \mathcal{A}_{4\,\mathrm{dec}}$, i.e., A is neither decomposable nor a hyperelliptic jacobian. A construction of Clemens associates a Γ_{00}-divisor to a double solid (see 2.1). We use this to associate a pencil $l_X = l_{X_\lambda}$ of Γ_{00}-divisors to $(\tilde{X}, X)$ in $P^{-1}(A)$ (after the choice of an isomorphism $P(\tilde{X}, X) \cong A$): Let $\Sigma(X)$ be the image of the second symmetric product of $\tilde{X}$ in A. We show that there is exactly one pencil $l_X = l_{X_\lambda}$ of Γ_{00}-divisors containing $\Sigma(X) \cup \Sigma(X_\lambda)$.

Then we define the threefold $T = T_A \subset |2\Theta|_{00}$ as the union of the pencils l_X for $(\tilde{X}, X) \in P^{-1}(A)$. We show that T is irreducible. In order to determine the degree of T, we study the geometry of A.

Let $h : A \to (|2\Theta|_{00})^*$ be the natural rational map. Let t be a generic element of A. We regard $h(t)$ as a hyperplane in $|2\Theta|_{00}$. We show that $h(t)$ contains a line l_X if and only if $\Theta \cap \Theta_t$ contains a Prym-embedding of either $\tilde{X}$ or $\tilde{X}_\lambda$. Here Θ_t is the translate of Θ by t. Then we show that there are exactly 27 Prym-embedded curves in $\Theta \cap \Theta_t$ which are not images of each other by translations or $\pm$ id; furthermore, these Prym-embedded curves are not images of each other by λ. It follows that a generic hyperplane section of T contains 27 distinct lines l_X. We use this to show that T is a cubic threefold.

Next we study the singularities of T. Denoting by θ_{null} the divisor of $\mathcal{A}_4$ parametrizing ppav's with vanishing theta-nulls, we show that

Theorem 2. *$T = T_A$ is smooth if $A \in \mathcal{A}_4 \backslash (\bar{\mathcal{J}}_4 \cup \theta_{\mathrm{null}})$;*

– *T has one ordinary double point if $A \in \mathcal{J}_4 \backslash (\theta_{\mathrm{null}} \cup \mathcal{J}_{\mathrm{hyp}})$;*

– *T has a finite number of double points if* $A \in \theta_{\text{null}} \backslash \bar{\mathscr{J}}_4$ *(one double point for each vanishing theta-null of A and one ordinary double point generically on* θ_{null});

– *T has one double point t such that the quadric tangent cone to T at t has rank* 3 *if* $A \in \theta_{\text{null}} \cap (\mathscr{J}_4 \backslash \mathscr{J}_{\text{hyp}})$.

Corollary. *Let* $\mathscr{T}_0$ *be the moduli space of cubic threefolds with one node; then we have a finite rational map*

$$\theta_{\text{null}} \to \mathscr{T}_0 \to \mathscr{J}_4$$

where the first map is defined by $A \mapsto T_A$ *and the second map associates its intermediate jacobian to (the blow up of) a cubic threefold (at its node).*

We also have a nice geometric characterization of T: Let $\tilde{A}$ be the blow up of A at 0 and let $\tilde{h}$ be the lift of h to $\tilde{A}$, we show

Theorem 3. *Let* $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. *Then* $\tilde{h}$ *is a morphism. The branch locus of* $\tilde{h}$ *is the union of the dual variety* T^* *of T and the image of the exceptional divisor of* $\tilde{A}$ *by* $\tilde{h}$.

Let $\tilde{P}(A)$ be the finite cover of $P^{-1}(A)$ parametrizing pairs consisting of a Prym-curve $(\tilde{X}, X)$ and an isomorphism $\phi : P(\tilde{X}, X) \xrightarrow{\cong} A$ well-determined up to translations and $-$id. Sending $(\tilde{X}, X, \phi)$ to l_X defines a morphism $l: \tilde{P}(A) \to F$ (recall that F is the variety of lines in T). We show that there is a dense open subset U of F such that $l|_{l^{-1}(U)}$ is finite of degree 2. When A is generic, we show that l is an unramified double cover. *We can then show that our cubic threefold T and involution* λ *are isomorphic to those introduced by Donagi for generic* ppav's.

The most important application of our construction is the solution to a geometric version of the Schottky problem in genus 4: Let $\tau|2\Theta|_{00}$ be the linear system of the quartic tangent cones to Γ_{00}-divisors. It is conjectured (see [GG] and [Do3]) that (here A can have any dimension)

– A is a jacobian if and only if the base locus of $|2\Theta|_{00}$ contains points other than 0;

– A is a jacobian if and only if the base locus of $\tau|2\Theta|_{00}$ is nonempty.

It has been proved by Welters [W1] that the base locus of $|2\Theta|_{00}$ is a surface for a jacobian (see 5.13). It is shown in [Iz1] that the base locus of $\tau|2\Theta|_{00}$ is the canonically embedded curve (with one well-determined exception) for a jacobian (see 6.20). Our results have the following consequence:

Theorem 4. *Let* $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$, *then*:

i) *The only base point of* $|2\Theta|_{00}$ *is* 0 *with multiplicity* 4^4. *Here "multiplicity" is as defined in* 6.18 *below.*

ii) *The base locus of* $\tau|2\Theta|_{00}$ *is empty. In particular,* $\tau|2\Theta|_{00}$ *always has dimension* $\geqq 3$.

For other applications see [Iz2].

The structure of the paper is as follows: We first recall in section 1 some well-known definitions and results which we will need to use. In section 2 we write down some unpublished results of Beauville, Debarre, Clemens and Donagi which we will use later on. On a first reading, the reader may want to skip sections 1 and 2 and start with section 3. In section 3 we study the structure and the dimension of the fibers of P for non-jacobians and we define the involution λ (3.11–3.13). In section 4 we determine the number of Prym-embedded curves in a generic intersection of two translates of Θ. We define, in 4.3, the involution λ on the fiber of P at a non-hyperelliptic jacobian. In sections 5 and 6 we only consider non-jacobians. Section 5 is devoted to the definition of the pencils l_X and the study of their incidence correspondence. Theorem 4i) is proved in 5.13 (which only uses *smooth* Prym-curves) and 6.19. The existence of the dense open subset $U \subset F$ above is a consequence of 5.7 and 5.11 once we construct T. In section 6 we define the threefold T, show its irreducibility and determine its degree and its singularities. We prove theorem 4ii) in 6.20. In 6.25 we determine the branch locus and the ramification locus of $\tilde{h}$. We prove theorem 2 for non-jacobians in 6.13, 6.14 and 6.26. In 6.27 we show that l is an unramified double cover for A generic. We make the connection with Donagi's work in 6.31. Finally, in section 7 we prove the results we announced for non-hyperelliptic jacobians. We include a list of notations at the end.

Acknowledgements. Most of the results in this paper were part of my PhD thesis directed by H. Clemens whom I especially wish to thank for his continual encouragement and his generosity and patience. I also wish to thank A. Beauville and O. Debarre for their careful reading of the numerous preliminary versions of this paper and many useful comments and suggestions.

1. Conventions, basic facts and definitions

By an *element* or *point* of a scheme S we mean a point over $\mathbb{C}$. By a component of S we mean irreducible component unless we specify otherwise. By a *generic* element of S we mean an element of some dense open subset of S.

1.1. The linear system $|2\Theta|_{00}$. For a ppav A of dimension g with symmetric theta divisor Θ we denote by Γ_{00} the space of sections of $\mathcal{O}_A(2\Theta)$ with multiplicity greater than or equal to 4 at $0: |2\Theta|_{00}$ is the projectivization of Γ_{00}. For an indecomposable ppav, i.e., one that is not the product of two ppav's of lower dimensions, the dimension of Γ_{00} is $2^g - g.(g+1)/2 - 1$ (see [Ig], p. 188).

1.2. Prym varieties and Prym-curves. Let $\mathscr{P}_{g+1}$ be the moduli space of stable curves X of arithmetic genus $g+1$ with an allowable double cover $\pi: \tilde{X} \to X$. By "allowable" we mean that $\pi: \tilde{X} \to X$ verifies condition (**) on page 173 of [B1]. By a Prym-curve of genus $g+1$ we mean an element of $\mathscr{P}_{g+1}$, by a smooth, irreducible, ..., Prym-curve $(\tilde{X}, X)$ we mean that X is smooth, irreducible, ... We denote by η the torsion-free rank one coherent sheaf on X such that $\pi_* \mathcal{O}_{\tilde{X}} \cong \mathcal{O}_X \oplus \eta$. If X is smooth, η is a point of order 2 in JX. In general we have an isomorphism $\eta \cong \mathrm{Hom}(\eta, \mathcal{O}_X)$ which induces a morphism $\eta \otimes \eta \to \mathcal{O}_X$. This morphism is not an isomorphism at the points of X where π is ramified. The covering involution $\sigma: \tilde{X} \to \tilde{X}$ induces an involution (still denoted by σ) on $J\tilde{X}$ (the generalized jacobian of $\tilde{X}$). The Prym variety $P(\tilde{X}, X) = P(X, \eta)$ is the image of $(\sigma - \mathrm{id}): J\tilde{X} \to J\tilde{X}$.

The subvariety $\mathrm{Ker}(\sigma + \mathrm{id})$ of $J\tilde{X}$ is the union of the "even part" $A = P(\tilde{X}, X)$ and the "odd part" A^-: a translate of A. There is a parametrization of A, A^- and Θ associated to $(\tilde{X}, X)$:

$$A \cong \{L \in \mathrm{Pic}\,\tilde{X} : \nu(L) \cong \omega_X, h^0(\tilde{X}, L) \text{ is even}\},$$
$$\Theta \cong \{L \in \mathrm{Pic}\,\tilde{X} : \nu(L) \cong \omega_X, h^0(\tilde{X}, L) \text{ is even and positive}\},$$
$$A^- \cong \{L \in \mathrm{Pic}\,\tilde{X} : \nu(L) \cong \omega_X, h^0(\tilde{X}, L) \text{ is odd}\},$$

where ω_X is the dualizing sheaf of X and $\nu : \mathrm{Pic}\,\tilde{X} \to \mathrm{Pic}\,X$ is the norm map. The Prym variety is a ppav of dimension g. The map which to a Prym-curve associates its Prym variety is called the Prym map and is denoted by P. The set

$$\{L_p - \sigma L_p : p \in \tilde{X}, L_p \text{ is an effective Cartier divisor of degree 1 supported at } p\}$$

is isomorphic to $\tilde{X}$ (when $\tilde{X}$ does not have a pencil of degree 2) and contained in A^-. This is easy to see if $\tilde{X}$ is smooth, if $\tilde{X}$ is not smooth, we need only remark that the isomorphism class of $L_p - \sigma L_p$ only depends on p and not on the choice of L_p with support p (use the description on page 158 of [B1] reproduced in the proof of lemma 3.1 below). A Prym-embedding of $\tilde{X}$ in A is, by definition, a translate of $\{L_p - \sigma L_p : p \in \tilde{X}\}$ by an element of A^-.

The cotangent space to A at 0 is canonically identified with $H^0(X, \omega_X \otimes \eta)$. If X is smooth, the image of X by the morphism associated to the linear system $|\omega_X \otimes \eta|$ in the projectivization $\mathbb{P}H^0(X, \omega_X \otimes \eta)^*$ is called the Prym-canonical model of X and denoted by χX. The curve χX is also the image of any Prym-embedding of $\tilde{X}$ by the Gauss map. (The proofs of the above facts can be found in [B1] and [M1].) We call $\Sigma(X, \eta)$ (or $\Sigma(X)$ when there is no ambiguity on η) the surface image of the symmetric product $\tilde{X}^{(2)}$ by the map

$$\tilde{X}^{(2)} \to A,$$
$$(p, q) \mapsto [p, q] = L_p - \sigma L_p + L_q - \sigma L_q.$$

For a fixed abelian variety A and $(\tilde{X}, X) \in P^{-1}(A)$, $\Sigma(X) = \Sigma(X, \eta, \phi)$ is well-defined in A only after the choice of an isomorphism $\phi : P(\tilde{X}, X) \xrightarrow{\cong} A$ up to translations and $-\mathrm{id}$.

From now on, whenever we mention $\Sigma(X)$ or $\Sigma(X, \eta)$, we implicitly assume that we have made a choice for ϕ up to translations and $-\mathrm{id}$.

The morphism $\Sigma_A = \bigcup_{(X,\eta,\phi) \in \tilde{P}(A)} \Sigma(X) \to A$ lifts to a morphism $\bigcup_{(X,\eta,\phi) \in \tilde{P}(A)} \tilde{X}^{(2)} \to \tilde{A}$.

1.3. Canonical curves of genus 5 and the singular quadrics containing them. For a curve X we denote by $\mathrm{Pic}^d X$ the principal homogeneous space on JX parametrizing invertible sheaves of degree d on X, by W_d^r the variety parametrizing linear systems of projective dimension r and degree d on X and by g_d^r an arbitrary element of W_d^r. We let $\mathrm{Div}^d X$ be the variety of Cartier divisors of degree d on X. Finally, we denote by K_X an arbitrary canonical divisor on X with nonsingular support.

For a smooth curve X of genus 5 there is a net N of quadrics containing its canonical model κX and, if X is non-trigonal, a plane quintic Q parametrizes the singular quadrics in the net. If X is trigonal, then every quadric containing κX is singular and a plane quintic Q parametrizes those that are singular at some point of κX; the plane quintic Q is the image of X by its unique g_5^2 hence is obtained from X by identifying two points.

By a suitable isomorphism $JX \cong \mathrm{Pic}^4 X$, each g_4^1 on X goes to a singular point of the theta divisor Θ' of JX. The tangent cone to the theta divisor at a g_4^1 is a singular quadric containing κX (singular at a point of κX if X is trigonal). This defines a finite morphism of degree 2 from the singular locus $\tilde{Q}'$ of Θ' onto the plane quintic Q. Under this morphism each g_4^1 and its opposite $|K_X - g_4^1|$ go to the same singular quadric q. A given q is of rank less than or equal to 4, if its rank is 4 it has two rulings by planes. One of the two rulings cuts on κX the divisors of g_4^1 and the other cuts on κX the divisors of $|K_X - g_4^1|$. The double cover $\tilde{Q}' \to Q$ is allowable and JX is its Prym variety. See [B2], p. 362 for X nontrigonal, [ACGH], p. 274 and [B1], Theorem 5.4(i) for X trigonal.

1.4. The tetragonal and the trigonal constructions. Consider a smooth curve C of genus g with a g_4^1, say g_C, and an étale double cover $\tilde{C}$. The curve $\tilde{C}$ is of genus $2g-1$. The set of liftings of divisors of g_C in $\tilde{C}^{(4)}$ is a curve which has two connected components $\tilde{C}'$ and $\tilde{C}''$ with involutions σ' and σ''. These involutions interchange complementary liftings of the same divisor of g_C. The quotients $C' = \tilde{C}'/\sigma'$ and $C'' = \tilde{C}''/\sigma''$ have genus g and come with g_4^1's, say $g_{C'}$ and $g_{C''}$: on C' for instance the points of a divisor of $g_{C'}$ are the classes modulo σ' of the liftings of a given divisor of g_C. The construction is symmetric and we say that $((\tilde{C}, C), (\tilde{C}', C'), (\tilde{C}'', C''))$ (or $((\tilde{C}, C, g_C), (\tilde{C}', C', g_{C'}), (\tilde{C}'', C'', g_{C''})))$ is a tetragonally related triple. If the ramification points of g_C are all simple, then σ', σ'' are fixed point free, $(\tilde{C}, C), (\tilde{C}', C'), (\tilde{C}'', C'')$ are smooth and their Prym varieties are isomorphic.

Given a g_4^1 on C, define $\tilde{W}$ as the set of elements $s+t$ of $C^{(2)} = \mathrm{Div}^2(C)$ such that $h^0(g_4^1 - s - t) :=$ dimension of $H^0(g_4^1 - s - t)$ is positive. Define an involution σ on $\tilde{W}$ by sending s, t to s', t' whenever $s + t + s' + t' \in g_4^1$. The curve $W = \tilde{W}/\sigma$ comes with a g_3^1: the points of a generic divisor of the g_3^1 are the 3 partitions of a divisor of g_4^1 into two sets of two points. The double cover $\tilde{W} \to W$ is allowable and its Prym variety is JC. We say that C is trigonally related to $(\tilde{W}, W)$.

The curves C, C', C'' above are trigonally related to a curve W (of genus $g+1$). The curve W comes with three double covers with Prym varieties JC, JC', JC''.

The tetragonal construction is due to Donagi [Do1] and the trigonal construction to Recillas [R] (see also [DS], pp. 47–48, for the case where W is singular). These two constructions have been generalized in [B3].

1.5. Double solids. The proofs of the following facts can be found in [C1].

A quartic double solid Z with six nodes is a double cover of $\mathbb{P}^3$ branched along a quartic surface B with 6 ordinary double points or nodes $p_1, \ldots, p_6$. By general position for the nodes we mean that they impose independent conditions on planes, quadrics and cubics in $\mathbb{P}^3$. Let $\tilde{Z}$ be the blow up of Z at its nodes. The intermediate jacobian

$$JZ = H^{2,1}(\tilde{Z})^* / H_3(\tilde{Z}, \mathbb{Z})$$

is an element of $\mathscr{A}_4$ and can be identified with the group of algebraic one-cycles in $\tilde{Z}$ which are algebraically equivalent to 0, modulo rational equivalence. Let $\mathscr{Z}$ be the moduli space of quartic double solids with six nodes in general position. The map $J : \mathscr{Z} \to \mathscr{A}_4$ which to Z associates JZ is dominant, hence its generic fibers have dimension 3.

A curve of degree d in $\tilde{Z}$ is, by definition, a curve C in $\tilde{Z}$ such that

- $C.\tilde{\pi}^* \mathscr{O}_{\mathbb{P}^3}(1) = d$ where $\tilde{\pi} : \tilde{Z} \to \mathbb{P}^3$ is the natural map;

- $C.\mathscr{Q}_i = 0$ for all i, where $\mathscr{Q}_i$ is the exceptional quadric in $\tilde{Z}$ above p_i.

A curve of degree d in Z is the image of a curve of degree d in $\tilde{Z}$. A twisted cubic in Z is one of the two components of the inverse image in Z of a twisted cubic in $\mathbb{P}^3$ which is everywhere tangent to B (with even multiplicity at all points of contact).

Fix a curve C_0 in $\tilde{Z}$. For any family $\mathscr{C} \to S$ of curves in $\tilde{Z}$ algebraically equivalent to C_0, we define the Abel-Jacobi map $AJ : S \to JZ$ (with base curve C_0) by sending a curve C to the rational equivalence class of $C - C_0$.

The threefold $\tilde{Z}$ admits 42 conic bundle structures over $\mathbb{P}^2$ which permit us to write $JZ \cong P(\tilde{X}, X)$ in 42 (generically) distinct ways. For 12 of these, say $(\tilde{X}_i, X_i)$, $(\tilde{X}'_i, X'_i)$ for $i \in \{1, \ldots, 6\}$, the curves $\tilde{X}_i$ and $\tilde{X}'_i$ are the normalizations of the curves parametrizing respectively lines through p_i and twisted cubis through p_j (for all $j \neq i$) in Z. To avoid introducing more notation, we will refer to $\tilde{X}_i$ and $\tilde{X}'_i$ as the curves parametrizing lines through p_i and twisted cubics through p_j (for all $j \neq i$) in Z.

Fixing $(\tilde{X}, X)$ there is a one-dimensional family of double solids Z such that $(\tilde{X}, X) \cong (\tilde{X}_i, X_i)$ for some i. For the double solid Z the Prym-curves $(\tilde{X}_i, X_i)$ and $(\tilde{X}_j, X_j)$ are tetragonally related whenever $i \neq j$. For fixed i and j, denote by $(\tilde{X}_{ij}, X_{ij})$ the third Prym-curve in the tetragonal relation. Then X_{ij} is (the normalization of) a discriminant curve for one of the other 30 conic bundle structures of $\tilde{Z}$ and $\tilde{X}_{ij}$ (is the normalization of the curve which) parametrizes pairs of incident lines in Z, one of which is an element of $\tilde{X}_i$ and the other an element of $\tilde{X}_j$. So 15 curves occur in this way.

Similarly, $(\tilde{X}_i, X_i)$ and $(\tilde{X}'_j, X'_j)$ are tetragonally related whenever $i \neq j$. The third Prym-curves $(\tilde{X}'_{ij}, X'_{ij})$ in the tetragonal relations give the other 15 conic bundle structures on $\tilde{Z}$. The two g^1_4's on X_i relating $(\tilde{X}_i, X_i)$ to $(\tilde{X}_j, X_j)$ and $(\tilde{X}'_j, X'_j)$ are opposite, i.e., if we denote them by g and h, then $|g + h| = |K_{X_i}|$.

The curves X_i are also (normalizations of) discriminant curves for the projections of B from p_i. For each i, X_i maps onto a plane sextic with five double points which are the images of p_j for $j \neq i$. The plane sextic admits an everywhere tangent conic C_{ti}: this is the image of the projectivized tangent cone to B at p_i under r_i. Here r_i is the lift of the projection from p_i to the blow up of $\mathbb{P}^3$ at p_i.

Let $\varrho : \mathbb{P}^3 \to \mathbb{P}^3$ be the rational map of degree 2 defined by the linear system of quadrics in $\mathbb{P}^3$ containing the p_i's. The map ϱ is the composition

$$\mathbb{P}^3 \xrightarrow{\mathscr{O}(2)} \mathbb{P}^9 \longrightarrow \mathbb{P}^3$$

where the arrow on the right is the projection with center the linear subspace generated by the images of the p_i's. We denote by ι the rational involution which interchanges the sheets of ϱ. The projectivized tangent space $\mathbb{P}T_0A$ to $A = JZ$ at 0 can be canonically identified with the dual of the linear system of quadrics in $\mathbb{P}^3$ through the p_i's. Hence the target space of ϱ can be canonically identified with $\mathbb{P}T_0A$.

Also, for each set $\mathscr{F}$ of cardinality 4 contained in $\{p_1, \ldots, p_6\}$, we denote by $\bar{\iota}_{\mathscr{F}}$ the birational map $\mathbb{P}^3 \to \mathbb{P}^3$ defined by the linear system of cubics in $\mathbb{P}^3$ which are singular at the elements of $\mathscr{F}$. The rational maps $\bar{\iota}_{\mathscr{F}}$ are compositions

$$\mathbb{P}^3 \xrightarrow{\mathcal{O}(3)} \mathbb{P}^{19} \longrightarrow \mathbb{P}^3$$

where the arrow on the right is the projection with center the linear subspace generated by the projective tangent spaces to the image of $\mathbb{P}^3$ in $\mathbb{P}^{19}$ at the points of $\mathscr{F}$. Let, for instance, $\mathscr{F}$ be the set $\{p_1, \ldots, p_4\}$, then, under $\bar{\iota}_{\mathscr{F}}$, the plane through p_j for $j \neq i$ $(1 \leqq i, j \leqq 4)$ is blown down to a double point p'_i of $\bar{\iota}_{\mathscr{F}} B$. The points p_5, p_6 go to two nodes p'_5, p'_6 of $\bar{\iota}_{\mathscr{F}} B$. It follows from [DO], pp. 126–130, that

– one can identify the source $\mathbb{P}^3$ with the target $\mathbb{P}^3$ in a unique way, sending $p_1, \ldots, p_6$ to $p'_1, \ldots, p'_6$;

– if we define $\iota_{\mathscr{F}}$ to be the birational involution of $\mathbb{P}^3$ obtained from $\bar{\iota}_{\mathscr{F}}$ after this identification, then the subgroup $\mathscr{G}$ of the Cremona group of $\mathbb{P}^3$ generated by

$$\{\iota_{\mathscr{F}} : \mathscr{F} \text{ of cardinality 4 and contained in } \{p_1, \ldots, p_6\}\}$$

contains the birational involution ι and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^5$. We denote by ιZ, $\iota_{\mathscr{F}} Z$ the double solids with respective branch loci ιB, $\iota_{\mathscr{F}} B$. We obtain in this way a set of 32 double solids which breaks into the union of 16 subsets $\{Z_i, \iota Z_i\}$ with, for instance, $Z = Z_1$. These double solids are all birationally equivalent and have canonically isomorphic intermediate jacobians because, for instance, they have (at least) one common discriminant curve with the same double cover (up to canonical isomorphism).

2. Generalities and background

Here we write down some unpublished results of Beauville, Clemens, Debarre and Donagi which we will use later on.

2.1. The Γ_{00}-divisor D_Z associated to a quartic double solid Z. Before writing down the construction of Clemens [C2], we prove a lemma which we will use here and also later.

Let $(\tilde{X}, X) = (X, \eta)$ be an element of $P^{-1}(A)$ such that X is smooth. Recall that ω_X is the invertible sheaf of regular differentials on X. Let Ω_A be the rank 4 free sheaf of regular differentials on A. Let $S^4 = S^4H^0(A, \Omega_A) \cong S^4H^0(X, \omega_X \otimes \eta)$ be the space of symmetric quadrilinear forms on T_0A, the tangent space to A at 0. We have a linear map

$$\tau : \Gamma_{00} \to S^4$$

which to each element of Γ_{00} associates the quartic term of its Taylor expansion at 0.

Lemma 2.1.1. *For $A = JC$ the jacobian of a smooth curve C of genus 4 with a unique g_3^1 (denoted by ξ), the map τ is injective.*

Proof. For $\zeta \in \mathrm{Pic}^3 C$, let $\Theta_\zeta = W_3 - \zeta$. Let θ_ζ be a nonzero section of $\mathcal{O}_A(\Theta_\zeta)$ and let

$$\theta_\xi = q + g + \dots$$

be the Taylor expansion of θ_ξ at 0, with q and g irreducible polynomials of respective degrees 2 and 4 ([ACGH], p. 232). The quadric

$$q \in S^2 H^0(A, \Omega_A) = S^2 H^0(C, \omega_C) \subset \mathrm{Hom}(T_0 A, (T_0 A)^*)$$

has rank 3 ([ACGH], p. 232). Let $D_0 \in T_0 A$ be a generator of its kernel and put $f = D_0 g$. Then ([BD2], p. 33) the canonical model of C is the complete intersection of $\bar{q} = \{q = 0\}$ and $\bar{f} = \{f = 0\}$. Lift θ_ξ to a theta function $\bar{\theta}_\xi$ on the universal covering space of A, i.e., $T_0 A$, then we can lift θ_ζ to $\bar{\theta}_\zeta$ with $\bar{\theta}_\zeta(z) = \bar{\theta}_\xi(z - \bar{\zeta} + \bar{\xi})$ where $\bar{\zeta}$, $\bar{\xi}$ are fixed preimages of ζ, ξ in $T_0 A$. For a constant vector field D on A, the derivative (with respect to $\bar{\zeta}$) $D\bar{\theta}_\zeta$ descends to a section of $\mathcal{O}_A(\Theta_\zeta)$ denoted by $D\theta_\zeta$. The space $T_0 A$ can be identified with the space of constant vector fields on A so D_0 can be regarded as a constant vector field on A. The space Γ_{00} is generated by

$$\{(\theta_\xi)^2\} \cup \{DD_0(\theta_\xi)^2 = 2(DD_0\theta_\xi . \theta_\xi - D_0\theta_\xi . D\theta_\xi) : D \in T_0 A\}$$

(see [BD2], p. 33). The quartic terms of the Taylor expansions at 0 of $(\theta_\xi)^2$ and $DD_0(\theta_\xi)^2$ are respectively

$$q^2 \quad \text{and} \quad 2(q.Df - f.Dq).$$

If τ is not injective, then there exist $D \in T_0 A$ and $r \in \mathbb{C}$ with $(r, D) \neq (0, 0)$ such that

$$r.q^2 + q.Df - f.Dq = 0$$

i.e.,

$$q.(r.q + Df) = f.Dq.$$

As q and f are irreducible, this implies

$$r.q + Df = Dq = 0.$$

Hence D is a multiple of D_0. We can suppose $D = D_0$. We get

$$r.q + D_0 f = 0$$

and applying D_0 to this equation we obtain $(D_0)^2 f = 0$. This would imply that the image of D_0 in $\mathbb{P}T_0 A$ is in $\bar{q} \cap \bar{f} = C$. As C has degree 6 and the ruling of $\bar{q}$ cuts the divisors of ξ on C, this is impossible. Q.E.D.

Remark 2.1.2. This result can be proved in a similar way for a curve C with two distinct g_3^1's. It can also be proved for a hyperelliptic curve in a different way.

Let $\tilde{\pi}$ and π be the projections $\tilde{Z} \to \mathbb{P}^3$ and $Z \to \mathbb{P}^3$. Consider the component $\mathcal{D}_Z$ of the Hilbert scheme of conics (see 1.5) in $\tilde{Z}$ whose generic element is a smooth rational curve (i.e., one component of the inverse image in $\tilde{Z}$ of a conic in $\mathbb{P}^3$ which is everywhere tangent (with even multiplicity) to the branch locus B of Z). Clemens defines $D_Z = AJ(\mathcal{D}_Z)$, where $AJ: \mathcal{D}_Z \to JZ$ is the Abel-Jacobi map with base curve the inverse image in $\tilde{Z}$ of a line in $\mathbb{P}^3$ which is tangent to B (all such conics are rationally equivalent because they are inverse images of lines in $\mathbb{P}^3$ ([C1], pp. 208–209), hence it does not matter which line we choose in $\mathbb{P}^3$). For a given ppav A, let $\mathcal{X}_A$ be the family of double solids with intermediate jacobian isomorphic to A.

Theorem 2.1.3. *Let A be generic and let Z be a generic element of $\mathcal{X}_A$. Then the divisor D_Z is an element of $|2\Theta|_{00}$. The tangent cone to D_Z at 0 is $\varrho(B)$: the image of B by the map ϱ given by the linear system of quadrics through the nodes of B.*

Proof. Let C be a conic in $\mathbb{P}^3$, everywhere tangent (with even multiplicity) to B and generic for this property. Let V be the plane containing C. The data of C everywhere tangent to the canonical curve $S = B \cap V$ of genus 3 is equivalent to the data of a divisor on S with square linearly equivalent to $2K_S$ or equivalent to the data of a nonzero point γ of order 2 in JS and a divisor in the *pencil* $|K_S + \gamma|$. Thus there is at least a pencil of curves rationally equivalent to C in $\tilde{Z}$. We want to find the reducible curves, i.e., the unions of two lines (bitangent to S) in this pencil. By [DO], p. 165, odd theta-characteristics on S are in one-to-one correspondence with bitangent lines to S in V. A simple computation shows that there are 12 theta-characteristics k on S such that k and $k+\gamma$ are odd. Thus there are $6 = 12/2$ unordered pairs of bitangent lines to S in V whose unions cut divisors of $|K_S + \gamma|$ on S, hence there are (at least) 6 unordered pairs of incident lines in $\tilde{Z}$ whose sums are rationally equivalent to C.

Let E_Z be the Fano variety of lines in $\tilde{Z}$. We just saw that if $I \subset E_Z \times E_Z$ is the divisor of ordered pairs of incident lines, then $AJ(I) = D_Z$. It follows in particular that D_Z is of dimension $\leqq 3$. The following implies that D_Z cannot have dimension less than 3.

Using the identification $\mathbb{P}T_0A \cong |\mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{I}|^*$ where $\mathcal{I}$ is the ideal sheaf of the nodes of B (see 1.5), one has, by [W1] (pp. 20–28 (especially 2.14) and appendix to section 3: everything works with $\tilde{Z} = X$ in [W1], the dimensions of some spaces are different but the commutative diagrams that we need are valid), that the projectivized kernel of the codifferential of $AJ: E_Z \to A$ at an irreducible (i.e., such that its image in Z does not pass through a node of Z) line $l \in E_Z$ is the set of quadrics in $\mathbb{P}^3$ containing the nodes of B and the points of contact of $\tilde{\pi}(l)$ with B. From this it immediately follows that the projectivized kernel of the codifferential of $AJ: E_Z \times E_Z \to A$ at an ordered pair (l, l') of irreducible lines is the set of quadrics which contain the points of contact of $\tilde{\pi}(l)$ and $\tilde{\pi}(l')$ with B (and the nodes of B). It is easily seen that, generically, the 4 points of contact of two incident lines with B are not contained in any quadric through the nodes of B, so $AJ: E_Z \times E_Z \to A$ is generically of maximal rank on I. Hence the image of I is 3-dimensional and $AJ: \mathcal{D}_Z \to A$ has generic rank 3.

Suppose, momentarily, that Z is smooth.

Then $A = JZ$ is a ppav of dimension 10. By [W1], p. 77, if

$$\phi : E_Z \times E_Z \to A$$

is induced by the Abel-Jacobi map AJ and addition, then $\phi^*\Theta$ is homologous to I modulo $\mathrm{Pic}\, E_Z \oplus \mathrm{Pic}\, E_Z$. By [W1], p. 70, the restrictions of $\phi^*\Theta$ to two fibers $\{l\} \times E_Z$ and $E_Z \times \{l\}$ have homology class $3D_l$, where $D_l = I.(\{l\} \times E_Z)$ or $(E_Z \times \{l\}).I$ respectively. The D_l all have the same homology class D in E_Z. Hence $\phi^*\Theta$ is homologous to

$$I + 2(D \times E_Z) + 2(E_Z \times D).$$

By [C1], the homology class of $AJ(E_Z)$ is $2[\Theta]^8/8!$. A straightforward computation, using Pontrjagin product and

$$(1) \qquad [\Theta]^{d-j}/(d-j)! = \sum \gamma_{i_1} \times \delta_{i_1} \times \ldots \times \gamma_{i_j} \times \delta_{i_j}$$

(where j is any number between 1 and $d = \dim A$ which is 10 in this case, γ_i, δ_i form a symplectic basis of $H^1(A, Z)$, $\{i_1, \ldots, i_j\}$ is any set of distinct integers between 1 and d and "$\times$" is the Pontrjagin product) shows then that the homology class of ϕ^*I in A is $24.[\Theta]^7/7!$. The map $I \to AJ(I)$ is of degree at least 12. So the reduced image of I has homology class $n.[\Theta]^7/7!$ where $n = 1$ or 2. Analogously to the computation of the homology class of E_Z in [C1], pp. 213–221, one degenerates to the case where Z has one node and sees that the homology class of D_Z is $n.[\Theta]^6/6!$. Degenerating one node at a time to the case where Z has 6 nodes in general position, one sees that D_Z has homology class $n.[\Theta]$ where $n = 1$ or 2.

Lemma 2.1.4. *The tangent cone to D_Z at 0 contains $\varrho(B)$.*

Assume the lemma for a moment. Then D_Z has a singularity of order $\geqq 4$ at the origin. As A is generic and Θ nonsingular, n has to be 2. Since, by 2.1.1, $|2\Theta|_{00}$ has no member with a singularity of order > 4 at 0, the singularity of D_Z is of order 4. This means $D_Z \in |2\Theta|_{00}$ and $\varrho(B)$ is equal to the tangent cone at 0 to D_Z.

Proof of the lemma. Let C and V be as before (proof of 2.1.3). Let C' and C'' be the two components of the inverse image of C in $\tilde{V} = \tilde{\pi}^{-1}(V)$. Let $|C'|$ be the complete linear system containing C' in $\tilde{V}$. Using the identification $\mathbb{P}T_0 A \cong |\mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{I}|^*$ (proof of theorem 2.1.3), the kernel of the codifferential of the Abel-Jacobi map

$$AJ : \mathcal{D}_Z \to D_Z$$

at C' contains all the quadrics through the nodes of B which also contain the points of intersection of $\tilde{\pi}_* N$ and B for some $N \in |C'|$. This can be seen by mimicking the computations in [W1], pp. 20–28, where the smooth curve in the double solid would be N (since containment is a closed condition, the result remains valid when N is not smooth). We saw above that, generically, this kernel is 1-dimensional. So there is a unique quadric Q through the nodes of B such that there is an element N of $|C'|$ with $N.\tilde{B} \subset \tilde{\pi}^* Q$ ($\tilde{B}$ is the strict transform of B in $\tilde{Z}$). Let B' be an element of the pencil spanned by $\tilde{B}$ and $\tilde{\pi}^* Q$ which contains N. Let l be a line in $\mathbb{P}^3$, tangent to B at a point t. The curve $\tilde{\pi}^* l$ has an embedded point at t. Going to the limit, one sees that when $N = \tilde{\pi}^* l$, B' has a singular point at t. Hence, B and Q are tangent at t. Hence, $\varrho(B)$ and the limits (at 0) of tangent planes to D_Z are tangent. Hence, $\varrho(B)$ is contained in the projectivized tangent cone at 0 to D_Z.

2.2. How to recover B from the plane representation of X_i. A result of Donagi that we will need is ([Do2])

Theorem 2.2.1. *Let B be a generic quartic in $\mathbb{P}^3$ with 6 nodes in general position (see 1.5). Let $X \subset \mathbb{P}^2$ be the discriminant curve of the projection of B from one of its nodes, say p_1. Then X has a unique everywhere tangent conic, say C_t, which is irreducible. Let $\bar{B}$ be the double cover of $\mathbb{P}^2$ branched along X. Let $\tilde{B}$ be the blow up of $\bar{B}$ at its nodes, let $\tilde{p}$ be the composition $\tilde{B} \to \bar{B} \to \mathbb{P}^2$ and let h be the hyperplane class in $\mathbb{P}^2$. Let C_1 and C_2 be the two components of the inverse image of C_t in $\tilde{B}$ and put $H_i = |\tilde{p}^*h + C_i|$. Then H_1 and H_2 are the only linear systems which map $\tilde{B}$ onto a quartic with discriminant curve projectively equivalent to X. If B_i is the image of $\tilde{B}$ in $\mathbb{P}^3$ by the map associated to H_i, then B_1 and B_2 are projectively equivalent and any quartic surface in $\mathbb{P}^3$ with discriminant curve projectively equivalent to X is projectively equivalent to B_1 and B_2.*

Proof. By an everywhere tangent conic to X we mean a reduced conic which has contact of even order at all of its points of intersection with X and which does not contain any of the double points of X. Let C_t be the everywhere tangent conic to X which is the image of the tangent cone to B at p_1.

We first prove the uniqueness of the everywhere tangent conic. Let $\tilde{B}$ be the blow up of B at its nodes. Generically, the Picard number of $\tilde{B}$ is 7 because, in the moduli space of K3 surfaces, the subvariety of those with Picard number > 7 has dimension less than 13 [S]. Let H be the linear system associated to the morphism $\tilde{B} \to \mathbb{P}^3$. Let p_i $(i = 1, \dots, 6)$ be the nodes of B. The exceptional curves E_i above p_i and H generate a sublattice of finite index of the Picard group of $\tilde{B}$. These verify (for all i and $j \neq i$)

$$H^2 = 4, \quad (E_i)^2 = -2, \quad H.E_i = E_i.E_j = 0.$$

Suppose that there exists another everywhere tangent conic to X, say C. Let $\tilde{C}$ be the inverse image of C in $\tilde{B}$. We claim that the double cover $\tilde{C} \to C$ is split. This is clear if C is irreducible. If C is the union of two lines l_1 and l_2, then both lines are also everywhere tangent to X: if, for instance, l_1 has odd order of contact with X at some point t, then, since the total intersection multiplicity of l_1 with X is 6, l_1 has to have odd order of contact with X at another point $t' \neq t$. Since $C = l_1 \cup l_2$ has even order of contact with X everywhere, both t and t' must be on l_2. However, $l_1 \neq l_2$ because C is reduced.

Put $\tilde{C} = C' \cup C''$ with $C' \cong C'' \cong C$. Let $\tilde{p} : \tilde{B} \to \mathbb{P}^2$ be the composition of $\tilde{B} \to \mathbb{P}^3$ with the projection from p_1. The following numerical computations yield a contradiction:

There exists an integer m such that in the Picard group of $\tilde{B}$

$$mC' \equiv aH + \sum b_j E_j \quad \text{for some } a, b_j \in \mathbb{Z}.$$

i) $C'.(H - E_1) = 2 = C''.(H - E_1)$. This is because $H - E_1$ is the linear system associated to $\tilde{p}$ and $C'.(H - E_1)$ (resp. $C''.(H - E_1)$) is the degree of $\tilde{p}_* C' = \tilde{p}_* C'' = C$ in $\mathbb{P}^2$.

ii) $C'.E_i = 0$ for all $i > 1$. Hence $mC' \equiv aH + bE_1$ (with $b = b_1$).

iii) As C is distinct from C_t, C' (resp. C'') does not contain E_1 and we have $C'.E_1 \geqq 0$ (resp. $C''.E_1 \geqq 0$).

Supposing $C'.E_1 \leqq C''.E_1$, as $C'.E_1 + C''.E_1 = 4$, we have three possible cases:

1) $C'.E_1 = 0$. In this case $C'.H = 2$ and $2C' \equiv H$,

2) $C'.E_1 = 1$, then $C'.H = 3$ and $4C' \equiv 3H - 2E_1$,

3) $C'.E_1 = 2$, then $C'.H = 4$ and $C' \equiv H - E_1$.

As C' is a curve of arithmetic genus 0 on the K3 surface $\tilde{B}$, the self-intersection of C' is -2. All three cases are ruled out by computing the self-intersections of the right hand sides of the linear equivalences.

Now we prove the second half of the theorem. We adopt the notation of the statement of the theorem.

Generically, $\bar{B}$ has exactly 5 nodes as singularities. Let B' be a quartic such that X is (projectively equivalent to) the discriminant curve of the projection of B' from one of its nodes, say q_1. Then the blow up of B' at q_1 is isomorphic to $\bar{B}$ and the minimal desingularization of B' is isomorphic to $\tilde{B}$. Let E' be the exceptional divisor of $\tilde{B}$ above q_1. Then, by the uniqueness of the everywhere tangent conic, $\tilde{p}(E') = C_t$ and so E' is equal to either C_1 or C_2. Then B' is the image of $\tilde{B}$ in $\mathbb{P}^3$ by a morphism associated to a linear subsystem of H_1 or H_2. We have

i) the degree of H_i is 4:

$$(\tilde{p}^*h + C_i)^2 = 2h^2 + C_i^2 + 2h.C_t = 6 + C_i^2 \quad \text{and} \quad C_i^2 = -2;$$

ii) the dimension of H_i is 3. Indeed, by Riemann-Roch, $h^0(\tilde{B}, \mathcal{O}_{\tilde{B}}(H_i)) \geqq 4$, so H_i has no fixed components and by general results on K3 surfaces (see e.g. [S]) $h^0(\tilde{B}, \mathcal{O}_{\tilde{B}}(H_i)) = 4$.

The morphism $\tilde{B} \to \mathbb{P}^3$ associated with H_i factors through $\bar{B}$, contracts C_i, and its image is a quartic surface B_i with exactly 6 nodes as singularities. The quartic B' is projectively equivalent to either B_1 or B_2.

The involution of the cover $\bar{B} \to \mathbb{P}^2$ exchanges C_1 and C_2. It lifts to an involution i of $\tilde{B}$ which exchanges H_1 and H_2 and induces a projective isomorphism between B_1 and B_2.

Finally, let H be a linear system which maps $\tilde{B}$ onto a quartic with discriminant curve (projectively equivalent to) X. Then (by the above) H must be the inverse image of H_1 by an automorphism s of $\tilde{B}$ which induces the identity on X because X has no nontrivial automorphisms (the normalization of X is a generic curve of genus 5). Therefore s preserves the fibers of $\tilde{p}$. It follows that $s = i$ or $s = \mathrm{id}$. Q.E.D.

2.3. The Fano surface E_Z is an intersection of translates of Θ. This is due to Beauville [C2].

Let Z be a double solid with $A = JZ$. Put $(\tilde{X}, X) = (\tilde{X}_1, X_1)$ (see 1.5 for the definition of $(\tilde{X}_1, X_1)$). From now on we identify E_Z with its image by the Abel-Jacobi map.

Proposition 2.3.1. *The surface E_Z is an intersection of translates of Θ.*

Proof. Let $g_6^2 = |K_X - p - q|$ be the linear system on X associated to the plane representation of X given by Z. Consider a line l in $\tilde{Z}$. The line l picks a lifting D_l of the divisor $r_1(\tilde{\pi}(l)) \cap X$ in g_6^2: the divisor D_l is the sum of the points of $\tilde{X}$ corresponding to lines incident to l (see 1.5 for the definition of r_1).

Let p', p'', q', q'' be the liftings of p, q in $\tilde{X}$. By [B3], p. 359, and the parametrizations of A and A^- in 1.2, there is a choice of these liftings, say p', q', such that $h^0(\tilde{X}, D_l + p' + q')$ is even as well as $h^0(\tilde{X}, D_l + p'' + q'')$. For an element x of A, let t_x be translation by x in A, i.e., $t_x(y) = x + y$ for all $y \in A$. Let $\Theta_x = (t_{-x})^* \Theta$ be the translate of Θ by x.

So in the parametrization of A associated to $(\tilde{X}, X)$ (1.2), we obtain an embedding

$$E_Z \subset \Theta . \Theta_{p'+q'-p''-q''} = \Theta . \Theta_{[p',q']},$$
$$l \mapsto D_l + p' + q'.$$

By [C1], p. 220, the homology class of E_Z is $[\Theta]^2$, so we have equality.

2.4. Embeddings of Prym-curves in intersections of two translates of Θ. We write down the result of [De].

Consider a tetragonally related triple of smooth, non-hyperelliptic and non-trigonal Prym-curves $((\tilde{X}, X), (\tilde{Y}, Y), (\tilde{U}, U))$ in $P^{-1}(A)$. Let $g_Y \in W_4^1(Y)$ relate $(\tilde{Y}, Y)$ to $(\tilde{X}, X)$ and $(\tilde{U}, U)$. If Y is bielliptic, suppose that g_Y is *not* the pullback of a g_2^1 on an elliptic curve. Then, for each p, q and r in $\tilde{Y}$ such that πp and πq are not images of each other by a bielliptic involution of Y and such that there exists $x \in \tilde{Y}$ with $\pi p + \pi q + \pi r + \pi x \equiv g_Y$, we have

Proposition 2.4.1. *The elements $[p, q]$ and $[p, r]$ are also in $\Sigma(X)$ and $\Sigma(U)$. The intersection*

$$\Theta . \Theta_{[p,q]} . \Theta_{[p,r]}$$

is the union of three Prym-embedded curves $\tilde{X}'$, $\tilde{Y}'$, $\tilde{U}'$.

Proof. Write

$$[p, q] = p + q + r + x - \sigma p - \sigma q - r - x = p + q + r + \sigma x - \sigma p - \sigma q - r - \sigma x$$

to see that $[p, q]$ (and $[p, r]$) are also elements of $\Sigma(X)$ and $\Sigma(U)$ (these are differences of liftings of divisors of g_Y in $\tilde{Y}$). By [BD1], p. 615, we have:

$$\Theta . \Theta_{[p,q]} . \Theta_{[p,r]} = S_{pqr} \cup W_p(Y),$$

where

$$W_p(Y) = \{D \in \operatorname{Pic}^8 \tilde{Y} : h^0(D(-p)) \geqq 2\},$$
$$S_{pqr} = \{D \in \operatorname{Pic}^8 \tilde{Y} : h^0(D(-p-q-r)) > 0\}.$$

We have

$$|K_Y - \pi p - \pi q - \pi r| = \pi x + |K_Y - g_Y|$$

as linear systems and, by the definition of the tetragonal construction in 1.4, S_{pqr} splits as the union of $p+q+r+x+\tilde{X}'$ and $p+q+r+sx+\tilde{U}'$ where $(\tilde{X}', X')$, $(\tilde{U}', U')$ are the Prym-curves tetragonally related to $(\tilde{Y}, Y)$ via $|K_Y - g_Y|$. The situation being symmetric, one can repeat this argument with $(\tilde{X}, X)$ or $(\tilde{U}, U)$ instead of $(\tilde{Y}, Y)$ and obtain that $\tilde{Y}' = W_p(Y)$ is also a Prym-embedded curve.

2.5. Prym-embeddings of $\tilde{X}_{ij}$ in E_Z. Let Z be as in 1.5. Recall (2.3) that we identify E_Z with its image by the Abel-Jacobi map. The following is due to Clemens.

Proposition 2.5.1. *With the notation of* 1.5, *for each* i, j, *there are* (*two*) *Prym-embeddings of* $\tilde{X}_{ij}$ *in* E_Z.

Proof. As we noted in 1.5, for all $i \neq j$, $((\tilde{X}_i, X_i), (\tilde{X}_j, X_j), (\tilde{X}_{ij}, X_{ij}))$ is a tetragonally related triple. Take $i = 1$, $j = 2$. Let g_6^2 be associated to the plane representation of X_1 obtained from Z. Let l_t be a line in Z through p_2, corresponding to a point $t \in \tilde{X}_2$. Let p_2' and p_2'' be the two points of X_1 above p_2. Then l_t projects to a line through the image of p_2 in $\mathbb{P}^2$ which cuts a divisor E of $|g_6^2 - p_2' - p_2''|$ on X_1. So we see that l_t picks, via incidence, a lifting, say $s_1 + s_2 + s_3 + s$, of E. The pair of incident lines (l_s, l_t) through p_1 and p_2 corresponding to $(s, t) \in \tilde{X}_{12} \subset \tilde{X}_1 \times \tilde{X}_2$ picks, via incidence, the divisor $s_1 + s_2 + s_3 + \sigma s$. Hence the image of $\tilde{X}_{12} \subset \mathscr{D}_Z$ by AJ is a Prym-embedding because it is a translate of one connected component of the set of liftings of a g_4^1 on X_1.

Let V be the plane spanned by $\tilde{\pi}(l_s)$ and $\tilde{\pi}(l_t)$. Then $\tilde{V} = \tilde{\pi}^{-1}(V)$ is a Del Pezzo surface isomorphic to $\mathbb{P}^2$ blown up at 7 points q_i ($0 \leqq i \leqq 6$) such that q_0, q_1, q_2 and q_0, q_3, q_4 are colinear. The plane V is the image of $\tilde{V}$ by the map defined by the linear system of strict transforms of cubics in $\mathbb{P}^2$ passing through the q_i's. The lines $\langle q_0, q_1 \rangle$ (generated by q_0 and q_1) and $\langle q_0, q_3 \rangle$ are blown down to p_1 and p_2. By [Dm], pp. 23–24, a basis of the Picard group of $\tilde{V}$ is given by

$$H, -E_0, \ldots, -E_6$$

where H is the pullback of the hyperplane class in $\mathbb{P}^2$ and E_i is the exceptional divisor above q_i. When the q_i's are in general position (with respect to lines and conics) the lines in $\tilde{V}$ are given by:

– the exceptional curves E_i (7 curves),

and the strict transforms of the following curves in $\mathbb{P}^2$:

– lines through 2 of the q_i's (21 curves),

– conics through 5 of the q_i's (21 curves),

– cubics through all the q_i's and with a double point at one of them (7 curves).

In our degenerate case, some of the above curves coincide.

The case of the other pairs of incident lines being analogous, let us suppose, for instance, that $l_s = L_{35}$ (L_{ij} is the strict transform of the line through q_i and q_j), and $l_t = E_3$, we have

$$l_s + l_t \equiv H - E_5 \equiv L_{56} + E_6 \equiv E_0 + L_{05} .$$

Notice that E_0 and L_{56} are the two lines in $\tilde{V}$ which project to the line $\langle p_1, p_2 \rangle$ in $\mathbb{P}^3$. So

$$l_{st} = E_6 \equiv l_s + l_t - L_{56}$$

and

$$l'_{st} = L_{05} \equiv l_s + l_t - E_0$$

are lines in $\tilde{Z}$. Thus we obtain two embeddings of $\tilde{X}_{12}$ in E_Z

$$(l_s, l_t) \mapsto l_{st} \quad \text{and} \quad (l_s, l_t) \mapsto l'_{st} .$$

These are Prym-embeddings of $\tilde{X}_{12}$ because they are translates of the image of $\tilde{X}_{12} \subset \mathscr{D}_Z$ by AJ which is a Prym-embedding.

3. The fibers of P over $\mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$

In this section we suppose A to be an element of $\mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. Let $(\tilde{X}, X)$ be an element of $P^{-1}(A)$. As products of ppav's of lower dimension are in $\bar{\mathscr{J}}_4$, we have, by [B1], Theorem 5.4,

(i) $\pi : \tilde{X} \to X$ is ramified exactly at the possible singular points of X,

(ii) at the singular points of $\tilde{X}$ the two branches are not exchanged under the covering involution σ,

(iii) the inverse image in $\tilde{X}$ of any irreducible component of X is irreducible.

Let g be a (base point free and) nonsingular g^1_4 on X, i.e., the generic divisors of g are supported on the smooth locus of X. Let ϕ be the morphism $X \to \mathbb{P}^1$ associated to g. Suppose that ϕ is finite (we will see in the discussion preceding 3.11 that X always has such a g^1_4). The following technical lemma extends the tetragonal construction to singular Prym-curves (its proof can be skipped on a first reading).

Lemma 3.1. *Prym-curves tetragonally related to $(\tilde{X}, X)$ through g are well-defined and their Prym varieties are isomorphic to A.*

Proof. Let $\tilde{L}$ be the set of liftings of the divisors of g in $\mathrm{Div}^4 \tilde{X}$. Fixing one lifting $\tilde{G}_0$ of some divisor G_0 of g, we can embed $\tilde{L}$ in $A \cup A^-$ by $\psi_0 : \tilde{G} \mapsto \mathcal{O}_{\tilde{X}}(\tilde{G} - \tilde{G}_0)$. So $\tilde{L}$ has two connected components $\tilde{Y}$ and $\tilde{U}$ with, for instance, $\psi_0(\tilde{Y}) \subset A$, $\psi_0(\tilde{U}) \subset A^-$. The curves $\tilde{Y}$ and $\tilde{U}$ each come with an involution, say σ_Y and σ_U, which interchange complementary liftings of the same divisor of g. Put $Y = \tilde{Y}/\sigma_Y$, $U = \tilde{U}/\sigma_U$. The curves Y and U come with finite morphisms of degree four $\phi_Y : Y \to \mathbb{P}^1$ and $\phi_U : U \to \mathbb{P}^1$ (induced by

ϕ) with associated g_4^1's g_Y and g_U. We will study $\pi_Y : \tilde{Y} \to Y$, g_Y and ϕ_Y, the study of $\pi_U : \tilde{U} \to U$, g_U and ϕ_U being analogous.

Let $\{t_i : 1 \leqq i \leqq n\}$ and $\{\tilde{t}_i : 1 \leqq i \leqq n\}$ be the singular points of X and $\tilde{X}$ respectively. Let K_i, $\tilde{K}_i$ be the groups of germs of rational functions at t_i and $\tilde{t}_i$, and let $\mathcal{O}_i^*$, $\tilde{\mathcal{O}}_i^*$ be the stalks of $\mathcal{O}_X^*$ and $\mathcal{O}_{\tilde{X}}^*$ at t_i and $\tilde{t}_i$. By [B1], p. 158, we have splittings of the groups of Cartier divisors with respective supports t_i and $\tilde{t}_i$:

$$K_i^*/\mathcal{O}_i^* \cong \mathbb{C}^* \times \mathbb{Z} \times \mathbb{Z}\,, \quad \tilde{K}_i^*/\tilde{\mathcal{O}}_i^* \cong \mathbb{C}^* \times \mathbb{Z} \times \mathbb{Z}$$

and the action of σ is given by $\sigma^*(z, m, n)_i = \big((-1)^{m+n} z, m, n\big)_i$ and hence

$$\pi_*(z, m, n)_i = \big((-1)^{m+n} z^2, m, n\big)_i\,.$$

For any point $s \in X$, we denote by $\tilde{s}$ a lifting of s in $\tilde{X}$.

Let G be a divisor of g and let $\tilde{G}$ be a lifting of G in $\tilde{X}$. We denote by a, b, c, d points on X near the points of G such that $D = a + b + c + d$ is a divisor of g. We choose $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ such that $\tilde{D}_1 = \tilde{a} + \tilde{b} + \tilde{c} + \tilde{d}$ approaches $\tilde{G}$ when D approaches G. We determine the local behaviour of π_Y and ϕ_Y by determining their monodromies around their ramification points. The data of a path in $\tilde{Y}$ from $\tilde{D}_1$ to $\tilde{E}_1 = \tilde{a}' + \tilde{b}' + \tilde{c}' + \tilde{d}'$ with $\pi_* \tilde{D}_1 = \pi_* \tilde{E}_1 = D$ which lifts a loop γ in $\mathbb{P}^1$ (with base point the common image of the points of $\tilde{D}_1, \tilde{E}_1$) around $p = \phi(G) = \phi_Y(\tilde{G})$ is equivalent to the data of liftings $\gamma_1, \ldots, \gamma_4$ of γ in $\tilde{X}$ going respectively from $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ to $\tilde{a}', \tilde{b}', \tilde{c}', \tilde{d}'$ (after a permutation of $\tilde{a}', \tilde{b}', \tilde{c}', \tilde{d}'$) whose images in X do not intersect (except, possibly, at the end points) and none of which passes through a point of $\tilde{G}$. *One can replace any lift γ_i of γ by $\sigma\gamma_i$ in order to connect different pairs of divisors.*

If the support of G is contained in the smooth locus of X, then, by [W1], pp. 103–108, $\tilde{Y}$ is singular at $\tilde{G}$ if and only if G and $\tilde{G}$ have the forms $G = 2s_1 + 2s_2$ and $\tilde{G} = \tilde{s}_1 + \sigma_Y \tilde{s}_1 + \tilde{s}_2 + \sigma_Y \tilde{s}_2$ for some possibly equal points s_1 and s_2 of X.

1a) First suppose $s_1 \neq s_2$. Then $\tilde{G}$ is the common limit of

$$\tilde{D}_1 = \tilde{a} + \tilde{b} + \tilde{c} + \tilde{d}\,, \quad \tilde{D}_2 = \sigma\tilde{a} + \sigma\tilde{b} + \tilde{c} + \tilde{d}\,,$$
$$\tilde{D}_3 = \tilde{a} + \tilde{b} + \sigma\tilde{c} + \sigma\tilde{d}\,, \quad \tilde{D}_4 = \sigma\tilde{a} + \sigma\tilde{b} + \sigma\tilde{c} + \sigma\tilde{d}\,.$$

Locally, near p, the monodromy group acts on a nearby fiber $\{a, b, c, d\}$ by the two transpositions (a, b) and (c, d). We see that we can lift γ to paths from $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ to, respectively, $\sigma\tilde{b}, \sigma\tilde{a}, \sigma\tilde{d}, \sigma\tilde{c}$. Hence the above divisors form two irreducible components of $\tilde{Y}$, locally near $\tilde{G}$, and the quotient by σ_Y gives two irreducible components for Y, locally near $t = \pi_Y \tilde{G}$, which meet at the branch point t. So π_Y is allowable near t. The divisor of g_Y containing t contains a divisor of bidegree $(1, 1)$ supported at t.

1b) Now suppose $s_1 = s_2 = s$. Then $\tilde{G}$ is the common limit of

$$\tilde{D}_1,\ \tilde{D}_2,\ \tilde{D}_3,\ \tilde{D}_4,\ \tilde{D}_5 = \sigma\tilde{a} + \tilde{b} + \tilde{c} + \sigma\tilde{d}\,, \quad \tilde{D}_6 = \tilde{a} + \sigma\tilde{b} + \sigma\tilde{c} + \tilde{d}\,.$$

Locally near s, the mondromy group acts on a, b, c, d by, say, the cyclic permutation (a, b, c, d). We can lift γ to paths from $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ to, respectively, $\sigma\tilde{b}, \sigma\tilde{c}, \sigma\tilde{d}, \sigma\tilde{a}$. Again we obtain locally two irreducible components for $\tilde{Y}$ mapping to two irreducible components of Y. The divisor of g_Y containing $t = \pi_Y \tilde{G} \in Y$ contains a divisor of bidegree $(1, 2)$ or $(2, 1)$ supported at t.

Now we need to analyze the cases where the support of G contains some singular points of X. As ϕ does not contract any irreducible component of X to a point, it is easily seen that both components of the bidegree of G at a singular point are positive, so at most two singular points of X can occur in G, say t_1, t_2 or only t_1. By symmetry, we can restrict our study to the following cases:

2a) $G = (u, 1, 1)_1 + s_1 + s_2, s_1 \neq s_2$.

Any lifting of G is of the form $\tilde{G} = (z, 1, 1)_1 + \tilde{s}_1 + \tilde{s}_2$ (where z is a square root of u): this is not a fixed point of σ_Y. We see that $\tilde{G}$ is the limit of $\tilde{D}_1$ and $\tilde{D}_2$ (see 1a) for the notation). We can lift γ to paths from $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ to, respectively, $\sigma\tilde{a}, \sigma\tilde{b}, \tilde{c}, \tilde{d}$. Hence, locally around $\tilde{G}, \tilde{Y}$ has only one irreducible component and so does Y: $t = \pi_Y(\tilde{G})$ and $\pi_Y((-z, 1, 1)_1 + \sigma\tilde{s}_1 + \tilde{s}_2)$ are two ramification points of ϕ_Y in its fiber at p.

2b) $G = (u, 1, 1)_1 + 2s$.

If $\tilde{G} = (-z, 1, 1)_1 + 2\tilde{s}$ or $\tilde{G} = (-z, 1, 1)_1 + 2\sigma\tilde{s}$ the argument is as in 2a) except that we lift γ to paths from $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ to $\sigma\tilde{a}, \sigma\tilde{b}, \tilde{d}, \tilde{c}$. Let $\tilde{G} = (z, 1, 1)_1 + \tilde{s} + \sigma\tilde{s}$. Then $\tilde{G}$ is the limit of $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3, \tilde{D}_4$. We lift γ to paths from $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ to $\sigma\tilde{a}, \sigma\tilde{b}, \sigma\tilde{d}, \sigma\tilde{c}$. Locally around $\tilde{G}$, π_Y is as in 1a). The fiber of ϕ_Y at p contains a double point of Y and a (smooth) ramification point of ϕ_Y.

2c) $G = (u, 1, 1)_1 + (u', 1, 1)_2$.

For instance let $\tilde{G} = (z, 1, 1)_1 + (z', 1, 1)_2$. Then $\tilde{G}$ is the limit of $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3, \tilde{D}_4$. We lift γ to paths from $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ to $\sigma\tilde{a}, \sigma\tilde{b}, \sigma\tilde{c}, \sigma\tilde{d}$. Again, locally around $\tilde{G}$, π_Y is as in 1a). The fiber of ϕ_Y at p contains two nodes of Y.

3) $G = (-u, 2, 1)_1 + s$.

Let $\tilde{G} = (z, 2, 1)_1 + \tilde{s}$. Then $\tilde{G}$ is the limit of $\tilde{D}_1, \tilde{D}_2, \tilde{D}_6, \tilde{D}_7 = \sigma\tilde{a} + \tilde{b} + \sigma\tilde{c} + \tilde{d}$. We lift γ to paths from $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ to $\tilde{c}, \sigma\tilde{b}, \sigma\tilde{a}, \tilde{d}$. We see that $\tilde{Y}$ (and hence Y) has only one irreducible component locally around $\tilde{G}$: t is a ramification point of index 3 of ϕ_Y.

4a) $G = (u, 2, 2)_1$.

Here $\tilde{G} = (z, 2, 2)_1$ is the limit of $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3, \tilde{D}_4, \tilde{D}_5, \tilde{D}_6, \tilde{D}_7, \tilde{D}_8 = \tilde{a} + \sigma\tilde{b} + \tilde{c} + \sigma\tilde{d}$. Lift γ to paths from $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ to $\sigma\tilde{c}, \sigma\tilde{d}, \sigma\tilde{a}, \sigma\tilde{b}$, then to paths from $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ to $\tilde{c}, \sigma\tilde{d}, \tilde{a}, \sigma\tilde{b}$ to see that, locally near $\tilde{G}$, $\tilde{Y}$ has two branches which are not exchanged under σ_Y and map to two branches of Y; g_Y has a divisor of the same type as G.

4b) $G = (u, 3, 1)_1$.

As above $\tilde{G} = (z, 3, 1)_1$ is the limit of $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3, \tilde{D}_4, \tilde{D}_5, \tilde{D}_6, \tilde{D}_7, \tilde{D}_8$. Lift γ to paths from $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ to $\sigma\tilde{b}, \sigma\tilde{c}, \sigma\tilde{a}, \sigma\tilde{d}$, then to paths from $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ to $\tilde{b}, \sigma\tilde{c}, \tilde{a}, \sigma\tilde{d}$ to see that, locally near $\tilde{G}$, $\tilde{Y}$ has two branches which are not exchanged under σ_Y and map to two branches of Y and g_Y has a divisor of the same type as G.

An easy deformation argument now gives that $P(\tilde{Y}, Y) \cong P(\tilde{X}, X)$. Q.E.D.

If we fix a generic divisor E of g_Y (resp. g_U), then a generic hyperplane in $|\omega_Y|^*$ (resp. $|\omega_U|^*$) containing the plane spanned by E does not intersect the singular locus of κY (resp. κU), hence $h_Y = K_Y - g_Y$ and $h_U = K_U - g_U$ are nonsingular.

Lemma 3.2. *One of the two morphisms ψ_Y, ψ_U associated respectively to h_Y, h_U is finite.*

Proof. Suppose, for instance, that Y has a component Y_0 such that $\psi_Y(Y_0)$ is a point. If we let $Y = Y_0 \cup Y_1$, then it is easy to see that the respective genuses of Y_0 and Y_1 are 0 and 2, that $\#(Y_0 \cap Y_1) = 4$ and $\deg(g_{Y_0}) = \deg(g_{Y_1}) = 2$ where $g_{Y_0} = g_Y|_{Y_0}$ and $g_{Y_1} = g_Y|_{Y_1}$. It then follows easily that $\tilde{X}$ and $\tilde{U}$ embed in $\tilde{Y}_0^{(2)} \times \tilde{Y}_1^{(2)}$ (their images do intersect). By the projection to $\tilde{Y}_0^{(2)}$, $\tilde{X}$ and $\tilde{U}$ both map 2-to-1 onto the curve $\bar{X}$ which parametrizes liftings of divisors of g_{Y_0}. The curve $\bar{X}$ comes with the involution which exchanges complementary liftings of the same divisor of g_{Y_0}: we let E_0 be the quotient of $\bar{X}$ by this involution. Reasoning as in the proof of 3.1, one sees that E_0 is a smooth elliptic curve isomorphic to $\tilde{Y}_0$ and that we have commutative diagrams

$$\begin{array}{ccc} \tilde{X} & \to & \bar{X} \\ \downarrow & & \downarrow \\ X & \to & E_0, \end{array} \qquad \begin{array}{ccc} \tilde{U} & \to & \bar{X} \\ \downarrow & & \downarrow \\ U & \to & E_0. \end{array}$$

We will say that Y (resp. $(\tilde{Y}, Y)$) is of type $(0, 2)$ if Y is the union of a curve of genus 0 and a curve of genus 2 meeting in 4 points. A curve of type $(0, 2)$ cannot be a double cover of a smooth elliptic curve. Hence $(\tilde{U}, U)$ is not of type $(0, 2)$ and ψ_U is finite. Q.E.D.

A Prym-curve $(\tilde{X}, X)$ is said to be of type $(1, 1)$ if X is the union of two curves of genus 1 meeting at 4 points, we call $\mathcal{P}_{1,1}$ the space of Prym-curves of type $(1, 1)$, and put $\mathcal{A}_{1,1} = P(\mathcal{P}_{1,1})$. We can now prove our first result on the structure of the fibers of P.

Theorem 3.3. *Suppose that A is in $\mathcal{A}_4 \backslash \mathcal{J}_4$. The family of singular Prym-curves in $P^{-1}(A)$ which are not of type $(1, 1)$ has dimension 1. There is a dense open subset of $P^{-1}(A)$ whose elements are either smooth or of type $(1, 1)$. If $A \in \mathcal{A}_{1,1}$, then the family $F_{1,1}$ of Prym-curves of type $(1, 1)$ in $P^{-1}(A)$ has pure dimension 2. There is a dense open subset $U_{1,1}$ of $F_{1,1}$ such that, for $(\tilde{X}, X) \in U_{1,1}$, $(\tilde{X}, X)$ is a union of two smooth elliptic curves and there is a dense open subset W of $W_4^1(X)$ such that the Prym-curves tetragonally related to $(\tilde{X}, X)$ through elements of W are smooth.*

Proof. We would like to reduce the proof of the theorem to a computation on cotangent spaces to our various moduli spaces. However, since our moduli spaces are coarse, their tangent spaces are not spaces of infinitesimal deformations and are difficult to compute. To go around this we pass to the spaces of formal versal deformations.

3.4. Let **R**, **A** and **M** be the spaces of formal versal deformations of $(\tilde{X}, X)$, A and X respectively. Let $\bar{\mathscr{M}}_5$ be the moduli space of stable curves of genus 5. Then there are natural maps $\mathbf{R} \to \mathscr{P}_5$, $\mathbf{A} \to \mathscr{A}_4$ and $\mathbf{M} \to \bar{\mathscr{M}}_5$ sending respectively $0 \in \mathbf{R}$ to $(\tilde{X}, X) \in \mathscr{P}_5$, $0 \in \mathbf{A}$ to $A \in \mathscr{A}_4$ and $0 \in \mathbf{M}$ to $X \in \bar{\mathscr{M}}_5$. All these maps have finite degree at the image of 0 (see [DS], p. 67, for the case of $\mathbf{M} \to \bar{\mathscr{M}}_5$, the other cases are analogous). The Prym map and the forgetful map $\mathscr{P}_5 \to \bar{\mathscr{M}}_5$ lift to **R**, **A** and **M** to give us commutative diagrams

$$\begin{array}{ccc} \mathbf{R} & \longrightarrow & \mathbf{M} \\ \downarrow & & \downarrow \\ \mathscr{P}_5 & \longrightarrow & \bar{\mathscr{M}}_5, \end{array} \qquad \begin{array}{ccc} \mathbf{R} & \xrightarrow{\bar{P}} & \mathbf{A} \\ \downarrow & & \downarrow \\ \mathscr{P}_5 & \longrightarrow & \mathscr{A}_4. \end{array}$$

Let $(\tilde{X}, X)$ be in $P^{-1}(A)$ and suppose that X has m nodes. The space **M** splits as $\mathbf{M}_{\text{local}} \times \mathbf{M}_m$ where $\mathbf{M}_m$ is the base for the deformations of X that are "as singular as" X and $\mathbf{M}_{\text{local}}$ is the base for the deformations of the localization of X at its nodes ([DS], p. 66). Similarly, **R** splits as $\mathbf{R}_{\text{local}} \times \mathbf{R}_m$ with $\{0\} \times \mathbf{R}_m$ mapping to $\{0\} \times \mathbf{M}_m$. Correspondingly, we have a map (finite at 0 again) $\mathbf{R}_m \to \mathscr{P}_{5,m}$ sending 0 to $(\tilde{X}, X)$, where $\mathscr{P}_{5,m}$ is the subspace of $\mathscr{P}_5$ parametrizing Prym-curves with at least m nodes. The restriction P_m of the Prym map to $\mathscr{P}_{5,m}$ lifts to $\mathbf{R}_m$ and we obtain a commutative diagram:

$$\begin{array}{ccc} \mathbf{R}_m & \xrightarrow{\bar{P}_m} & \mathbf{A} \\ \downarrow & & \downarrow \\ \mathscr{P}_{5,m} & \longrightarrow & \mathscr{A}_4. \end{array}$$

Since the vertical maps are finite at the images of 0, the fiber of $\mathscr{P}_{5,m} \to \mathscr{A}_4$ at $(\tilde{X}, X)$ has the same dimension as the fiber of $\mathbf{R}_m \to \mathbf{A}$ at 0. To prove the theorem, we compute the cotangent space to $\mathbf{R}_m$ and the codifferential of $\bar{P}_m$ at 0.

a) The tangent space to $\mathbf{R}_m$: It follows from [DS], p. 68, 2.6(i), that **R** is a (ramified) two sheeted cover of **M**: we can choose coordinates $x_1, \dots, x_m$ on $\mathbf{M}_{\text{local}}$ and $y_1, \dots, y_{m'}$ on $\mathbf{M}_m$ such that the cover $\mathbf{R} \to \mathbf{M}$ is given by $x_i \mapsto (x_i)^2$ and $y_i \mapsto y_i$. Hence the induced map $\{0\} \times \mathbf{R}_m \to \{0\} \times \mathbf{M}_m$ is an isomorphism and we can identify $T_0\mathbf{R}_m$ with $T_0\mathbf{M}_m$. Notice that, as $\mathbf{R} \to \mathbf{M}$ is ramified, we cannot identify $T_0\mathbf{R}$ with $T_0\mathbf{M}$.

b) The cotangent space to $\mathbf{M}_m$: Let Ω_X be the sheaf of Kähler differentials on X. Then the cotangent space $T_0^*\mathbf{M}$ is naturally isomorphic to $H^0(X, \Omega_X \otimes \omega_X)$ (see [DS], p. 65). Recall that $\{t_i, 1 \leqq i \leqq m\}$ is the set of nodes of X. We have an exact sequence

$$0 \to \bigoplus_i \mathbb{C}_i \to \Omega_X \to \omega_X$$

where the skyscraper sheaf $\mathbb{C}_i$ is the torsion subsheaf of Ω_X supported on t_i. Tensoring this sequence with ω_X, we obtain an exact sequence of cohomology groups:

$$0 \to \mathbb{C}^m \to H^0(X, \Omega_X \otimes \omega_X) \to H^0(X, (\omega_X)^2).$$

The vector space $\mathbb{C}^m$ is the space of sections of $\Omega_X \otimes \omega_X$ with support on the singular locus of X and, by [DS], p. 66, it can be canonically identified with $T_0^*\mathbf{M}_{\text{local}}$. Hence the image $\mathbb{T}^*$ of $H^0(X, \Omega_X \otimes \omega_X)$ in $H^0(X, (\omega_X)^2)$ can be canonically identified with $T_0^*\mathbf{M}_m$.

c) The codifferential of $\bar{P}_m$: Consider the composition

$$S^2 H^0(X, \omega_X \otimes \eta) \to T_0^* \mathbf{R} \to \mathbb{T}^* \subset H^0(X, (\omega_X)^2)$$

where the first map is the codifferential of $\bar{P}$ and the second map projection. This gives a map of bundles over $\mathbf{R}$ from the bundle with fibers $S^2 H^0(X, \omega_X \otimes \eta)$ to the bundle with fibers $H^0(X, (\omega_X)^2)$: this map agrees with multiplication generically (see [B2], p. 381), hence, by continuity, it is multiplication everywhere. Equivalently, the codifferential of $\bar{P}_m$ is multiplication.

d) Interpretation in terms of the normalizations of $\tilde{X}$ and X: Let $\tilde{X}_n$ and X_n be the respective normalizations of $\tilde{X}$ and X. Let $\pi_n : \tilde{X}_n \to X_n$ be the ramified cover obtained from π. The variety $P(\tilde{X}_n, X_n)$ is isogenous to $P(\tilde{X}, X)$. Let δ be a divisor on X_n such that

$$\pi_{n*} \mathcal{O}_{\tilde{X}_n} \cong \mathcal{O}_{X_n} \oplus \mathcal{O}_{X_n}(-\delta)$$

and let Δ be the branch divisor of π_n ($\Delta \equiv 2\delta$), pullback induces an isomorphism

$$S^2 H^0(X, \omega_X \otimes \eta) \cong S^2 H^0(X_n, \omega_{X_n}(\delta))$$

and a surjection

$$H^0(X, \Omega_X \otimes \omega_X) \twoheadrightarrow H^0(X_n, (\omega_{X_n})^2(\Delta)).$$

The kernel of this last map is clearly the space of global sections of $\Omega_X \otimes \omega_X$ with support on the singular locus of X. Now, pushforward induces an injection

$$H^0(X_n, (\omega_{X_n})^2(\Delta)) \hookrightarrow H^0(X, (\omega_X)^2).$$

So, as $\mathbb{T}^*$ is the quotient of $H^0(X, \Omega_X \otimes \omega_X)$ by the space of sections supported on the singular locus of X, we deduce that $\mathbb{T}^*$ can be identified with $H^0(X_n, (\omega_{X_n})^2(\Delta))$.

As pullback and pushforward commute to multiplication, it follows that the projectivization of the kernel of the codifferential of $\bar{P}_m$ is the space of quadrics in $T_0 A$ which contain the image χX_n of X_n by the map associated to the linear system $|\omega_{X_n}(\delta)|$. (This is the projectivization of the kernel of multiplication

$$S^2 H^0(X_n, \omega_{X_n}(\delta)) \to H^0(X_n, (\omega_{X_n})^2(\Delta)).)$$

3.5. We consider reducible Prym-curves in $P^{-1}(A)$.

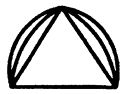
The curve X has at most 4 components because A has dimension 4: for each component C of X, if $\tilde{C}$ is its inverse image in $\tilde{X}$, then the Prym variety of the double cover $\tilde{C} \to C$ embeds in A and is positive dimensional since A is not a jacobian (i.e., there are no "elliptic tails", see [DS], pp. 73–85). Also, C has to intersect the union of the other components of X in an even number of points since $\tilde{C} \to C$ is ramified exactly at those points and at the singular points of C (because A is not a jacobian). Since A is not a product of lower dimensional ppav's, the curve X cannot be written as $C_1 \cup C_2$ with $\#(C_1 \cap C_2) \leqq 2$ ([B1], Lemma 4.11).

As in [B1] we associate a graph to X: the vertices of the graph are the irreducible components of X and each edge between two vertices represents a point of intersection of the two irreducible components. We investigate the various possibilities for the graphs:

– We first see that there are at most a finite number of Prym-curves with four components in $P^{-1}(A)$: from the considerations of the preceding paragraph it follows that any graph with 4 vertices has at least 8 edges so that all the components X_i $(1 \leqq i \leqq 4)$ of X are of genus 0, their double covers will be elliptic curves E_i $(1 \leqq i \leqq 4)$ which embed in A. Elliptic curves are rigid in an abelian variety. Also, the points where two elliptic curves meet are determined by A and the elliptic curves because they are the ramification points of the double covers $E_i \to X_i \cong \mathbb{P}^1$. Hence there is only a finite number of Prym-curves of this type in $P^{-1}(A)$.

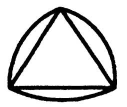
– Suppose now that X has three components. The possibilities are:

Case 1: The graph of X is



The genus of a curve with such a graph is at least $7 - 2 = 5$. So X has three smooth rational components, $\tilde{X}$ has two elliptic components and one component of genus 2. The elliptic components embed in A and do not move. The jacobian of the genus 2 component also embeds in A and does not move either because it is an abelian subvariety of A. The points where the components meet are determined as before. So there is only a finite number of such Prym-curves in $P^{-1}(A)$.

Case 2: The graph of X is



Here the genus of the associated curve is at least $6 - 2 = 4$. So X has two components C_1, C_2 of genus 0 and one component C_0 of genus 1, $\tilde{X}$ is the union of two elliptic curves and a curve of genus 3. The elliptic components embed in A and are rigid in A. The Prym variety of the ramified cover $\tilde{C}_0 \to C_0$ embeds in A. As abelian subvarieties of A are rigid in A, $P(\tilde{C}_0, C_0)$ does not move either. So if we show that there is exactly one one-dimensional family of Prym-curves $(\tilde{C}_0, C_0)$ with a given Prym variety, it will follow that there is exactly one one-dimensional family of Prym-curves with the above graph in $P^{-1}(A)$.

For each i, let t_i' and t_i'' be the two points of X_n mapping to t_i. Suppose that for all i between 1 and 4, t_i' is an element of C_0. The abelian variety $P(\tilde{C}_0, C_0)$ is not principally polarized. It is isogenous, for instance, to $P(\tilde{C}_0', C_0')$ where $\tilde{C}_0'$ and C_0' are obtained respectively from $\tilde{C}_0$ and C_0 by identifying t_1' with t_2' and t_3' with t_4' on both curves. This isogeny is of degree 2 (see [B1], p. 159). So it is enough to show that there is exactly one one-dimensional family of such Prym-curves $(\tilde{C}_0', C_0')$ with given Prym variety. As $P(\tilde{C}_0', C_0')$ is principally polarized, we can reason as in 3.4 with $(\tilde{C}_0', C_0')$ instead of $(\tilde{X}, X)$ to conclude that we have to show that χC_0 is not contained in any quadric (then the fiber of the map induced by the Prym map on the space of formal versal deformations of $(\tilde{C}_0', C_0')$ will be smooth of dimension 1 at 0). Let δ_0 be the divisor class on C_0 such that

$$\tilde{C}_0 = \mathrm{Spec}_{C_0}(\mathscr{O}_{C_0} \oplus \mathscr{O}_{C_0}(-\delta_0)) .$$

The map $C_0 \to \chi C_0 \subset \mathbb{P} T_0 P(\tilde{C}_0, C_0)$ is associated to the linear system $|\omega_{C_0}(\delta_0)|$: this is of degree 2 and is onto from C_0 to $\mathbb{P}^1 = \mathbb{P} T_0 P(\tilde{C}_0, C_0)$.

– Suppose now that X has two components. We have the following possibilities:

Case 1: The graph of X is

 .

Any curve associated to this graph has genus at least $6 - 1 = 5$. So both components of X have to be rational. As before, there is only a finite number of such Prym-curves in $P^{-1}(A)$.

Case 2: The graph of X is

 .

The genus of a curve with this graph is at least $4 - 1 = 3$. In this case

– either X is the union of a curve C_1 of genus 0 and a curve C_0 of genus 2, i.e., $(\tilde{X}, X)$ (resp. X) is of type $(0, 2)$,

– or X is the union of two curves of genus 1, i.e., $(\tilde{X}, X)$ (resp. X) is of type $(1, 1)$.

We first show that the fibers of the restriction of P to some dense open subset of the locus of Prym-curves of type $(0, 2)$ have dimension 1: arguing as in the last case where X has three components we have to show that the multiplication map

$$S^2 H^0(C_0, \omega_{C_0}(\delta_0)) \to H^0(C_0, (\omega_{C_0}(\delta_0))^2)$$

is injective. The image of C_0 in $\mathbb{P}^2 = \mathbb{P} T_0 P(\tilde{C}_0, C_0)$ is a quartic curve. Multiplication fails to be injective exactly when this quartic is twice a conic, i.e., $\omega_{C_0}(\delta_0) \cong \mathscr{O}_{C_0}(2 g_0)$, where g_0 is the unique g_2^1 of C_0. Hence $|\delta_0|$ is equal to g_0 since $\mathscr{O}_{C_0}(g_0) \cong \omega_{C_0}$. So the four branch points t_i' $(1 \leqq i \leqq 4)$ of $\tilde{C}_0 \to C_0$ verify (for instance) $t_i' + t_{i+1}' \in g_0$ $(i = 1, 3)$.

Now, using the tetragonal construction, we show that $P^{-1}(A)$ cannot contain a 2-dimensional family of such Prym-curves unless $A \in \mathscr{A}_{1,1}$.

The canonical embedding κX of X is the union of a nondegenerate sextic of genus 2 (the image of C_0) and a conic meeting it in 4 points, the image of the genus 0 component C_1 of X. We identify C_0 and C_1 with their images in the canonical space of X. The sums, three by three, of divisors of g_0 form the linear subsystem L of $|3 g_0|$ corresponding to the point p_0 in $|\omega_X|^*$ which is the center of the projection $|3 g_0|^* \to L^*$. Hence p_0 is on all the secants of C_0 spanned by divisors of g_0. In particular, since $t'_i + t'_{i+1} \in g_0$ (for $i = 1, 3$), p_0 is in the plane of C_1 but not on C_1 itself. So the image of κX by the projection from p_0 is the union of a twisted cubic and a line meeting it at two points. So there is a pencil of quadrics of (generic) rank 4 containing κX with singular locus p_0.

One ruling of any quadric of this pencil cuts a g^1_4 on X whose restrictions to C_0 and C_1 are respectively g_0 and g_1, where g_1 is the pencil of divisors on C_1 spanned by $t''_1 + t''_2$ and $t''_3 + t''_4$. Let g be a generic such g^1_4. As we saw in the proof of 3.2, the Prym-curves $(\tilde{Y}, Y), (\tilde{U}, U)$ tetragonally related to $(\tilde{X}, X)$ through g cannot be of type (0, 2). By 2c) in the proof of 3.1, both $(\tilde{Y}, Y)$ and $(\tilde{U}, U)$ have (at least) 4 nodes. So, as we have a 2-dimensional family of them, they can only be of type (1, 1) (we will see in 3.7 that they cannot be irreducible).

3.6. Now we show that for $A \in \mathscr{A}_{1,1}$, $P^{-1}(A)$ does not contain a 2-dimensional family of Prym-curves of type (0, 2).

Arguing as before, we see that the fibers of the restriction of P to $\mathscr{P}_{1,1}$ have pure dimension 2. Let $(\tilde{X}, X)$ be a generic Prym-curve of type (1, 1) in $P^{-1}(A)$ (i.e., $(\tilde{X}, X)$ is an element of some dense open subset of the family of Prym-curves of type (1, 1) in $P^{-1}(A)$). By the same type of argument as in the cases where X has 3 or 4 components, one sees that the family of Prym-curves of type (1, 1) with (at least) one singular component has dimension $\leqq 1$ in $P^{-1}(A)$. Hence X is the union of two smooth elliptic curves. The canonical embedding κX of X is the union of two elliptic quartic curves C_1, C_2 which span two projective spaces P_1 and P_2 of dimension 3 and meet in 4 points. Restricting to P_1 and P_2, we see that $P_1 \cup P_2$ is the only quadric of rank 2 containing κX. As in the smooth case (1.3) there is a net of quadrics containing κX. The generic such quadric is smooth: Otherwise, its rulings will cut nonsingular g^1_4's on X. So X would have a two-dimensional family of nonsingular g^1_4's. This implies, by the proof of lemma 4.8 in [B1], that for a generic such g^1_4, $h^0(\omega_X(-2g^1_4)) > 0$. This is clearly impossible. (So a plane quintic Q parametrizes the singular quadrics containing κX.) Let g be a g^1_4 on X, cut by one ruling of a quadric q of rank 4. We wish to show that, for g in some dense open subset W of $W^1_4(X)$, the Prym-curves $(\tilde{Y}, Y), (\tilde{U}, U)$ tetragonally related to $(\tilde{X}, X)$ through g are smooth. The restrictions of q to P_1 and P_2 vary with q (for q in any component of Q), so the restrictions of g to C_1 and C_2 vary with g (for g in any component of W^1_4). So, for a good choice of W, the divisors of g with singular support are of type 2a) in 3.1, hence they do not give rise to any singular points of Y or U by 3.1. So, by 3.1, we have to rule out the possibility that every element of some component of $W^1_4(X)$ has a divisor of the type $2s_1 + 2s_2$ with s_1 and s_2 smooth points of X. If this is the case, then every line l_t tangent to C_1 at t meets some line l_s tangent to C_2 at s: $l_t \cap l_s$ will then be in $V = P_1 \cap P_2$. It is easily seen that the maps $C_i \to V$ given by $t \mapsto l_t \cap V$ are birational onto their images (for instance, look at l_t for $t = t_1$). Hence C_1 and C_2 are isomorphic and t'_i corresponds to t''_i under this isomorphism. As in the previous cases, there is exactly one 1-dimensional family of this type of Prym-curves in $P^{-1}(A)$ and $(\tilde{X}, X)$ cannot be generic.

3.7. Now we eliminate the cases where X is irreducible with at least 3 nodes. Let $\mathscr{P}'_{5,m}$ be the space parametrizing irreducible Prym-curves with *exactly m* nodes.

If $m = 3$, for all $(\tilde{X}, X)$, there is exactly one quadric containing χX_n. Hence $\bar{P}_3$ has maximal rank at 0 for all $(\tilde{X}, X)$. So the fibers of P_3 are finite at irreducible Prym-curves and the closure of the image of $\mathscr{P}'_{5,3}$ is a divisor in $\mathscr{A}_4$.

If $m = 4$, for all $(\tilde{X}, X)$, there is exactly one pencil of quadrics containing χX_n. Hence $\bar{P}_4$ also has maximal rank at 0 for all $(\tilde{X}, X)$. The fibers of P_4 are finite at irreducible Prym-curves and the closure of the image of $\mathcal{P}'_{5,4}$ has codimension 2 in $\mathcal{A}_4$.

If $m = 5$, then χX_n is a twisted cubic and there is exactly one net of quadrics containing it. So $\bar{P}_5$ has maximal rank at 0 for all $(\tilde{X}, X)$. The fibers of P_5 are finite at irreducible Prym-curves and the closure of the image of $\mathcal{P}'_{5,5}$ has codimension 3 in $\mathcal{A}_4$.

3.8. We consider the case where X is irreducible with $m = 2$ nodes.

If χX_n is of degree 6, then there is at most one quadric containing χX_n, so that, locally at $(\tilde{X}, X)$, the fibers of P_2 and $\bar{P}_2$ have dimension at most 1.

If χX_n is of degree 3, then X_n is hyperelliptic and $|\omega_{X_n}(\delta)| = |3g_0|$ where g_0 is the unique g_2^1 of X_n. As $|\omega_{X_n}| = |2g_0|$, it follows that $|\delta| = g_0$ and

$$|t_1' + t_1'' + t_2' + t_2''| = |2\delta| = |2g_0|\,.$$

Any divisor of $|2g_0|$ is the sum of 2 divisors of g_0, hence

- either $t_1' + t_1'' \in g_0$ and $t_2' + t_2'' \in g_0$
- or, replacing t_1'' by t_1' if necessary, $t_1' + t_2' \in g_0$ and $t_1'' + t_2'' \in g_0$.

The first case is excluded as it would imply that A is a hyperelliptic jacobian by [B1], p. 171. In the second case, if we let p' and p'' be the images of t_1' and t_1'' in $\mathbb{P}^1$ by the morphism ϕ_0 associated to g_0, we see that ϕ_0 descends to a 2-to-1 morphism from X to the curve E of genus 1 obtained from $\mathbb{P}^1$ by identifying p' and p''. The pullback on X of any g_2^1 on E, has 1 divisor of type 2c) in 3.1 and 2 divisors of type 1a) or 1b). Let g be such a g_4^1. By 3.1, a divisor of type 2c) gives 2 nodes on each of the Prym-curves tetragonally related to $(\tilde{X}, X)$ through g and a divisor of type 1a) or 1b) gives one node on one of them. If we have a 2-dimensional family of such curves, generically, the only possibility is that one of them be of type $(1, 1)$: but a Prym-curve tetragonally related to a generic element of type $(1, 1)$ of $P^{-1}(A)$ through a generic g_4^1 is smooth by 3.6.

Now we consider the case where X is irreducible with one node. We first need

Lemma 3.9. *Let F_0 be a component of $P^{-1}(A)$. Let $(\tilde{X}, X)$ be a generic element of F_0 and suppose that $(\tilde{X}, X)$ is smooth or has one node. Then a Prym-curve tetragonally related to $(\tilde{X}, X)$ through a generic g_4^1 on X (i.e., an element of some dense open subset of $W_4^1(X)$) is either smooth or of type* $(1, 1)$.

Proof. Let g be a generic element of $W_4^1(X)$, then it is easily seen that the divisor of g with singular support is of type $(u, 1, 1) + s + t$ (see 3.1 for the notation) with $s \neq t$ and by 3.1, 2a) it does not give rise to singular points on the Prym-curves $(\tilde{Y}, Y)$, $(\tilde{U}, U)$ tetragonally related to $(\tilde{X}, X)$ through g. So, if $(\tilde{Y}, Y)$ or $(\tilde{U}, U)$ is not smooth, g has divisors of the type $2s + 2t$ (s and t distinct or not). Suppose that this is the case for every element g of some component W_0 of $W_4^1(X)$ and let $\mathcal{X}$ be the curve parametrizing divisors of the type $2s + 2t$ in elements of W_0. The natural morphism $\gamma : \mathcal{X} \to W_4^1$ is finite onto its image.

Since X is not trigonal, the components of W_4^1 are stable under $g \mapsto |K_X - g|$: This is in [ACGH], p. 274, if X is smooth. If X has a node, then X is obtained from X_n by identifying two points, say p_0 and q_0. Every nonsingular g_4^1 on X is obtained from a g_4^1 on X_n such that $h^0(g_4^1 - p_0 - q_0) > 0$. Write $g_4^1 = |K_{X_n} - p - q|$ and $|K_{X_n} - p_0 - q_0| = g_0$. The previous condition is equivalent to $h^0(g_0 - p - q) > 0$. Now, two g_4^1's are opposite if and only if the corresponding pairs of points $\{p, q\}$, $\{p', q'\}$ on X_n verify $p + q + p' + q' \in g_0$. Let $(\tilde{W}, W)$ be the Prym-curve trigonally related to X_n through g_0. It follows from the above that there is a birational map $W_4^1 \to \tilde{W}$ which descends to a birational map $Q \to W$. By [DS], p. 48, the double cover $\tilde{W} \to W$ is ramified with unexchanged branches at all the nodes of $\tilde{W}$. Hence all the components of $\tilde{W}$ are preserved by the involution of the cover $\tilde{W} \to W$. Hence the same is true of W_4^1 for the involution $g \mapsto |K_X - g|$.

For each $x \in \mathscr{X}$, let $2s_x + 2t_x$ be the corresponding divisor. By the above, if $\gamma(x) = g$, there exists y such that $\gamma(y) = |K_X - g|$. Then $\mathcal{O}_X(s_x + t_x + s_y + t_y)$ is an effective theta-characteristic on X. As X has only a finite number of theta-characteristics, it follows that for each component $\mathscr{Y}$ of $\mathscr{X}$ there is a component $\mathscr{Y}'$ of $\mathscr{X}$ and a vanishing theta-null g_0 on X such that

$$g_0 = \{s_x + t_x + s_y + t_y : x \in \mathscr{Y}, y \in \mathscr{Y}'\}.$$

Let $D = a + b + c + d$ be a generic divisor of g_0. Suppose that D can be written as $s_x + t_x + s_y + t_y$ in at least 2 different ways. By Riemann-Roch, this means that for any point occurring in D, for instance a, the tangent line $\langle 2a \rangle$ to κX at (the image of) a intersects at least 2 of the tangent lines $\langle 2b \rangle$, $\langle 2c \rangle$, $\langle 2d \rangle$. As no three of the tangent lines are in a plane (otherwise X would have a g_6^3 and hence a g_2^1), it follows that all 4 tangent lines intersect in a point $p \in |\omega_X|^*$. So we can associate to D six g_4^1's:

$$\begin{aligned} g_1 &= |2a + 2b|, & h_1 &= |K_X - g_1| = |2c + 2d|, \\ g_2 &= |2a + 2c|, & h_2 &= |K_X - g_2| = |2b + 2d|, \\ g_3 &= |2a + 2d|, & h_3 &= |K_X - g_3| = |2b + 2c|. \end{aligned}$$

It follows that we have an embedding of $\tilde{V} = \{D_2 \in \mathrm{Div}^2 X : h^0(g_0 - D_2) > 0\}$ in W_4^1. As both curves have the same genus, we can identify them and identify Q with the quotient of $\tilde{V}$ by the involution exchanging complementary divisors. This implies that Q (and not just a partial normalization of Q) has a 3-to-1 morphism to $\mathbb{P}^1$ which is impossible.

Thus a generic divisor of g_0 can be written in a unique way as $s_x + t_x + s_y + t_y$ with $x, y \in \mathscr{X}$. It is now easily seen that $s_x \mapsto t_x$ defines an involution on X with quotient a curve E of genus 1. The pullback on X of any g_2^1 on E is a g_4^1 with at least 4 divisors of the type $2s + 2t$. Hence at least one of $(\tilde{Y}, Y)$, $(\tilde{U}, U)$, say $(\tilde{Y}, Y)$, has two nodes or more. This is possible generically on a 2-dimensional family of Prym-curves only if $A \in \mathscr{A}_{1,1}$ in which case $(\tilde{Y}, Y)$ is of type (1, 1) and $(\tilde{X}, X)$, $(\tilde{U}, U)$ are smooth by 3.6. This concludes the proof of the lemma.

Now, suppose that the generic elements of F_0 have one node. Let F_1 be a component of $P^{-1}(A)$ such that every element of F_1 is tetragonally related to some element of F_0. Then, by 3.9, the generic elements of F_1 are smooth or of type (1, 1). Conversely, since the tetragonal relation is symmetric, every element of F_0 is tetragonally related to some element of F_1. However, by 3.6, since generic elements of F_0 are tetragonally related to generic elements of

F_1 through generic g_4^1's, this implies that the generic elements of F_0 are either smooth or of type $(1, 1)$ which is a contradiction. This concludes the proof of theorem 3.3. Q.E.D.

Remark 3.10. *The space* $\mathscr{P}_{1,1}$ *of Prym-curves of type* $(1, 1)$ *is irreducible of dimension* 8. *In particular,* $\mathscr{A}_{1,1}$ *is irreducible of dimension* 6. *A generic element of* $\mathscr{A}_{1,1}$ *has* 4 *vanishing theta-nulls.*

Proof. The irreducibility of $\mathscr{P}_{1,1}$ is easy. The last assertion is proved in [De1], p. 268: our $\mathscr{A}_{1,1}$ is equal to Debarre's $\mathscr{A}_{2,2}^2 \subset \mathscr{A}_4$.

We will now proceed to define the involution λ *on the set of Prym-curves whose Prym varieties are not jacobians, later, in* 4.3, *we will define* λ *for Prym-curves whose Prym varieties are non-hyperelliptic jacobians.*

In the parametrization of A associated to $(\tilde{X}, X)$ in 1.2, any Prym-embedding of $\tilde{X}$ in A is given by $\tilde{X}_E = \{E(L_p - \sigma L_p) : p \in \tilde{X}\}$ for some invertible sheaf E on $\tilde{X}$ such that $\nu(E) \cong \omega_X$ and $h^0(E)$ is odd. As

$$\tilde{X}_E \cap \Theta = \{p : h^0(E(L_p - \sigma L_p)) > 0\}$$

it is immediately seen that $\tilde{X}_E$ is in Θ if and only if $h^0(E) \geqq 3$. So the family of Prym-embeddings of $\tilde{X}$ in any symmetric theta divisor Θ is parametrized by

$$\tilde{X}_\lambda = \{E \in \operatorname{Pic} \tilde{X} : \nu(E) \cong \omega_X, h^0(\tilde{X}, E) \text{ is odd and } \geqq 3\}\,.$$

For each $E \in \tilde{X}_\lambda$, the images of the divisors of $|E|$ in $|\omega_X|^*$ form a plane. The union of these planes is a threefold because a generic canonical divisor is not the pushforward of some D with $h^0(D) \geqq 3$ (this follows easily from the "uniform position theorem" in [ACGH], chapter 3). Also, the map $D \mapsto \pi_* D$ is finite on divisors. Hence, $\tilde{X}_\lambda$ is always a curve and we cannot have $h^0(E) \geqq 5$ for any E.

Considering the different types of Prym-curves whose Prym varieties are in $\mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$ (see 3.5–3.9), one sees easily that X always has a finite morphism of degree 4 onto $\mathbb{P}^1$. Letting g, $(\tilde{Y}, Y)$, g_Y and h_Y be as in 3.2, we can suppose by 3.2 that h_Y does not map any component of Y to a point. Considering now the parametrization of A associated with $(\tilde{Y}, Y)$, we identify $\tilde{X}$ with a connected component of the set $\{D \in \operatorname{Div}^4 \tilde{Y} : \pi_* D \equiv g_Y\}$.

Proposition 3.11. *The curve* $\tilde{X}_\lambda$ *is isomorphic to the connected component* $\tilde{X}'$ *of the set*

$$\{E \in \operatorname{Div}^4 \tilde{Y} : \pi_* E \equiv h_Y\}$$

such that for all $D \in \tilde{X}$ *and* $E \in \tilde{X}'$, $h^0(\tilde{Y}, E + D)$ *is even.*

Proof. Clearly $\tilde{X}' \subset \tilde{X}_\lambda$. Choose a generic point $p \in \tilde{X}$. Then the morphism $E \mapsto E(p - \sigma p)$ defines an isomorphism of $\tilde{X}_\lambda$ with

$$W_p = W_p(X) = \{D \in \operatorname{Pic} \tilde{X} : \nu(D) \cong \omega_X, h^0(D(-p)) \geqq 2\}$$

(the inverse isomorphism of the morphism is simply $D \mapsto D(\sigma p - p)$ whose image is $\tilde{X}_\lambda$ because $h^0(D(\sigma p - p))$ is odd and $\geqq 2$). Choose two generic points $q, r \in \tilde{X}$. As p and q are generic we see that the natural map $f : X \to |K_X - \pi p - \pi q|^*$ is a birational morphism onto its image X_0 which has only nodes as singularities. As $S = \Theta . \Theta_{[p,q]}$ is a complete intersection in the smooth variety A, it is Cohen-Macaulay. By [B3], p. 365,

$$\{E \in \mathrm{Div}^6 \tilde{X} : \pi_* E \equiv K_X - \pi p - \pi q\}$$

has two connected components and in a way analogous to the proof of [BD1], proposition 1, S is isomorphic to one of them. We claim that S is irreducible (and normal): it is enough to show that it is smooth in codimension 1. For a divisor E ($\in S, \pi_* E \equiv K_X - \pi p - \pi q$), we denote by l_E the line spanned by $\pi_* E$ in $\mathbb{P}^2 = |K_X - \pi p - \pi q|^*$. In codimension 1, l_E can at worst either be tangent to X_0 at a (unique) smooth point or contain a (unique) ramification point of f or contain a (unique) singular point of X_0. If E has nonsingular support, the proof of the corollary p. 365 in [B3] applies and S is smooth at E. Otherwise l_E contains $f(t)$ for some unique singular point t of X: a proof analogous to that of 3.1, 2a) shows that S is smooth at E.

A computation analogous to that of [BD1] (proof of proposition 2) shows that the only translates of Θ containing S are Θ and $\Theta_{[p,q]}$. Hence, as S is irreducible, $\Theta . \Theta_{[p,q]} . \Theta_{[p,r]} = S . \Theta_{[p,r]}$ has dimension 1. So $\Theta . \Theta_{[p,q]} . \Theta_{[p,r]}$ is the union of W_p and one connected component of

$$\{E \in \mathrm{Pic}^5 \tilde{X} : \nu(E) \cong \omega_X(-\pi p - \pi q - \pi r)\}$$

which has homology class $2[\Theta]^3/3$ by [B3] and specialization from the case where X is smooth. So W_p is the support of a subscheme $\tilde{W}_p$ of $\Theta . \Theta_{[p,q]} . \Theta_{[p,r]}$ with homology class $[\Theta]^3/3$. As $\tilde{X}' \subset \tilde{X}_\lambda \cong W_p \subset \tilde{W}_p$ and the homology class of $\tilde{X}'$ is $[\Theta]^3/3$, we have $\tilde{X}' = \tilde{X}_\lambda = W_p = \tilde{W}_p$. Q.E.D.

Remark 3.12. For $(\tilde{X}, X)$ generic, the fact that the curve parametrizing Prym-embeddings of $\tilde{X}$ in Θ is a special subvariety (in the sense of [B3]) associated to $|K_Y - g_Y|$ is stated in [C2] with an indication of one possible proof (different from ours).

The curve $\tilde{X}_\lambda$ has an involution $\sigma_\lambda : L \mapsto \omega_{\tilde{X}} \otimes L^{-1} \cong \sigma^* L$. Let X_λ be the quotient of $\tilde{X}_\lambda$ by σ_λ. The curve $\tilde{X}'$ comes with the involution induced by σ_Y. Under the identification $\tilde{X}' = \tilde{X}_\lambda$ the two involutions coincide and we obtain

Definition and Proposition 3.13. *The double cover $\pi_\lambda : \tilde{X}_\lambda \to X_\lambda$ is allowable with Prym variety A. If $(\tilde{X}, X)$ is tetragonally related to $(\tilde{Y}, Y)$ through g^1_4 on Y, then $(\tilde{X}_\lambda, X_\lambda)$ is tetragonally related to $(\tilde{Y}, Y)$ through $|K_Y - g^1_4|$. For $A \notin \mathcal{J}_4$, we define $\lambda(\tilde{X}, X)$ to be $(\tilde{X}_\lambda, X_\lambda)$. The morphism $\lambda : (\tilde{X}, X) \mapsto (\tilde{X}_\lambda, X_\lambda)$ is an involution.*

Proof. The first assertion is a consequence of 3.1 and the second assertion. The second assertion is a consequence of 3.11. That $\lambda : (\tilde{X}, X) \mapsto (\tilde{X}_\lambda, X_\lambda)$ is an involution follows from the definition $\tilde{X}_\lambda = \{a : \tilde{X} + a \subset \Theta\}$ and the fact that $\tilde{X}' = \tilde{X}_\lambda$ is a Prym-embedding by 2.4. Q.E.D.

Recall that η is the torsion free coherent sheaf on X associated to $\pi : \tilde{X} \to X$. By an odd (resp. even) vanishing theta-null on X we mean a vanishing theta-null $E \in W_4^1(X)$ such that $h^0(\eta(E))$ is odd (resp. even). Consider the parametrization of A associated with X as in 2.4. Recall that $\tilde{X}_\lambda$ can be identified with

$$\{E' \in \operatorname{Pic}^8 \tilde{X} : \nu(E') \cong \omega_X, h^0(E') = 3\} .$$

The following implies that λ is not the identity.

Lemma 3.14. *Suppose X is smooth. Then, for $E \in \operatorname{Pic}^4 X$, $\pi^* E$ is a singular point of $\tilde{X}_\lambda$ if and only if E is an odd vanishing theta-null on X.*

Proof. As we saw at the beginning of this section, the singular points of X_λ (resp. $\tilde{X}_\lambda$) are exactly the branch (resp. ramification) points of π_λ. Suppose that E is an odd vanishing theta-null and let $E' = \pi^* E$. Then E' verifies

$$\nu(E') \cong E^2 \cong \omega_X, \quad h^0(E') = h^0(E) + h^0(\eta \otimes E) = 3, \quad E' \cong \omega_{\tilde{X}} \otimes E'^{-1} \cong \sigma^* E' .$$

So E' is an element of $\tilde{X}_\lambda$. As $\sigma_\lambda(E')$ coincides with $\sigma^* E' \cong E'$, π_λ is ramified at E'.

For the converse just go backwards. Q.E.D.

3.15. We have a surface in Θ traced by the Prym-embeddings of $\tilde{X}$ (or $\tilde{X}_\lambda$) in Θ. Choose an isomorphism $P(\tilde{X}, X) \cong A$ up to translations and $-\mathrm{id}$. The isomorphism between $P(\tilde{X}, X)$ and $P(\tilde{X}_\lambda, X_\lambda)$ is canonical up to translations and $-\mathrm{id}$ since it is induced by the identification $\tilde{X}_\lambda = \{E \in \operatorname{Pic} \tilde{X} : \nu(E) \cong \omega_X, h^0(\tilde{X}, E) \text{ is odd and } \geq 3\}$. Therefore we also obtain a choice of isomorphism $P(\tilde{X}_\lambda, X_\lambda) \cong A$. Considering A embedded in $J\tilde{X}$ (resp. $J\tilde{X}_\lambda$) via our choice of isomorphism, these embeddings are translates of

$$\{L_p - \sigma L_p : p \in \tilde{X}\}, \quad (\text{resp. } \{L_p - \sigma_\lambda L_p : p \in \tilde{X}_\lambda\})$$

in $J\tilde{X}$ (resp. $J\tilde{X}_\lambda$). There exist fixed elements a of $J\tilde{X}$ and b of $J\tilde{X}_\lambda$ such that $a + (L_p - \sigma L_p)$ and $b + (L_q - \sigma_\lambda L_q)$ are in A for all $p \in \tilde{X}$, $q \in \tilde{X}_\lambda$ and the above surface is the set of elements $a + (L_p - \sigma L_p) + b + (L_q - \sigma_\lambda L_q)$ for $p \in \tilde{X}$, $q \in \tilde{X}_\lambda$.

Suppose that we have a Prym-embedding of $\tilde{X}_\lambda$ in $\Theta . \Theta_x$ for $x \in A$. Then there exists $p \in \tilde{X}$ such that

$$a + (L_p - \sigma L_p) + b + (L_q - \sigma_\lambda L_q) \in \Theta . \Theta_x \quad \text{for all } q \in \tilde{X}_\lambda ,$$

that is

$$a + (L_p - \sigma L_p) + b + (L_q - \sigma_\lambda L_q) - x \in \Theta \quad \text{for all } q \in \tilde{X}_\lambda .$$

This implies that there exists $p' \in \tilde{X}$ such that in A

$$a + (L_p - \sigma L_p) + b + (L_q - \sigma_\lambda L_q) - x = a + (L_{p'} - \sigma L_{p'}) + b + (L_q - \sigma_\lambda L_q) \quad \text{for all } q \in \tilde{X}_\lambda .$$

So we obtain $x = [p, \sigma p'] \in \Sigma(X)$ and in fact we have exactly 2 Prym-embeddings of $\tilde{X}_\lambda$ in $\Theta . \Theta_x$ (these correspond to p and $\sigma p'$, hence coincide if $p = \sigma p'$).

Conversely, for all $x \in \Sigma(X)$ and $q \in \tilde{X}_\lambda$ the Prym-embedding of $\tilde{X}$ in Θ corresponding to q and its translate by x intersect in 2 points which trace 2 copies of $\tilde{X}_\lambda$ in $\Theta . \Theta_x$ as q varies.

Summarizing, we obtain

Proposition 3.16. *Suppose that $A \in \mathcal{A}_4 \backslash \mathcal{J}_4$. Let x be an element of A. The intersection $\Theta . \Theta_x$ contains a Prym-embedding of $\tilde{X}_\lambda$ if and only if $x \in \Sigma(X)$. In that case, $\Theta . \Theta_x$ contains (at most) 2 copies of $\tilde{X}_\lambda$ which are those that correspond to the points s and t of $\tilde{X}$ such that $x = [s, t]$.*

Notice that, with the notation of 2.4 and 3.11, the Prym-embeddings of $\tilde{X}_\lambda$ in $\Theta . \Theta_{[s,t]}$ for $[s, t] \in \Sigma(X)$ are just W_s and W_t. Also, if we have a Prym-embedding $\tilde{X}_E$ of $\tilde{X}$ in $\Theta . \Theta_a$, then $t^*_{-a}(-\tilde{X}_E) = \{a - x : x \in \tilde{X}_E\}$ is the other Prym-embedding of $\tilde{X}$ in $\Theta . \Theta_a$.

Recall that Σ_A is the disjoint union of the surfaces $\Sigma(X)$ for all $(\tilde{X}, X) \in P^{-1}(A)$. The following result on the structure of the fibers of P will, in particular, permit us to determine the dimension of the fibers of P (see 3.18).

Theorem 3.17. *Let $(\tilde{X}, X)$ be an element of $P^{-1}(A)$ and let $\mathbf{P}(\tilde{X}, X)$ be the fiber of $\bar{P} : \mathbf{R} \to \mathbf{A}$ at 0 (see 3.4). Suppose that $A \in \mathcal{A}_4 \backslash \mathcal{J}_4$. Then:*

i) *If $(\tilde{X}, X)$ is fixed by λ, then $\Sigma(X)$ is contained in some symmetric theta divisor which is then singular at 0. Conversely, if $\Sigma(X)$ is contained in some translate of Θ, then $(\tilde{X}, X)$ is fixed by λ.*

ii) *If $\mathbf{P}(\tilde{X}, X)$ is not smooth of dimension 2 at 0 for a smooth $(\tilde{X}, X)$, then $(\tilde{X}, X)$ belongs to the 10-dimensional family of smooth Prym-curves with two even vanishing theta-nulls g_1 and g_2 such that $\eta = \mathcal{O}_X(g_1 - g_2)$, $A \in \theta_{\text{null}}$ and $(\tilde{X}, X)$ is fixed by λ.*

iii) *Let F_0 and F_1 be two components of $P^{-1}(A)$ such that every element of F_0 is tetragonally related to some element of F_1 (and vice versa). Then $\bigcup_{(\tilde{X}, X) \in F_0} \Sigma(X) = \bigcup_{(\tilde{Y}, Y) \in F_1} \Sigma(Y)$.*

iv) *Suppose that $\mathbf{P}(\tilde{X}, X)$ is smooth of dimension 2 at 0 for a Prym-curve $(\tilde{X}, X)$ which is either smooth or of type (1, 1). Then the component of Σ_A containing $\Sigma(X)$ surjects onto A. If X is of type (1, 1) and X_1 and X_2 are the components of X, let $\Sigma_{12}(X)$ be the image of $X_1 \times X_2$ in $\Sigma(X)$. Then the component of Σ_A containing $\Sigma_{12}(X)$ surjects onto A.*

v) *If $(\tilde{X}, X)$ is in some component F_0 of $P^{-1}(A)$ such that $\bigcup_{(\tilde{X}, X) \in F_0} \Sigma(X) = A$ and $(\tilde{X}, X)$ is smooth or generic in F_0, then $\mathbf{P}(\tilde{X}, X)$ is smooth of dimension 2 at 0.*

Summarizing the above (and using 3.6), we can write $P^{-1}(A) = P^\lambda \cup P^s$ where P^λ is the union of the components of $P^{-1}(A)$ pointwise fixed by λ and P^s is the union of the components F_0 such that $\bigcup_{(\tilde{X}, X) \in F_0} \Sigma(X) = A$. Let $R(A)$ be the subvariety of $P^{-1}(A)$ parametrizing Prym-curves such that $T_0 \mathbf{P}(\tilde{X}, X)$ is not of dimension 2 and let $S(A)$ be the complement of $R(A)$ in $P^{-1}(A)$. Then $R(A)$ contains P^λ and $S(A)$ is contained in P^s. If $\mathrm{Sm}(A) \subset P^{-1}(A)$ is the subvariety parametrizing smooth Prym-curves, then $R(A) \cap \mathrm{Sm}(A) = P^\lambda \cap \mathrm{Sm}(A)$ and $S(A) \cap \mathrm{Sm}(A) = P^s \cap \mathrm{Sm}(A)$.

Proof. i) If $(\tilde{X}, X)$ is fixed by λ, then some translate $t_{-x}^*\Sigma(X)$ of $\Sigma(X)$ is contained in Θ, so $x \in \Theta$ and the projectivized Zariski tangent space to Θ at x contains the tangent cone at 0 to $\Sigma(X)$ which generates $\mathbb{P}T_0 A$ because it contains χX_n. Hence Θ is singular at x. By [B1], the subvariety of $\mathscr{A}_4$ parametrizing ppav's with singular theta divisor is the union of $\bar{\mathscr{J}}_4$ and θ_{null}, hence $A \in \theta_{\text{null}}$. By 3.3 there are smooth Prym-curves $(\tilde{W}, W)$ in $P^{-1}(A)$; as, by [M2], pp. 344–347, all singular points of Θ come from even vanishing theta-nulls on $(\tilde{W}, W)$, the singular points have order 2. Hence $t_x^*\Theta$ is a symmetric theta divisor which is singular at 0 and contains $\Sigma(X)$.

The last assertion is a consequence of the definition of $\tilde{X}_\lambda$.

ii) If $\mathbf{P}(\tilde{X}, X)$ is not smooth of dimension 2 at 0 for $(\tilde{X}, X)$ smooth, then χX is contained in a quadric ([B2], p. 381).

First consider the case where χX is contained in a singular quadric q. Then, as X is not trigonal, the ruling of q cuts a g_4^1 on X which we denote by g_0. So $|2g_0| = |K_X + \eta|$ by assumption. If we put $h_0 = |K_X - g_0|$, then $|2h_0| = |K_X + \eta|$ and $h_0 = |g_0 + \eta| \neq g_0$. So we obtain a second singular quadric $q' \neq q$ which contains χX and whose ruling cuts the divisors of h_0 on X. It follows that $E = \chi X$ is an elliptic quartic curve and $\chi : X \to E$ is of degree 2. Let δ be the divisor class on E associated to the double cover χ. By adjunction, $\chi^*|\delta| = |K_X|$. As K_X and $K_X + \eta$ are linearly equivalent to pullbacks of divisors from E and $\chi^* : JE \to JX$ is injective, η is the pullback of a point $\bar{\eta}$ of order 2 in JE. Choose $\bar{g}_1 \in W_2^1(E)$ such that $|2\bar{g}_1| = |\delta|$ and put $\bar{g}_2 = |\bar{g}_1 + \bar{\eta}|$. Then $g_1 = \chi^*\bar{g}_1$ and $g_2 = \chi^*\bar{g}_2$ are two even vanishing theta-nulls on X with $g_1 - g_2 \equiv \eta$ and $|g_1 + g_2| = |K_X + \eta|$. So there is a nonsingular quadric containing E whose rulings cut the divisors of $\bar{g}_1, \bar{g}_2$ and g_1, g_2 on E and X respectively. The linear system $|\pi^* g_1| = |\pi^* g_2|$ on $\tilde{X}$ gives a vanishing theta-null on A.

Now suppose that χX is contained in a smooth quadric q. If χX has degree 4, we are in the previous case, so suppose that χX has degree 8. As X is not trigonal, the rulings of q cut two distinct g_4^1's on X, say g_1 and g_2. So

$$|K_X + \eta| = |g_1 + g_2|.$$

Write $h_i = |K_X - g_i|$ for $i = 1, 2$. Then

$$|h_1 + h_2| = |2K_X - g_1 - g_2| = |K_X + \eta| = |g_1 + g_2|.$$

Hence divisors of h_i give quadrisecants to χX which have to be contained in q. It follows that they are cut by the rulings of q. Hence $h_i = g_i$ because $h_i \neq g_j$ for $i \neq j$ as $|K_X + \eta| = |g_1 + g_2| \neq |K_X|$. So X has two vanishing theta-nulls which are also even as $|g_i + \eta| = g_j$ $(i \neq j)$. As before $A \in \theta_{\text{null}}$.

We have $|\pi^* g_1| = |\pi^* g_2|$ and $h^0(\tilde{X}, \pi^* g_1) = 4$. So, in the parametrization of A and Θ given in 1.2, we have $\pi^* g_1 + \Sigma(X) \subset \Theta$. So, by i), $(\tilde{X}, X)$ is fixed by λ.

iii) By 3.3, we only need to consider the case where the generic elements of F_0 and F_1 are smooth and the case where, for instance, the generic elements of F_0 are of type $(1, 1)$ and the generic elements of F_1 are smooth. We will prove iii) in the second case, the first case is similar and easier and is left to the reader. Choose a generic $(\tilde{X}, X) \in F_0$. There is a component W_0 of $W_4^1(X)$ such that, for $g \in W_0$, one of the Prym-curves tetragonally related to $(\tilde{X}, X)$ through g, say $(\tilde{Y}, Y)$, is in F_1. Choosing g generic, we can assume that Y is smooth by 3.6. As in 2.4 and because, by 3.6, every g_4^1 on X restricts to g_2^1's on the components of X, the intersection of every component $\Sigma_0(X)$ of $\Sigma(X)$ with $\Sigma(Y)$ contains a curve: the union (over W_0) of these curves is $\Sigma(Y)$. Hence $\Sigma(Y) \subset \bigcup_{F_0} \Sigma_0(X)$. We claim that we also have $\Sigma(X) \subset \bigcup_{F_1} \Sigma(Y)$. For $i = 1$ or 2, the image of $X_i^{(2)}$ in A is contained in $\bigcup_{F_1} \Sigma(Y)$ because the g_2^1's to which g restricts on the components of X vary with g. Now a moment of reflexion will convince the reader that the intersection of $\Sigma(Y)$ with $\Sigma_{12}(X)$ actually determines g hence must vary with g as well. So $\Sigma_{12}(X)$ is also contained in $\bigcup_{F_1} \Sigma(Y)$. This is true for every generic pair of tetragonally related curves $\{(\tilde{X}, X), (\tilde{Y}, Y)\} \in F_0 \times F_1$ so $\bigcup_{F_0} \Sigma(X) = \bigcup_{F_1} \Sigma(Y)$.

iv) In this part, let $\Sigma'(X)$ be $\Sigma(X)$ if $(\tilde{X}, X)$ is smooth and $\Sigma_{12}(X)$ if $(\tilde{X}, X)$ is of type $(1, 1)$. It is enough to show that there is a point $[p, q] \in \Sigma'(X)$ such that the tangent space to $\Sigma'(X)$ at $[p, q]$ is transverse to the image $T_{\mathbf{P},X}$ in $T_0 A$ of $T_0 \mathbf{P}(\tilde{X}, X)$. This is equivalent to: there is a point $[p, q] \in \Sigma'(X)$ such that $\langle p, q \rangle$ does not intersect the line $\mathbb{P} T_{\mathbf{P},X}$.

When $(\tilde{X}, X)$ is smooth, this is clear since there is no line in $\mathbb{P} T_0 A$ which intersects all the secants to χX.

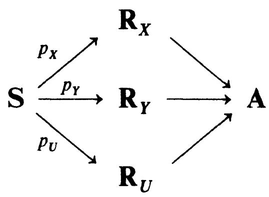
When $(\tilde{X}, X)$ is of type $(1, 1)$, χX_n is the union of two lines l_1, l_2 in $\mathbb{P} T_0 A$ (the images of X_1 and X_2) which do not intersect each other. There is no line in $\mathbb{P} T_0 A$ which intersects all the lines which intersect both l_1 and l_2.

v) Suppose that $(\tilde{X}, X) \in F_0$ for some component F_0 of $P^{-1}(A)$ with

$$\bigcup_{(\tilde{U}, U) \in F_0} \Sigma(U) = A .$$

If X is smooth, it follows from i) and ii) that $\mathbf{P}(\tilde{X}, X)$ is smooth of dimension 2 at 0. Otherwise, $(\tilde{X}, X)$ is generic hence of type $(1, 1)$ by 3.3. Let F_1 be as before. The generic elements of F_1 are smooth (3.6) and, by ii) and iii), $\mathbf{P}(\tilde{Y}, Y)$ is smooth of dimension 2 at 0 for smooth Prym-curves $(\tilde{Y}, Y)$ in F_1. We will show that this implies that $\mathbf{P}(\tilde{X}, X)$ is smooth of dimension 2 at 0.

Let $\{(\tilde{X}, X, g_X), (\tilde{Y}, Y, g_Y), (\tilde{U}, U, g_U)\}$ be a tetragonally related triple. Suppose that $((\tilde{X}, X), (\tilde{Y}, Y))$ is an element of $F_0 \times F_1$ such that $(\tilde{Y}, Y)$ is generic in F_1 and g_Y is a generic element (i.e., in some dense open subset) of $W_4^1(Y)$. Let $\mathbf{R}_X, \mathbf{R}_Y, \mathbf{R}_U$ be the versal formal deformation spaces of, respectively, $(\tilde{X}, X, g_X), (\tilde{Y}, Y, g_Y)$ and $(\tilde{U}, U, g_U)$ and let $\mathbf{S} \subset \mathbf{R}_X \times \mathbf{R}_Y \times \mathbf{R}_U$ be the subspace parametrizing tetragonally related triples of Prym-curves with g_4^1's. The projections induce morphisms $p_X : \mathbf{S} \to \mathbf{R}_X, p_Y : \mathbf{S} \to \mathbf{R}_Y, p_U : \mathbf{S} \to \mathbf{R}_U$ which are quasi-finite of degree 2 and finite on an open set containing the origin. Consider the commutative diagram



where the maps on the right are induced by the Prym map and $\mathbf{A}$ is as in 3.4. Then the differential of the map $\mathbf{R}_Y \to \mathbf{A}$ is surjective with 3-dimensional kernel at 0. Also, p_Y is unramified at 0 because $(\tilde{U}, U)$ is smooth by 3.6 hence distinct from $(\tilde{X}, X)$ which is singular. So the differential of the composition $\mathbf{S} \to \mathbf{A}$ of the above maps is surjective with 3-dimensional kernel at 0. This implies that the differential of $\mathbf{R}_X \to \mathbf{A}$ is surjective at 0. Hence the differential of $\mathbf{R} \to \mathbf{A}$ (where $\mathbf{R}$ is as in 3.4) is surjective at 0 and $\mathbf{P}(\tilde{X}, X)$ is smooth of dimension 2 at 0. Q.E.D.

Corollary 3.18. *Suppose that* $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. *Then* $P^{-1}(A)$ *is* 2-*dimensional.*

Proof. If $P^{-1}(A)$ has a component F_0 of dimension $\geqq 3$, then the generic element $(\tilde{X}, X)$ of F_0 is smooth (3.3) and, by 3.17, $(\tilde{X}, X) = (\tilde{X}_\lambda, X_\lambda)$ and $\Sigma(X) \subset \Theta$. Take $x \in \Theta$ generic. There is a 2-dimensional family of elements $(\tilde{X}, X)$ of F_0 with $x = [p, q] \in \Sigma(X)$. Hence, by 3.2, there is a 2-dimensional family of Prym-embedded curves in $\Theta . \Theta_x$. By [BD1], prop. 1, $\Theta . \Theta_x$ is isomorphic to a connected component of the variety of liftings of the divisors of $|K_X - \pi p - \pi q|$ in $\tilde{X}$. By the genericity assumption on x, the image of X in $\mathbb{P}^2$ given by $|K_X - \pi p - \pi q|$ has no triple tangents and no cusps, so, by [W1], pp. 103–108, $\Theta . \Theta_x$ is smooth. The following computation shows that the family of deformations of $\tilde{X}$ in $\Theta . \Theta_x$ is at most 1-dimensional and hence yields a contradiction.

Let $\mathscr{N}$ be the normal bundle of $\tilde{X}$ in $\Theta . \Theta_x$ and let $\mathscr{T}$ be the tangent bundle of $\Theta . \Theta_x$. By the exact sequence

$$0 \to (\omega_{\tilde{X}})^* \to \mathscr{T} \to \mathscr{N} \to 0$$

and adjunction for $\Theta . \Theta_x$, we have $\mathscr{N} \cong \omega_{\tilde{X}}(-\Theta - \Theta_x)$: this is a line bundle of degree 0 hence $h^0(\mathscr{N}) \leqq 1$. Q.E.D.

4. Intersections of 2 translates of Θ

In this section our aim is to compute the number of Prym-embedded curves in a generic intersection of 2 translates of Θ for $A \in \mathscr{A}_4 \backslash (\bar{\mathscr{J}}_{\text{hyp}} \cup \mathscr{A}_{4\text{dec}})$.

Proposition 4.1. *For* $A = JC$ *the jacobian of a non-hyperelliptic smooth curve* C *of genus* 4, *a generic intersection of two translates of* Θ *contains* 27 *Prym-embedded curves which are not images of each other by a translation or* $-$ id.

Proof.

4.2. The fiber of the Prym map at $A = JC$ splits into two components each isomorphic to the quotient of $C^{(2)}$ by the automorphism group of C (after blowing down the locus of "elliptic tails" to the diagonal of $C^{(2)}$, see [B1], pp. 170–175, 183 and [DS], § 4):

One component is the space whose elements $(\tilde{X}, X)$ are trigonally related to C (see 1.4). The elements of the other component (off the inverse image of the diagonal of $C^{(2)}$) are of the form $(\tilde{C}_{pq}, C_{pq})$, where C_{pq} is obtained from C by identifying the two points $p, q \in C$ and $\tilde{C}_{pq}$ is obtained by taking 2 copies of C and identifying p on one copy with q on the other.

4.3. View A as $\mathrm{Pic}^3 C$ and take $\Theta = W_3$ (the effective divisor classes). Then, for each trigonal Prym-curve $(\tilde{X}, X)$ obtained via Recillas' construction from $g_4^1 \in W_4^1(C)$, we see immediately that we have the following Prym-embeddings of $\tilde{X}$ in Θ and only these

$$\{s, t\} \mapsto |s + t + p_0|$$

or

$$\{s, t\} \mapsto |K_C - (s + t + p_0)|$$

for all $p_0 \in C$. The family of these embeddings is parametrized by $\tilde{C}_{pq}$. Analogously, the embeddings of $\tilde{C}_{pq} = C_1 \cup_{p=q} C_2 (C_i \cong C)$ are given by:

$$C_1 \ni s \mapsto |s_1 + s_2 + s|,$$
$$C_2 \ni t \mapsto |K_C - t - t_1 - t_2|$$

for all s_1, s_2, t_1, t_2 such that $|s_1 + s_2 + t_1 + t_2| = |K_C - p - q|$. So we can define λ on $P^{-1}(A)$ in the same way as in 3.11 and, for $(\tilde{X}, X)$ trigonal, $(\tilde{X}_\lambda, X_\lambda) = (\tilde{C}_{pq}, C_{pq})$ where $p + q = K_C - g_4^1$.

4.4. We have a surjective morphism of generic degree 6:

$$C^{(2)} \times C^{(2)} \to A,$$
$$(a + b, a' + b') \mapsto |a + b - a' - b'|.$$

If we fix two points a, b on C, the homology class of the image of $\{a + b\} \times C^{(2)}$ is $[\Theta]^2/2$. The claim easily follows now by using Pontrjagin product and formula (1) in 2.1.

4.5. We will now compute the number of Prym-embedded curves in $\Theta . \Theta_x$ for x generic in A. We will first consider trigonal curves. We use the notation of 4.2 and 4.3.

The set of effective divisor classes in $\mathrm{Pic}^1 C$ is 1-dimensional ($= C$) and, for x generic, $C + x$ does not intersect it. Hence we can assume $h^0(p_0 + x) = 0$ for all $p_0 \in C$. Equivalently, $h^0(K_C - p_0 - x) = 2$ for all $p_0 \in C$. Let $g_5^1 = |K_C - p_0 - x|$. By 4.3, we need to know when we will have: for all $s, t \in \tilde{X}$,

1) either $h^0(s + t + p_0 + x) > 0$,

2) or $h^0(K_C - s - t - p_0 + x) > 0$.

By Riemann-Roch, case 2 is equivalent to case 1 with x replaced by $-x$. By Riemann-Roch again, case 1 is equivalent to:

For all s, t, if $h^0(g_4^1 - s - t) > 0$, then $h^0(g_5^1 - s - t) > 0$.

This is possible if and only if $g_5^1 = g_4^1 + t_0$ for some t_0 in C. Or if and only if there is an element t_0 of C such that $|t_0 + p_0 + x| = |K_C - g_4^1|$.

Taking into consideration the cases 1 and 2, we see that for each of the 6 representatives $a_i + b_i - a_i' - b_i'$ $(1 \leqq i \leqq 6)$ of x in $C^{(2)} \times C^{(2)}$ we have (two) Prym-embeddings of the trigonal Prym-curves obtained from $|K_C - a_i - b_i|$ and $|K_C - a_i' - b_i'|$ in $\Theta . \Theta_x$.

Now, $a_i + b_i - a_i' - b_i' \equiv a_j + b_j - a_j' - b_j'$ means

$$a_i + b_i + a_j' + b_j' \equiv a_j + b_j + a_i' + b_i' \equiv K_C - p_{ij} - q_{ij}$$

for some $p_{ij}, q_{ij} \in C$.

We see that for all $i \neq j$ the line $l_{ij} = \langle p_{ij}, q_{ij} \rangle$ encounters simultaneously $\langle a_i, b_i \rangle$, $\langle a_i', b_i' \rangle$, $\langle a_j, b_j \rangle$ and $\langle a_j', b_j' \rangle$. In particular, the number of lines which are secant to κC and incident to $\langle a_i, b_i \rangle$, $\langle a_i', b_i' \rangle$ for some i (or incident to $\langle a_i, b_i \rangle$, $\langle a_j', b_j' \rangle$ for some i, j) is 15.

By 4.3, in the singular case, $\tilde{C}_{pq}$ embeds in $\Theta . \Theta_x$ if and only if for some s_1, s_2, t_1, t_2 such that $|s_1 + s_2 + t_1 + t_2| = g_4^1 = |K_C - p - q|$:

$$h^0(t + t_1 + t_2 + x) > 0 \quad \text{and} \quad h^0(K_C - t - s_1 - s_2 + x) > 0 \quad \text{for all } t \in C .$$

Equivalently $h^0(t_1 + t_2 + x) > 0$ and $h^0(s_1 + s_2 - x) > 0$.

So we must have $t_1 + t_2 = a_i' + b_i'$, $s_1 + s_2 = a_j + b_j$ and $\langle p, q \rangle = l_{ij}$ for some $i \neq j$.

Counting everything we obtain 27 Prym-embedded curves in $\Theta . \Theta_x$. Q.E.D.

Proposition 4.6. *If A is generic, then a generic element of A lies on exactly* 27 *non-isomorphic surfaces $\Sigma(X)$. In particular, the map $\Sigma_A \to A$ is onto.*

Proof. Let $(\tilde{X}, X)$ be a Prym-curve which is smooth or of the form $(\tilde{C}_{pq}, C_{pq})$ (see 4.2). Let $\bar{P} : \mathbf{R} \to \mathbf{A}$ be as in 3.4 and $\mathbf{P}(\tilde{X}, X)$ be as in 3.17. Choose a point a in $\Sigma(X)$. Let $\mathscr{A}$ be the formal versal deformation space of the pair (a, A) such that $a \in A$ and let $\mathscr{C}$ be the formal versal deformation space of the pair $\big(a, (\tilde{X}, X)\big)$ such that $a \in \Sigma(X)$. Pulling back we have a diagram:

$$\begin{array}{ccccc} \mathscr{C} & \subset & \mathbf{F} & \to & \mathbf{R} \\ & \searrow^{p} & \downarrow & & \bar{P} \downarrow \\ & & \mathscr{A} & \to & \mathbf{A} . \end{array}$$

The number we are interested in is the generic degree of p. Noticing that the proof of 3.16 only depends on the fact that λ is well-defined, by 4.1, we need to show that p is unramified at 0 when A is a generic jacobian and a is a generic point of A. The cotangent space to $\mathbf{R}$ at 0 is canonically isomorphic to $H^0(X, (\omega_X)^2)$ ([DS], p. 68). We have the following exact sequences of cotangent spaces:

$$0 \to H^0(X, (\omega_X)^2) \to T_0^* \mathscr{C} \to T_a^* \Sigma(X) \to 0 ,$$
$$0 \to S^2 H^0(X, \omega_X \otimes \eta) \to T_0^* \mathscr{A} \to H^0(X, \omega_X \otimes \eta) \to 0 .$$

We have a surjection:

$$H^0(X, \omega_X \otimes \eta) \twoheadrightarrow T_a^* \Sigma(X)$$

and multiplication:

$$S^2 H^0(X, \omega_X \otimes \eta) \to H^0(X, (\omega_X)^2)$$

(this is injective if and only if $\bar{P}$ has maximal rank (see [B2], p. 381)).

In case $(\tilde{X}, X)$ is a *generic* element of $\mathscr{P}_5$, then $\bar{P}$ has maximal rank at 0 and we have an isomorphism between $H^0(X, (\omega_X)^2)$ and the kernel of the composition

$$T_0^* \mathscr{A} \twoheadrightarrow H^0(X, \omega_X \otimes \eta) \twoheadrightarrow T_a^* \Sigma(X).$$

Or an isomorphism between the kernel of

$$H^0(X, \omega_X \otimes \eta) \twoheadrightarrow T_a^* \Sigma(X)$$

and

$$\frac{H^0(X, (\omega_X)^2)}{S^2 H^0(X, \omega_X \otimes \eta)}.$$

This corresponds to the deformations of χX in $\mathbb{P} T_0 A$ and means that χX moves in two directions transversal to the line $\langle \pi p, \pi q \rangle$ where $a = [p, q] \in \Sigma(X)$. Equivalently, the deformations of χX fill $\mathbb{P} T_0 A$ (because a can take any generic value in $\Sigma(X)$).

These remain valid when $A = JC$ is a generic jacobian. For $(\tilde{X}, X) = (\tilde{C}_{st}, C_{st})$ a generic singular Prym-curve, the projectivized kernel of the codifferential of $\bar{P}$ at 0 is the set of quadrics in $\mathbb{P} T_0 A$ containing $\kappa C \cup \langle s, t \rangle$ ([DS], p. 72). For s and t generic no quadric in $\mathbb{P} T_0 A$ contains $\langle s, t \rangle \cup \kappa C$, hence $\bar{P}$ has maximal rank. The lines $\langle s, t \rangle$ fill $\mathbb{P}^3$, hence the deformations of $\chi C_{st} = \langle s, t \rangle \cup \kappa C$ fill $\mathbb{P}^3$. Letting λ act, we obtain the same result for $(\tilde{X}, X)$ a generic trigonal element of $P^{-1}(JC)$.

The fact that the map $\Sigma_A \to A$ is onto follows from the fact that it is generically finite and that Σ_A is complete by the properness of P (see [B1]). Q.E.D.

From now on in this section, we suppose that A is any element of $\mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$.

Corollary 4.7. *Suppose that* $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. *Let* $(\tilde{X}, X)$ *be an element of* $P^{-1}(A)$. *Suppose that* $(\tilde{X}, X)$ *is smooth or of type* $(1, 1)$ *and that* $\mathbf{P}(\tilde{X}, X)$ *is smooth of dimension* 2 *at* 0. *The map* p *in* 4.6 *is unramified generically on* $\Sigma'(X)$ (*see the proof of* 3.17iv) *for the notation*).

Proof. First suppose $(\tilde{X}, X)$ smooth. As in the proof of 4.6, we need to show that, if F_0 is the component of $P^{-1}(A)$ containing $(\tilde{X}, X)$, then $\bigcup_{F_0} \chi Y = \mathbb{P} T_0 A$. This is a consequence of 3.17iv) and the fact that χX is the tangent cone at 0 to $\Sigma(X)$.

If $(\tilde{X}, X)$ is of type $(1, 1)$, the proof is analogous to that of 3.17v).

Corollary 4.8. *Suppose that $A \in \mathscr{A}_4 \backslash \mathscr{J}_4$. Then a generic element of A lies on exactly 27 surfaces $\Sigma(X)$ associated to elements $(\tilde{X}, X)$ of P^s (see 3.17) which are not images of each other by λ.*

Proof. Since $\Sigma_A \to A$ is onto for A generic by 4.6, it is onto for all A. Hence a generic element x of A lies on some surface of the form $\Sigma(X)$. Since x is generic, all the Prym-curves $(\tilde{X}, X)$ such that $x \in \Sigma(X)$ are (generic) elements of P^s and are not fixed by λ. Hence, by 3.17, $\mathbf{P}(\tilde{X}, X)$ is smooth of dimension 2 at 0. So, by 4.7, since x is generic, p is unramified at $x \in \Sigma(X)$. Since x is generic in A, the pair (x, A) is automorphism-free, therefore the coarse moduli space parametrizing pairs (y, B) with $y \in B \in \mathscr{A}_4$ is smooth at (x, A). Also, for each $(\tilde{X}, X)$ such that $x \in \Sigma(X)$, the coarse moduli space parametrizing pairs $(y, (\tilde{Y}, Y))$ with $y \in \Sigma(Y) \in \mathscr{P}_5$ is smooth at $(x, (\tilde{X}, X))$: If $(\tilde{X}, X)$ is smooth, this is clear since x is generic in $\Sigma(X)$ and therefore $(x, (\tilde{X}, X))$ is automorphism-free. If $(\tilde{X}, X)$ is of type $(1, 1)$, then by 3.17iv) x is a generic element of $\Sigma_{12}(X)$ and again $(x, (\tilde{X}, X))$ is automorphism-free. So we can deduce from 4.6 that x lies on exactly 27 surfaces $\Sigma(X)$ with $(\tilde{X}, X) \in P^s$. Since $P^{-1}(A)$ is two-dimensional by 3.18, the union of the sets $\Sigma(X) \cap \Sigma(X_\lambda)$ is at most a divisor in A so a generic x does not lie on it. Therefore, since x is generic, the surfaces $\Sigma(X)$ on which x lies are not images of each other by λ. Q.E.D.

Corollary 4.9. *Suppose that $A \in \mathscr{A}_4 \backslash \mathscr{J}_4$. Then a generic intersection of two translates of Θ contains 27 Prym-embedded curves which cannot be obtained from each other by a translation,* $-\mathrm{id}$ *or by λ.*

Proof. By 3.16, for $a \in A$ and $(\tilde{X}, X) \in P^{-1}(A)$, $\tilde{X}$ admits a Prym-embedding in $\Theta . \Theta_a$ if and only if $a \in \Sigma(X_\lambda)$. Now the corollary follows from 4.8.

5. The pencil l_X of Γ_{00}-divisors associated to $(\tilde{X}, X)$

In 5.1–5.5, we prove a few preliminaries on double solids. We will use these to define the pencils l_X and to study their incidence correspondence.

Lemma 5.1. *Suppose that Z is a double solid such that $X = X_1$ is the discriminant curve of the projection of Z from a node p_1 of B (cf. 1.5), then*

$$\Sigma(X) \subset D_Z$$

(*cf.* 2.1).

Proof. Consider two generic elements p, q of $\tilde{X}$. Let l_p, l_q be the strict transforms in $\tilde{Z}$ of the lines in Z corresponding to p, q; L^1, L^2 the two rulings of the exceptional quadric $\mathscr{Q}_1$ in $\tilde{Z}$ above p_1 and L_p^1 the line of the ruling L^1 which passes through $\mathscr{Q}_1 \cap l_p$. The restriction of the ideal sheaf of $\mathscr{Q}_1$ to $\mathscr{Q}_1$ is $\mathscr{O}_{\mathscr{Q}_1}(1, 1)$. So to obtain a conic (cf. 1.5) in $\tilde{Z}$ we have to add the two rulings of $\mathscr{Q}_1$ to $l_p \cup l_q$. The surface $\Sigma(X)$ is equal to the image by AJ (with base curve the inverse image in $\tilde{Z}$ of a line in $\mathbb{P}^3$ tangent to B) of

$$\{l_p \cup L_p^1 \cup L_q^2 \cup l_q : p, q \in \tilde{X}\}$$

because both surfaces are equal to the union of the Prym-embeddings of $\tilde{X}$ which pass through the origin.

Let V be the plane spanned by $\tilde{\pi}(l_p)$, $\tilde{\pi}(l_q)$ in $\mathbb{P}^3$ and $L_{pq} = \mathscr{Q}_1 \cap \tilde{\pi}^{-1}(V)$. Then L_{pq} is an element of the linear system $\mathcal{O}_{\mathscr{Q}_1}(1,1)$. Hence

$$AJ(l_p \cup L_p^1 \cup L_q^2 \cup l_q) = AJ(l_p \cup L_{pq} \cup l_q).$$

We claim that $l_p \cup L_{pq} \cup l_q$ is in the linear equivalence class of a smooth rational conic in $\tilde{\pi}^{-1}(V)$: this will prove 5.1.

Indeed, as in 2.5, $\tilde{V} = \tilde{\pi}^{-1}(V)$ is a Del Pezzo surface isomorphic to the blow up of $\mathbb{P}^2$ at 7 points q_i. The 7 points are not in general position but, as $\pi^{-1}(V)$ has a node, 3 of them, say q_1, q_2, q_3, lie on a line whose strict transform in $\tilde{V}$ is precisely L_{pq}. The arithmetic genus of $C = l_p \cup L_{pq} \cup l_q$ is 0. The morphism $\tilde{V} \to V \cong \mathbb{P}^2$ is given by the anticanonical sheaf $\omega_{\tilde{V}}^{-1}$ of $\tilde{V}$ ([Dm], pp. 67–68). So, by adjunction, the restriction of $\mathcal{O}_{\tilde{V}}(C)$ to each component of C is the trivial sheaf, hence $h^0(C, \mathcal{O}_C(C)) = 1$. From the cohomology sequence associated to the exact sequence

$$0 \to \mathcal{O}_{\tilde{V}} \to \mathcal{O}_{\tilde{V}}(C) \to \mathcal{O}_C(C) \to 0$$

one then deduces that C moves in a complete linear system of (projective) dimension 1. Since the arithmetic genus of C is 0, the generic member of this pencil is a smooth rational conic in $\tilde{Z}$. Q.E.D.

Lemma 5.2. *With the notation of* 1.5, *for a generic double solid Z and any discriminant curve X_{ij} for Z*:

$$\Sigma(X_{ij}) \not\subset D_Z.$$

Proof. Suppose that for every $Z \in \mathscr{X}$, $\Sigma(X_{12}) \subset D_Z$. Take Z generic in $\mathscr{X}$. Then, in particular, the tangent cone at 0 to $\Sigma(X_{12})$, i.e., the Prym-canonical embedding χX_{12} of X_{12}, is contained in the tangent cone at 0 to D_Z, i.e., by 2.1.3, $\varrho(B)$. Let $\tilde{\mathscr{X}}$ be the space parametrizing triples (Z, p_1, p_2) such that $Z \in \mathscr{X}$ and $p_1 \neq p_2$ are two nodes of B. Then $\tilde{\mathscr{X}}$ projects dominantly to $\mathbb{P}^3 \times \mathbb{P}^3$ with fibers birational to $\mathbb{P}^{11}$, hence $\tilde{\mathscr{X}}$ is irreducible. Since the condition "χX_{12} is contained in $\varrho(B)$" is a closed condition and $\tilde{\mathscr{X}}$ is irreducible, it is enough to check that it cannot hold for one particular choice of A and $Z \in \mathscr{X}_A$.

5.3. The double solids Z for which B is the ramification locus of ϱ have the same intermediate jacobian A ([V], p. 945). They form an irreducible component $\mathscr{W}$ of $\mathscr{X}_A$. Let Z be generic in $\mathscr{W}$. By [V], p. 935, $X_1 \subset B \subset \mathbb{P}^3$ is embedded by the linear system $|K_{X_1} - p_1|$ (where we identify $p_i \in \mathbb{P}^3$ with the corresponding point on X_1). Denote by $\bar{X}_1$ the image of X_1 in $\mathbb{P}^3$ by the linear system $|K_{X_1} - p_2|$. Then, by [V], p. 945, there exists a unique element Z' of $\mathscr{W}$ such that $\bar{X}_1$ is a discriminant curve for Z'. The branch locus B' of $Z' \to \mathbb{P}^3$ is not projectively equivalent to B: As $\bar{X}_1$ and X_1 are not projectively equivalent, if B were projectively equivalent to B', then a discriminant curve X_i $(i \neq 1)$ of B would have to be projectively equivalent to $\bar{X}_1$. Hence X_i would be abstractly (as opposed to projectively) isomorphic to X_1. However, by [V], p. 943, each discriminant curve X_j of B is determined, up to abstract isomorphism by the moduli in C of the nodes p_k $(k \neq j)$ of B (here C is the twisted cubic through $p_1, \dots, p_6$). Hence, generically on $\mathscr{W}$, the discriminant curves of B are nonisomorphic to each other and X_i cannot be isomorphic to X_1.

Now we show that the assumption $\chi X_{12} \subset B$ (for all Z in $\mathscr{W}$) implies that B and B' are projectively equivalent and thus obtain a contradiction.

The g^1_4 of X_1 relating $(\tilde{X}_1, X_1)$ to $(\tilde{X}_2, X_2)$ is $|K_{X_1} - 2p_1 - 2p_2|$. Also, one of the discriminant curves of B' is tetragonally related to $\bar{X}_1$ through $|K_{X_1} - 2p_1 - 2p_2|$ because the image of p_1 is a cusp of the plane representation of X_1 obtained by projection from $p_2 \in B'$. So one of the discriminant curves of B' is X_2 or X_{12}. Hence $\chi X_1, \chi X_2$ and χX_{12} are also contained in $\varrho'(B')$ (ϱ' is the map given by the linear system of quadrics through the nodes of B'). Hence the intersection $\varrho(B) \cap \varrho'(B')$ has degree greater than 16 and $\varrho(B) = \varrho'(B')$ as these are irreducible. However, B is determined, up to projective equivalence, by $\varrho(B)$:

For $Z \in \mathcal{W}$, $\varrho(B)$ is the Kummer variety of the curve of genus 2 obtained as the double cover of C branched at $p_1, \dots, p_6$. Hence, up to projective equivalence, the data of $\varrho(B)$ is equivalent to that of $p_1, \dots, p_6$. Given $K = \varrho(B)$, construct the minimal desingularization ([C1], pp. 112–113) $\hat{\mathbb{P}}^3$ of the double cover of $\mathbb{P}^3$ branched along K. Let $\hat{\varrho} : \hat{\mathbb{P}}^3 \to \mathbb{P}^3$ be the lift of the (branched) covering map. The surface K contains 16 conics. Through each double point t of K, 6 conics $C_{t1}, \dots, C_{t6}$ pass. It is easily seen that we obtain $B \subset \mathbb{P}^3$ by blowing down the inverse images of $C_{t1}, \dots, C_{t6}$ in $\hat{\varrho}^{-1}(K)$ (the 6 curves are blown down to $p_1, \dots, p_6$ and, by the uniqueness result of [V] (p. 945), up to projective equivalence, the surface B that we obtain does not depend on the choice of t).

Analogously, B' is determined by $\varrho'(B')$ up to projective equivalence. Hence B and B' are projectively equivalent. Contradiction. Q.E.D.

Recall that in 1.5 we introduced the group $\mathcal{G}$ of birational transformations of $\mathbb{P}^3$ and described its orbits in $\mathcal{Z}$. We have the following:

Proposition 5.4. *Let $\{Z_1, \iota Z_1, \dots, Z_{16}, \iota Z_{16}\}$ be the orbit under $\mathcal{G}$ of an element $Z = Z_1$ of $\mathcal{Z}$. Then*

i) *$E_{Z_j} = E_{\iota Z_j}$ for all j and*

ii) *if Z is generic, then $E_{Z_j} \neq E_{Z_k}$ for all $j \neq k$.*

*(By "=" (resp. "$\neq$") we mean that they are (resp. are not) images of each other by a translation or $-$*id *in $A = JZ$.)*

Proof. We refer to 1.5 for the definitions of $\varrho, \iota, \iota_{\mathcal{F}}$.

i) For a line l in $\mathbb{P}^3$ (bitangent to ιB but otherwise generic) $\iota(l)$ is a curve of degree 7 with double points at the p_i's and everywhere tangent to B (with even order of contact).

So it is sufficient to show that, for each generic line l' in $\tilde{Z}$, there is a curve C' in $\tilde{Z}$ (birational, via $\tilde{p}$, to its projection C in $\mathbb{P}^3$, such that C is a septic everywhere tangent to B with even order of contact and has double points at the p_i's) such that $l' \cup C'$ is a complete intersection in $\tilde{Z}$. (Then the image of $l' \cup C'$ under AJ is constant when l' varies.)

Let l be $\pi(l')$, q the unique quadric in $\mathbb{P}^3$ containing l and the p_i's. Let $q' \neq q$ be an element of the pencil of quadrics containing the p_i's and the two points of contact of l with B. In the pencil of quartics spanned by B and $(q')^2$ there exists a quartic B' containing l. Let B, B' and q' also denote the equations of B, B' and q' and of their strict transforms in $\tilde{Z}$. Write $B' = B - r^2.(q')^2$ for a complex number r. Then, as $B = M^2$ in $\tilde{Z}$, we have $B' = (M - r.q').(M + r.q')$ in $\tilde{Z}$. Hence the strict transform of B' in $\tilde{Z}$ is the union of two surfaces, one of which, say Q, contains l'. Then $Q \cap \tilde{p}^{-1}(q)$ is a complete intersection of the required type.

ii) Consider a subset $\mathscr{F} \subset \{p_1, \ldots, p_6\}$ of cardinality 4, say $\mathscr{F} = \{p_1, \ldots, p_4\}$, and let $Z' = \iota_{\mathscr{F}} Z$. The plane spanned by p_2, p_3, p_4 is blown down by $\iota_{\mathscr{F}}$ to a double point p'_1 of Z' and lines through p_1 in Z go to lines through p'_1 in Z'. So, if X is the discriminant curve for the projection of B from p_1, then X is also the discriminant curve for the projection of $B' = \iota_{\mathscr{F}} B$ from p'_1.

Projecting from p_1 and p'_1 we have 2 plane representations of X as a sextic with 5 nodes.

Suppose that these plane representations are given by the linear systems $|K - p - q|$ and $|K - p' - q'|$. By 2.3, for some choice of liftings of p, q, p', q' say $\bar{p}, \bar{q}, \bar{p}', \bar{q}'$ in $\tilde{X}$, we have:

$$E_Z = \Theta . \Theta_{[\bar{p}, \bar{q}]}, \quad E_{Z'} = \Theta . \Theta_{[\bar{p}', \bar{q}']} .$$

If $\{p, q\} \neq \{p', q'\}$, then E_Z and $E_{Z'}$ cannot be images of each other by a translation or $-$ id. We show below that the two plane representations are not projectively equivalent, hence $\{p, q\} \neq \{p', q'\}$.

The two quartics B and B' are birationally equivalent. Denote their minimal desingularizations by $\tilde{B}$ and $\tilde{B}'$ respectively. Then, as $\tilde{B}$ and $\tilde{B}'$ are K3 surfaces, in particular they are minimal with unique minimal model. Hence, as they are birationally equivalent, they are isomorphic. Let E_i, E'_i denote the (-2)-curves in $\tilde{B}$ blown down to the nodes of B and B' respectively and let H and H' be the linear systems on $\tilde{B}$ associated to the compositions

$$\tilde{B} \to B \subset \mathbb{P}^3,$$
$$\tilde{B} \to B' \subset \mathbb{P}^3.$$

Then, for all four-tuples of indices (i, j, k, l) such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$, we have

$$H \equiv E'_i + E_j + E_k + E_l \quad \text{and} \quad H' \equiv E_i + E'_j + E'_k + E'_l .$$

Suppose that the two plane representations of X are projectively equivalent. Then, by 2.2, and with the notation of 2.2

– either $H = H'$ and $E_1 = E'_1$. So from the equations above, we get

$$H \equiv E_1 + E_2 + E_3 + E_4$$

which is impossible because $H.H = 4$ and $H.E_i = 0$ for all i.

– or E_1 and E'_1 are the two components of the inverse image of C_t in $\tilde{B}$. This implies $E_1 . E'_1 = 6$. From the equations above we get $E_1 . E'_1 = 0$ so this case is ruled out as well. Q.E.D.

As a first consequence we have

Proposition 5.5. *For any double solid Z and discriminant curve X_i for Z, we have $((\tilde{X}_i)_\lambda, (X_i)_\lambda) = (\tilde{X}_i', X_i')$. For a generic double solid Z in $\mathscr{Z}$, the Prym-embedded curves in E_Z are Prym-embeddings of*

$$\tilde{X}_i, (\tilde{X}_i)_\lambda = \tilde{X}_i', \tilde{X}_{ij}.$$

Proof. First, as $\{(\tilde{X}_i, X_i), (\tilde{X}_{ij}, X_{ij})\}$ and $\{(\tilde{X}_i', X_i'), (\tilde{X}_{ij}', X_{ij}')\}$ are tetragonally related to $(\tilde{X}_j, X_j)$ through opposite g_4^1's, we have, by 3.13,

- either $((\tilde{X}_i)_\lambda, (X_i)_\lambda) = (\tilde{X}_i', X_i')$
- or $((\tilde{X}_i)_\lambda, (X_i)_\lambda) = (\tilde{X}_{ij}', X_{ij}')$.

The second case is ruled out as, generically, for $j \neq k$, $X_{ij}' \not\cong X_{ik}'$.

Secondly, we have a Prym-embedding of $\tilde{X}_i$ in E_Z for each choice of a ruling of the exceptional quadric above p_i in $\tilde{Z}$. Looking at the action of the birational involution ι (1.5) we see that the (X_i')'s are discriminant curves for the projections of ιZ from its nodes. As $E_Z = E_{\iota Z}$, we have two Prym-embeddings of $(\tilde{X}_i)_\lambda = \tilde{X}_i'$ in E_Z. By 2.5, we also have 2 Prym-embeddings of $\tilde{X}_{ij}$. So we have 27 nonisomorphic Prym-embedded curves in E_Z. By 3.16, $\tilde{X}_\lambda$ has a Prym-embedding in E_Z if and only if $(\tilde{X}, X)$ is in the fiber of the map p (in the proof of 4.6) at $x \in A$ such that $\Theta . \Theta_x = E_Z$. Since $\mathscr{C}$ (4.6) is irreducible and $\mathscr{A}$ (4.6) is smooth, the fibers of p are either positive-dimensional or finite of cardinality at most 27. So either there is a one-dimensional family of Prym-embedded curves in E_Z or their number is less than or equal to 27.

Let $\mathscr{E}_A \subset A = JZ$ be the closure of the subvariety of points x such that $\Theta . \Theta_x = E_{Z'}$ for some element Z' of $\mathscr{Z}_A$. The variety $\mathscr{E}_A$ is a divisor because $\mathscr{Z}_A \to \mathscr{E}_A / \pm 1$ is generically finite: by the irreducibility of $\mathscr{Z}$ (and $\mathscr{Z}_A$ for A generic), an easy way to see this is to specialize to the case considered in the proof of 5.3. The map $\mathscr{W} \to \mathscr{E}_A$ is generically finite because the closure of the image of $\mathscr{W}$ is the closure of the union of the diagonals of $\Sigma(X)$ (for $(\tilde{X}, X)$ a smooth discriminant (Prym-)curve of some element of $\mathscr{W}$) which is a divisor as it is the image of $\bigcup \{p - \sigma p : p \in \tilde{X}\}$ by multiplication by 2.

Generically, E_Z cannot contain a one-dimensional family of curves: otherwise the inverse image of $\mathscr{E}_A$ in Σ_A (1.2) would be of dimension $\geqq 4$, hence equal to Σ_A because Σ_A is irreducible of dimension 4. This would mean that the map $\Sigma_A \to A$ is not surjective, which is impossible by 4.6. Q.E.D.

We can now define l_X for every element of $P^{-1}(A)$ when A is generic. By 3.17i), for every $(\tilde{X}, X) \in P^{-1}(A)$, $(\tilde{X}, X) \not\cong (\tilde{X}_\lambda, X_\lambda)$. Also, X has at most two nodes because the family of Prym-curves with more singularities has dimension less than 10 and cannot surject onto $\mathscr{A}_4$.

Definition and Proposition 5.6. *For A generic and for each $(\tilde{X}, X) \in P^{-1}(A)$, the set of divisors of $|2\Theta|$ containing $\Sigma(X) \cup \Sigma(X_\lambda)$ is a pencil $l_X = l_{(\tilde{X}, X)}$ contained in $|2\Theta|_{00}$. The base locus of l_X is $\Sigma(X) \cup \Sigma(X_\lambda)$. The only Prym-curves $(\tilde{Y}, Y)$ with $l_Y = l_X$ are $(\tilde{X}, X)$ and $(\tilde{X}_\lambda, X_\lambda)$. If $T = T_A$ is the subvariety of $|2\Theta|_{00}$ swept by these pencils and $D : \mathscr{Z}_A \to |2\Theta|_{00}$ is the map which to a double solid Z associates D_Z (2.1), then T is the closure of the image of D and D is constant on the orbits of the group $\mathscr{G}$ (1.5).*

Proof. First suppose $(\tilde{X}, X)$ generic. There is a 1-dimensional family of double solids Z such that $X = X_1$ is a discriminant curve for the projection of Z from its node p_1. Recall (5.5) such $(\tilde{X}'_1, X'_1) = (\tilde{X}_\lambda, X_\lambda)$ so that X_λ is the discriminant curve for the projection of ιZ from p_1.

Consider the Γ_{00}-divisor D_Z associated to Z (2.1), then by 2.1.1, since A is generic, the map τ is injective. As the tangent cone at 0 to D_Z is $\varrho(B) = \varrho(\iota B)$ (2.1) we have $D_Z = D_{\iota Z}$. Hence, by 5.1, for all Z as above, D_Z contains $\Sigma(X) \cup \Sigma(X_\lambda)$.

Using Pontrjagin product and formula (1) (in 2.1) we see that the homology class of $\Sigma(X)$ is $2[\Theta]^2$. As A is generic, all Γ_{00}-divisors are irreducible and so the base locus of a pencil of Γ_{00}-divisors is a surface of homology class $4[\Theta]^2$. Hence, in particular, the set $l_X = l_{X_\lambda}$ of Γ_{00}-divisors D_Z containing $\Sigma(X) \cup \Sigma(X_\lambda)$ is at most a pencil. Generically, the set l_X is exactly a pencil because otherwise it would be a point and D_Z would contain $\Sigma(Y)$ for every $(\tilde{Y}, Y)$ tetragonally related to $(\tilde{X}, X)$: this is ruled out by 5.2. For $(\tilde{X}, X)$ not necessarily generic, by (semi-)continuity from the generic case, the pencil l_X is well-defined because $(\tilde{X}_\lambda, X_\lambda)$ is well-defined and not isomorphic to $(\tilde{X}, X)$. The second assertion is now clear.

To prove the last assertion notice that, for all $i \in \{1, \ldots, 16\}$, D_{Z_i} (1.5) contains $\Sigma(X_j) \cup \Sigma((X_j)_\lambda)$ for $j \in \{1, \ldots, 6\}$. Hence $D_{Z_i} = D_Z$. So D is constant on the orbits of $\mathscr{G}$. Q.E.D.

We do not assume A to be generic anymore. From now on in this section and unless otherwise stated, A will be an arbitrary element of $\mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. We will now extend the definition of l_X to part of $P^{-1}(A)$.

Definition and Proposition 5.7. *Let $(\tilde{X}, X)$ be an element of $P^{-1}(A)$ which is either fixed by λ or such that X and X_λ are irreducible. Fix an isomorphism $A \cong P(\tilde{X}, X)$ (which determines an isomorphism $A \cong P(\tilde{X}_\lambda, X_\lambda)$ up to translations and* $-$ id, *see* 3.15). *Then there is exactly one pencil l_X of Γ_{00}-divisors containing $\Sigma(X) \cup \Sigma(X_\lambda)$ ($= \Sigma(X)$ if $(\tilde{X}, X) = (\tilde{X}_\lambda, X_\lambda)$). If $(\tilde{X}, X) \neq (\tilde{X}_\lambda, X_\lambda)$ and $(\tilde{X}, X)$ is smooth, then the base locus of l_X is $\Sigma(X) \cup \Sigma(X_\lambda)$. If $(\tilde{X}, X) = (\tilde{X}_\lambda, X_\lambda)$, then the base locus of l_X is set-theoretically $\Sigma(X) = \Sigma(X_\lambda)$ and, for any two elements D, D' of l_X, the cycle on A associated to the scheme $D.D'$ is equal to twice the cycle associated to $\Sigma(X)$.*

Proof. From 5.6 we know that there is at least one pencil of Γ_{00}-divisors containing $\Sigma(X) \cup \Sigma(X_\lambda)$.

First suppose that $(\tilde{X}, X) = (\tilde{X}_\lambda, X_\lambda)$. Then, by 3.17i), there is a point ε of order 2 such that $\Sigma(X) \subset \Theta_\varepsilon$. The kernel of the restriction map $H^0(A, 2\Theta) \to H^0(\Theta_\varepsilon, 2\Theta|_{\Theta_\varepsilon})$ is generated by a nonzero section s with divisor $2\Theta_\varepsilon$ ($\in |2\Theta|_{00}$). Hence, if two sections have the same restriction to Θ_ε, they differ by a multiple of s. It is now easy to prove the claim in this case.

Now suppose that $(\tilde{X}, X) \neq (\tilde{X}_\lambda, X_\lambda)$, so that, by assumption, X and X_λ are irreducible. Choose a pencil l of Γ_{00}-divisors containing $\Sigma(X) \cup \Sigma(X_\lambda)$. If the base locus of l is $\Sigma(X) \cup \Sigma(X_\lambda)$, then one can show as in the proof of proposition 1 in [BD1] that no other Γ_{00}-divisor contains $\Sigma(X) \cup \Sigma(X_\lambda)$ and we will be done.

Suppose that the base locus of l contains points that are not in $\Sigma(X)\cup\Sigma(X_\lambda)$. As the homology class of $\Sigma(X)\cup\Sigma(X_\lambda)$ is $4[\Theta]^2=[(2\Theta)].[(2\Theta)]$, we deduce that no two elements of l intersect properly. So the base locus of l is $\Sigma(X)\cup\Sigma(X_\lambda)\cup D_1$ with D_1 an effective divisor. Any element of l can then be written as $D=D_1+D_2$ with D_2 an effective divisor. As in formula (1) (in 2.1), choose a symplectic basis $\{\gamma_j,\delta_j\}_{1\leqq j\leqq 4}$ of $H_1(A,\mathbb{Z})$ such that

$$D\equiv 2\sum \gamma_j\times\delta_j\times\gamma_k\times\delta_k\times\gamma_l\times\delta_l$$

where the sum is taken over distinct indices $j,k,l\in\{1,\ldots,4\}$. Then write

$$D_i\equiv \sum a^i_m\gamma_j\times\delta_j\times\gamma_k\times\delta_k\times\gamma_l\times\delta_l$$

(summing over indices such that $\{j,k,l,m\}=\{1,2,3,4\}$). So $a^1_m+a^2_m=2$ for all m. An easy computation using the fact that the intersection numbers $\Theta^d.D_i^{4-d}$ $(0\leqq d\leqq 4)$ are all nonnegative shows that the coefficients a^i_m are all nonnegative. Hence $0\leqq a^i_m\leqq 2$ for all m. We divide the remainder of our proof in several steps:

a) For D generic, D_2 cannot contain $\Sigma(X)\cup\Sigma(X_\lambda)$:

Otherwise, let E be another generic element of l and write $E=D_1+E_2$. Then D_2 and E_2 intersect properly. It is then easily seen that $D_2.E_2$ contains $\Sigma(X)\cup\Sigma(X_\lambda)$ only if $a^2_m=2$ for all m. This implies $D_1=0$ which is contrary to our hypotheses.

b) Let D_0 be an irreducible component of D_1 which contains, for instance, $\Sigma(X)$. Let K_0 be the connected component of 0 in the kernel of the homomorphism $A\to \mathrm{Pic}^0 A$ determined by D_0. Let B_0 be the quotient of A by K_0 and $\pi_0: A\to B_0$ the projection. Let M_0 be the ample divisor on B_0 such that $\pi_0^*M_0=D_0$. We show that $\dim B_0=3$:

Since the canonical class of $\tilde{X}^{(2)}$ is ample, $\tilde{X}^{(2)}$, and hence $\Sigma(X)$, does not have an elliptic fibration (nor is an abelian subvariety of A). So the restriction of π_0 to $\Sigma(X)$ is generically finite. Since $\pi_0(\Sigma(X))\subset M_0$, it follows that $\dim M_0\geqq 2$ and $\dim B_0\geqq 3$.

Now we show that we cannot have $B_0=A$. It is easily seen that $h^0(D_2)$ is equal to the product of the a^2_m's which are nonzero. Therefore, since $h^0(D_2)\geqq 2$, there exists m such that $a^2_m\geqq 2$. Let

$$\sum a^0_m\gamma_j\times\delta_j\times\gamma_k\times\delta_k\times\gamma_l\times\delta_l$$

be the homology class of D_0. If $B_0=A$, then $a^0_m\geqq 1$ for all m. Hence $a^2_m\leqq 1$ for all m. Contradiction.

Hence there is exactly one m such that $a^0_m=0$, say $m=4$, and $a^0_m\geqq 1$ for $m=1,2,3$. Then $a^2_4=2$, $a^2_m\leqq 1$ for $m=1,2,3$ and $h^0(D_2)=2$. Also, since $a^1_m\leqq 2$ for all m, there are at most two irreducible components of D_1, say D_0 and D_0', containing either $\Sigma(X)$ or $\Sigma(X_\lambda)$. As before (with D_2), $h^0(D-D_0)=h^0(D-D_0')=2$.

c) We show that $\Sigma(X)\cup\Sigma(X_\lambda)$ is contained in D_1.

Otherwise, suppose, for instance, that $\Sigma(X_\lambda)\subset D_2$ and $\Sigma(X)\subset D_1$. Let D and E be generic as in a). It is then easily seen that $D_2.E_2$ contains $\Sigma(X_\lambda)$ only if $a^2_m.a^2_n\geqq 2$ for all $m\neq n$. This is impossible since by b), $a^2_m\leqq 1$ for $m\neq 4$.

d) We show that *only the elements of l contain $\Sigma(X)\cup\Sigma(X_\lambda)$*. Suppose that there is a divisor $G\in|2\Theta|_{00}\backslash l$ which contains $\Sigma(X)\cup\Sigma(X_\lambda)$. Then, since, by b),

$$h^0(D-D_0)=h^0(D-D_0')=2\,,$$

G cannot contain D_0 or D_0' and intersects them properly. An easy computation shows that this implies $a_m^1\geqq 2$ for all m, which contradicts the fact that $D_2\neq 0$.

e) Finally, we show that, for $(\tilde{X},X)$ smooth, the base locus of $l=l_X$ is $\Sigma(X)\cup\Sigma(X_\lambda)$.

Suppose that the base locus of l_X is a divisor and let our notation be as before. Since X is smooth and $(\tilde{X},X)$ is not fixed by λ, it follows from 3.17 that $\mathbf{P}(\tilde{X},X)$ is smooth of dimension 2 at 0. This is equivalent to the fact that χX is not contained in any quadric ([B2], p. 381). So the image X' of χX in $\mathbb{P}T_0B_0$ by the projection from $\mathbb{P}T_0K_0$ is not contained in any conic. This implies, since the degree of χX is 8, that the degree of X' is either 4 or 8. Hence, extending π_0 to a morphism from the blow up of A at 0 to the blow up of B_0 at 0, we see that the generic degree d of the restriction of π_0 to $\Sigma(X)$ is either 1 or 2. Since the homology class of $\Sigma(X)$ is $4\sum\gamma_j\times\delta_j\times\gamma_k\times\delta_k$, we see that d cannot be 1 because otherwise M_0 will be 4 times a principal polarization on B_0 and we would have $a_m^0=4$ for $m\neq 4$. Hence $d=2$ and M_0 is twice a principal polarization on B_0. It follows that D_0 has homology class $2\sum_{\{j,k,l\}\neq\{1,2,3\}}\gamma_j\times\delta_j\times\gamma_k\times\delta_k\times\gamma_l\times\delta_l$ and is equal to D_1, D_2 has homology class $2\gamma_1\times\delta_1\times\gamma_2\times\delta_2\times\gamma_3\times\delta_3$. Now, reasoning as in b) with D_2 instead of D_0, we see that D_2 is the inverse image in A of a divisor of degree 2 on an elliptic curve B_2. The sum of the inverse image in A of a point on B_2 and the inverse image of a theta divisor in B_0, will then be homologically equivalent to $\Theta:\Theta$ will be reducible. Contradiction. Q.E.D.

In order to extend further the definition of l_X, we need

Lemma 5.8. *The set of smooth elements $(\tilde{X},X)$ of $P^{-1}(A)$ such that $(\tilde{X}_\lambda,X_\lambda)$ is smooth is a dense open subset of the subvariety* Sm(A) *of $P^{-1}(A)$ parametrizing smooth Prym-curves.*

Proof. Let $(\tilde{X},X)$ be an element of Sm(A). Choose a g_4^1 on X and let $(\tilde{Y},Y)$, $(\tilde{U},U)$ be the Prym-curves tetragonally related to $(\tilde{X},X)$ through it. Represent these Prym-curves and their images by λ as the vertices of an octahedron (figure 5 at the end of section 6). If we paint the face with vertices X, Y, U in white and the other faces alternatively in black and white, then, by 3.13, the vertices of each white face correspond to the Prym-curves of a tetragonal triple. In particular, $((\tilde{X}_\lambda,X_\lambda),(\tilde{Y},Y),(\tilde{U}_\lambda,U_\lambda))$ is a tetragonally related triple, as well as $((\tilde{X}_\lambda,X_\lambda),(\tilde{Y}_\lambda,Y_\lambda),(\tilde{U},U))$.

For $(\tilde{X},X)$ in some dense open subset of Sm(A) and g_4^1 in some dense open subset of $W_4^1(X)$ we have the following: if $(\tilde{X}_\lambda,X_\lambda)$ is not smooth, then it is of type (1, 1) and, by 3.6 and 3.13, $(\tilde{Y},Y)$, $(\tilde{U}_\lambda,U_\lambda)$ and their images by λ are smooth.

As $(\tilde{X}_\lambda,X_\lambda)$ is not smooth, by the proof of 3.9, Y is bielliptic and the g_4^1 on Y relating $(\tilde{Y},Y)$ to $(\tilde{U},U)$ and $(\tilde{X}_\lambda,X_\lambda)$ is the pullback of a g_2^1 on an elliptic curve. Then the opposite g_Y of this g_4^1 is also the pullback of a g_2^1 on an elliptic curve and has divisors of type 1a) or 1b) in 3.1. As $(\tilde{X}, X)$, $(\tilde{U}, U)$ are tetragonally related to $(\tilde{Y}, Y)$ through g_Y, one of the curves X or U cannot be smooth by 3.1 and we have a contradiction.

Again, to extend the definition of l_X, we need to prove that the natural rational map $h: A \to (|2\Theta|_{00})^*$ is generically finite; to prove this we need the lemma below. For tetragonally related Prym-curves $(\tilde{X}, X)$ and $(\tilde{Y}, Y)$, the isomorphism $P(\tilde{X}, X) \cong P(\tilde{Y}, Y)$ is canonical up to translations and $-\mathrm{id}$ since it is obtained by identifying $\tilde{Y}$ (resp. $\tilde{X}$) with a special subvariety of $J\tilde{X}$ (resp. $J\tilde{Y}$). We have

Lemma 5.9. *Choose an isomorphism $A \cong P(\tilde{X}, X)$ up to translations and* $-\mathrm{id}$. *Suppose that $(\tilde{X}, X)$ and $(\tilde{Y}, Y)$ are tetragonally related by g_4^1 on X, that l_X and l_Y are well-defined with base loci as in 5.7 and that $\Sigma(X)$ and $\Sigma(Y)$ have no common components. Then l_X and l_Y intersect and the intersection of the base locus of $\langle l_X, l_Y \rangle \subset |2\Theta|_{00}$ with $\Sigma(X)$ is*

$$S := \{[p, q] \in \Sigma(X) \colon h^0(g_4^1 - \pi p - \pi q) > 0 \textit{ or } h^0(K_X - g_4^1 - \pi p - \pi q) > 0\}.$$

If $(\tilde{X}, X)$ or $(\tilde{Y}, Y)$ is not fixed by λ, then

$$\Sigma(X) \cap \Sigma(Y) = \{[p, q] \in \Sigma(X) \colon h^0(g_4^1 - \pi p - \pi q) > 0\} \cup \textit{finite set}.$$

If $(\tilde{X}, X)$ and $(\tilde{Y}, Y)$ are both fixed by λ, then

$$\Sigma(X) \cap \Sigma(Y) = \{[p, q] \in \Sigma(X) \colon h^0(g_4^1 - \pi p - \pi q) > 0 \textit{ or } h^0(K_X - g_4^1 - \pi p - \pi q) > 0\}.$$

Proof. The fact that l_X and l_Y intersect is a consequence of the corresponding fact for $(\tilde{X}, X)$ generic in $\mathcal{P}_5$ and $(\tilde{Y}, Y)$ tetragonally related to it through a generic g_4^1 (this follows from 5.1 and 5.6). Let $S' = \{[p, q] \colon h^0(g_4^1 - \pi p - \pi q) > 0\} \subset \Sigma(X)$. An easy computation gives that the (reduced) homology class of S' is $2[\Theta]^3$. As in 2.4, we have $S' \subset \Sigma(X) \cap \Sigma(Y)$, so it follows that $S \subset D$ for any $D \in l_X \cup l_Y$ (use 3.13). The intersection of the base locus of $\langle l_X, l_Y \rangle$ with $\Sigma(X)$ and S have the same homology class so they are equal.

If, for instance, $(\tilde{X}, X)$ is not fixed by λ, then the homology class of $\Sigma(X) \cap \Sigma(Y)$ is half that of $D \cap \Sigma(Y)$ for any $D \in l_X$, i.e., $2[\Theta]^3$, and so $S' = \Sigma(X) \cap \Sigma(Y)$ up to a finite set.

If $(\tilde{X}, X)$ and $(\tilde{Y}, Y)$ are both fixed by λ, then $\Sigma(X) \cap \Sigma(Y)$ is set-theoretically equal to the intersection of the divisors of l_X and l_Y which is equal to S in this case. As S and $\Sigma(X) \cap \Sigma(Y)$ have the same homology class and are reduced, they are equal as varieties.

Lemma 5.10. *Suppose that $A \in \mathcal{A}_4 \backslash \bar{\mathcal{J}}_4$. The map h is generically finite. Also, a generic hyperplane in $|2\Theta|_{00}$ is spanned by lines associated to Prym-curves.*

Proof. To show that h is generically finite, it is enough to prove that one fiber of h is finite. As, by 3.6, the Prym-curves tetragonally related to a generic (see the beginning of section 1) Prym-curve of type (1, 1) through generic g_4^1's are smooth, we can find a generic smooth Prym-curve $(\tilde{X}, X)$ and a smooth Prym-curve $(\tilde{Y}, Y)$ which is tetragonally related to it through a generic g_4^1. Then $(\tilde{Y}, Y)$ will also be generic in $P^{-1}(A)$ and by 5.7, 5.8, 5.9, after the choice of an isomorphism $A \cong P(\tilde{X}, X)$ up to translations and $-\mathrm{id}$, the pencils l_X, l_Y are well-defined with base loci as in 5.7 and the base locus of $\langle l_X, l_Y \rangle$ is as in 5.9. Let S be as in 5.9 and $x \in S$ be generic. If we further suppose that $(\tilde{X}, X)$ is not fixed by λ, then, by our genericity assumptions and by 3.17, x is also generic in A and, in particular, h is well-defined at x. Now choose another (generic) g_4^1, say g_1, on X such that

$$S_1 := \{[p,q] \in \Sigma(X): h^0(g_1 - \pi p - \pi q) > 0 \text{ or } h^0(K_X - g_1 - \pi p - \pi q) > 0\}$$

contains x. We have $h^{-1}(x) \subset \Sigma(X) \cup \Sigma(X_\lambda)$ and $h^{-1}(x) \cap \Sigma(X) \subset S \cap S_1$ which is a finite set as the g_4^1's of X have no base points. Similarly, since X_λ is smooth by 5.8, $h^{-1}(x) \cap \Sigma(X_\lambda)$ is a finite set.

To prove the second assertion, notice that, since h is generically finite, a generic hyperplane H in $|2\Theta|_{00}$ is of the form $h(x)$ for some generic x in A. With the above notation, let $(\tilde{Y}_1, Y_1)$ be smooth and tetragonally related to $(\tilde{X}, X)$ through g_1. Then, as the base locus of $\langle l_X, l_Y, l_{Y_1}\rangle$ is finite, $\langle l_X, l_Y, l_{Y_1}\rangle$ has dimension $\geqq 3$. It is contained in H, hence equal to it.

We extend the definition of l_X again:

Definition and Proposition 5.11. *Suppose that $A \in \mathcal{A}_4 \setminus \bar{\mathcal{J}}_4$. Let U_l be the subset of $P^{-1}(A) \setminus P^\lambda$ such that for $(\tilde{X}, X) \in U_l$ the set of Γ_{00}-divisors containing $\Sigma(X) \cup \Sigma(X_\lambda)$ is a pencil l_X with base locus $\Sigma(X) \cup \Sigma(X_\lambda)$ (after the choice of an isomorphism $A \cong P(\tilde{X}, X)$ up to translations and* $-$ id). *Then U_l is a dense open subset of $P^{-1}(A) \setminus P^\lambda$.*

Proof. By 5.7 and 5.8, U_l contains a dense open subset of the set $\mathrm{Sm}(A)$ of smooth Prym-curves in $P^{-1}(A)$.

Suppose that $(\tilde{X}, X)$ is of type $(1,1)$. By 3.17, since $(\tilde{X}, X)$ is not fixed by λ, the component of Σ_A containing $\Sigma_{12}(X)$ maps onto A. Therefore, for $(\tilde{X}, X)$ in a dense open subset of $\mathcal{P}_{1,1} \cap P^{-1}(A)$, the generic elements of $\Sigma_{12}(X)$ are generic in A. So, as h is generically finite (5.10), $h(\Sigma(X))$ is 2-dimensional. The closure of $h(\Sigma(X))$ is a plane because it is so for A generic. Hence there is exactly one pencil l_X of Γ_{00}-divisors containing $\Sigma(X) \cup \Sigma(X_\lambda)$.

Let F_0 be a component of $\mathcal{P}_{1,1} \cap P^s$ (3.17). Suppose that the base locus of l_X contains a divisor for every $(\tilde{X}, X)$ in F_0. The restriction of h to any such divisor is not finite hence there is a divisor D_1 such that, for $(\tilde{X}, X)$ in some open set of F_0, the base locus of l_X is $\Sigma(X) \cup \Sigma(X_\lambda) \cup D_1$. As in the proof of 5.7, D_1 contains some component of $\Sigma(X)$ or $\Sigma(X_\lambda)$. Let F_1 be a component of $P^{-1}(A)$ parametrizing Prym-curves tetragonally related to elements of F_0 and let $(\tilde{Y}, Y) \in F_1$ be generic. Then $(\tilde{Y}, Y)$ is smooth by 3.6 and by 3.17 iii), $\Sigma(Y) \subset D_1$. This is true for every $(\tilde{Y}, Y) \in F_1$ so D_1 contains a translate of Θ (see 3.17). Now, with the notation of the proof of 5.7, $h^0(A, D_2)$ is the product of the coefficients a_m^2 which are nonzero, so we cannot have $h^0(A, D_2) \geqq 2$. Contradiction.

Whenever we mention $\Sigma(X)$ or l_X for a Prym-curve $(\tilde{X}, X)$ we suppose that we are given an isomorphism $A \cong P(\tilde{X}, X)$ up to translations and $\pm$ id. So far we have defined the pencil l_X on the fixed locus of λ, for all $(\tilde{X}, X)$ such that X and X_λ are irreducible and on an open dense subset of $P^{-1}(A)$ for A in $\mathcal{A}_4 \setminus \bar{\mathcal{J}}_4$. We will prove in 6.23 that l_X is defined on all of $P^{-1}(A)$. We can now count the number of lines l_X in a generic hyperplane in $|2\Theta|_{00}$ (5.12) and prove theorem 4i) of the introduction.

Lemma 5.12. *For x generic in A the hyperplane $h(x) \in (|2\Theta|_{00})^*$ contains exactly* 27 *lines l_X.*

Proof. The hyperplane $h(x)$ is the set of Γ_{00}-divisors which contain x. A line l_X is in $h(x)$ if and only if the Γ_{00}-divisors in l_X contain x. Or, by 5.7 and 5.11, if and only if

$$x \in \Sigma(X) \cup \Sigma(X_\lambda)\,.$$

So, by 4.8, we have 27 such lines. Q.E.D.

Theorem 5.13. *Suppose that $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. Then the only base point of $|2\Theta|_{00}$ is 0.*

Proof. Suppose that $x \in A$ is a base point of $|2\Theta|_{00}$ distinct from the origin. Then, for all $(\tilde{X}, X) \in P^{-1}(A)$, we must have $x \in \Sigma(X) \cup \Sigma(X_\lambda)$ (5.7, 5.11). By 5.9 this implies that for all $(\tilde{X}, X)$ in $P^{-1}(A)$,

$$x \in \{[p,q] \in \Sigma(X)\colon h^0(g_4^1 - \pi p - pq) > 0 \text{ or } h^0(K_X - g_4^1 - \pi p - \pi q) > 0\}\,.$$

Thus the line $\langle \pi p, \pi q\rangle$ in the canonical space of X is contained in all the singular quadrics containing κX. However, supposing X *smooth* (we can do so by 3.3), the intersection of these quadrics is X unless X is trigonal [ACGH] in which case A would be the jacobian of a curve. Q.E.D.

Together with the result of [W2] (see also [Iz1]), 5.13 gives:

Let A be in $\mathscr{A}_4 \backslash \mathscr{A}_{4\text{dec}}$, then A is a jacobian if and only if the base locus $V(\Gamma_{00})$ of $|2\Theta|_{00}$ contains another point besides 0, in that case

$$V(\Gamma_{00}) = C - C \cup \{\mathcal{O}_C(\pm(K_C - 2g_3^1))\}$$

(where $C - C = \{\mathcal{O}_C(p-q)\colon p, q \in C\}$).

Note that in the proof of 5.13 we only used *smooth* Prym-curves.

6. The cubic threefold over $\mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$

In this section we always suppose A to be an element of $\mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. In 6.4–6.11 and 6.27–6.31 we suppose A generic. We define the threefold T and study its basic properties. In particular, we prove theorems 2 and 3 of the introduction as well as theorem 4ii).

Definition 6.1. Suppose that $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. Define $T = T_A$ to be the smallest closed subvariety of $|2\Theta|_{00}$ containing the lines l_X whenever they are defined for $(\tilde{X}, X) \in P^{-1}(A)$.

Proposition 6.2. *Suppose that $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. Then T has pure dimension 3.*

Proof. If a component V of T corresponding to a component F_0 of $P^{-1}(A)$ has dimension 2, then it is a plane because it contains a 2-dimensional family of lines. The polar of this plane is a line $l \subset (|2\Theta|_{00})^*$. The base locus of the linear system V has a (1-dimensional) component l_0 which is contained in $\Sigma(X)$ for every $(\tilde{X}, X) \in F_0$. Hence, by 5.9, at most a finite number of elements of F_0 are tetragonally related to a fixed generic $(\tilde{X}, X) \in F_0$. So there is a component $F_1 \neq F_0$ of $P^{-1}(A)$ which parametrizes Prym-curves tetragonally related to elements of F_0. By specialization from the generic case (see 5.1), every divisor $D \in l_X$ contains $\Sigma(Y) \cup \Sigma(Y_\lambda)$ for some Prym-curve $(\tilde{Y}, Y) \in F_1$. Hence V is contained in the union of the lines corresponding to elements of F_1 (these lines are not all in V since, by 5.9, the divisors which belong to them do not all contain the base locus of V and cannot be a component of T.

Recall that $\mathscr{E}_A \subset A$ is the closure of the subvariety of points x such that $\Theta . \Theta_x = E_Z$ for some element Z of $\mathscr{Z}_A$. Let $\mathscr{E}' = \bigcup_A \mathscr{E}_A \subset \mathbb{A}|_{\mathscr{A}_4 \setminus \mathscr{J}_4}$ where $\mathbb{A}$ is the universal family of abelian varieties over the Siegel half space $\mathscr{H}$ of 4×4 complex matrices with positive definite imaginary part.

Proposition 6.3. *At a point* $x \in \mathscr{E}_A$, *the hyperplane* $h(x)$ *is tangent to* T.

Proof. First suppose that x is generic in $\mathscr{E}'$. As D_Z contains $\Sigma(X_i) \cup \Sigma((X_i)_\lambda)$, D_Z is the intersection point of the six lines $l_i = l_{X_i}$. Since $x \in \Sigma(X_i)$ (see 2.3), the hyperplane $h(x)$ contains the lines l_i. Generically, the l_i's are not in a plane:

Otherwise, the plane would contain also the 15 lines $l_{ij} = l_{X_{ij}}$ because

– l_{ij} intersects l_i and l_j: to see this take a double solid Z such that the discriminant curves for the projections of Z from two of its nodes are X_i and X_{ij}, then D_Z gives a point on l_i and l_{ij}.

– l_i, l_j and l_{ij} do not all pass through the same point by 5.2.

Now consider two of these planes P_x, P_y corresponding to $x, y \in \mathscr{E}_A$, both containing l_1. A hyperplane containing P_x and P_y contains more than 27 lines l_X. However, by 5.10, such a hyperplane is generic if l_1 and P_x, P_y are so because l_1 moves in a 2-dimensional family and each of P_x, P_y moves in a 1-dimensional family (obtained from the 1-dimensional family of double solids in $\mathscr{Z}_A$ for which X_1 is a discriminant curve), a total of 4. This contradicts 5.12.

So $h(x)$ is spanned by the l_i's. The lines l_i are in T hence they are also in $\mathbb{P}T_t T$. So $h(x)$ is contained in $\mathbb{P}T_t T$. For a nongeneric point of $\mathscr{E}'$ the statement stays valid by continuity.

In 6.4–6.11, we suppose A generic; we determine the incidence configuration of the lines l_X in a generic hyperplane section of T and deduce that T has degree 3. By continuity, it will follow that T has degree ≤ 3 whenever it is defined.

In the same way as 5.12, using 5.5, we obtain

Lemma 6.4. *Suppose that* A *is generic.*

i) *For* $x \in \mathscr{E}_A$ *generic, the hyperplane* $h(x)$ *contains exactly* 21 *lines* l_X, 6 *of which "count twice".*

ii) *Let* $\{Z_i, \iota Z_i\}_{1 \leq i \leq 16}$ *be a generic orbit of* $\mathscr{G}$, *then, if* $\Theta . \Theta_{x_i} = E_{Z_i} = E_{\iota Z_i}$ *(see* 5.4), *we have* $h(\pm x_i) = h(\pm x_j)$ *for all* i, j *(by* 5.6). *Also, by* 6.3, $h(x_i)$ *(regarded as a hyperplane in* $|2\Theta|_{00}$) *is tangent to* T *at* $D = D_{Z_i}$.

Corollary 6.5. *Suppose that A is generic. Only 6 lines l_X pass through a generic point of T.*

Proof. First notice that, generically, by 5.2, none of the l_{ij}'s passes through the intersection point of the l_i's. If there are more than six lines l_i passing through $D \in T$, then all these lines (and also the lines incident to any two of them) have to be in the tangent space at D to T. This contradicts 6.4.

Corollary 6.6. *Suppose that A is generic. At a generic point D of T, $(\tilde{X}_i, X_i)$ and $((\tilde{X}_i)_\lambda, (X_i)_\lambda)$ are the only elements $(\tilde{X}, X)$ of $P^{-1}(A)$ verifying $\Sigma(X) \cup \Sigma(X_\lambda) \subset D$.*

Proof. If $\Sigma(X) \cup \Sigma(X_\lambda) \subset D$, then $D \in l_X$ (5.6). The only lines passing through D are the l_i's and the only curves corresponding to l_i are $(\tilde{X}_i, X_i)$ and $((\tilde{X}_i)_\lambda, (X_i)_\lambda)$.

Corollary 6.7. *Suppose that $\{Z_i, \iota Z_i\}$ is the orbit, under $\mathcal{G}$, of a generic element of $\mathcal{Z}$ (see 1.5). Then the Z_i, ιZ_i are the only double solids with associated Γ_{00}-divisor $D = D_{Z_i}$.*

Proof. If Z is a double solid corresponding to D, then by 6.6, the discriminant curves for Z are X_i or $(X_i)_\lambda$ (these being the curves corresponding to the lines l_i through D).

Letting $\mathcal{G}$ (1.5) act we can suppose that the curves X_i are the discriminant curves for the projections from the nodes p_i of Z. It has been proven by Clemens and Donagi that the curves $X_2, \ldots, X_6$ determine the plane representation of X_1. Our proof is close to Donagi's:

By 1.5, the Prym-curves $(\tilde{X}_2, X_2), \ldots, (\tilde{X}_6, X_6)$ are tetragonally related to $(\tilde{X}_1, X_1)$. Thus they determine 5 g^1_4's (see 5.9), say $g_2, \ldots, g_6$, on X_1. These have the property (see 2.5):

$$|g_i + p_i' + p_i''| = g^2_6$$

where p_i', p_i'' are the points of X_1 above p_i and g^2_6 gives the plane representation of X_1 as discriminant curve for Z. Hence, if $g^2_6 = |K_X - p - q|$, then, for $i = 2, \ldots, 6$,

$$h^0(K_X - g_i - p - q) > 0 .$$

We see (1.3) that the singular quadrics q_i corresponding to the g_i's are on a line l in the net of quadrics containing κX_1. We need to show that the g_i's determine $\{p, q\}$ uniquely. This is a consequence of the following.

Put $X = X_1$ and recall that $\tilde{Q}' \cong W^1_4$ (1.3). The g_i's give a lift in $\tilde{Q}'$ of the divisor $D_l \in g^2_5$ cut on Q by l. Let S be a special subvariety of $P(\tilde{Q}', Q) = JX$ (1.3) corresponding to g^2_5, i.e., a translate of one connected component of the set

$$\{D \in \operatorname{Pic}^5 \tilde{Q} : h^0(D) > 0,\ \nu_Q(D) \cong \mathcal{O}_Q(g^2_5)\}$$

the two connected components of which are isomorphic by [B3] (here ν_Q is the norm map for the cover $\tilde{Q}' \to Q$). For each secant $\langle s, t\rangle$ to κX we can consider the five (counted with multiplicities) quadrics of rank ($\leq$) 4 containing κX and $\langle s, t\rangle$. Then $\langle s, t\rangle$ will be contained in one ruling of each quadric and hence will pick a g^1_4 above it. This defines a map $X^{(2)} \to S$

which is an isomorphism because the homology class of S is minimal [B3] and the image of $X^{(2)}$ by this map is exactly its image by the Abel-Jacobi map.

Therefore, by 2.2, B is well-determined up to projective equivalence. Q.E.D.

Corollary 6.8. *Suppose that A is generic and that $(\tilde{X}, X)$ is a generic element of $P^{-1}(A)$. Let $(\tilde{Y}, Y)$ be an element of $P^{-1}(A)$ such that l_Y passes through a generic point of l_X. Then $(\tilde{X}, X)$ and $(\tilde{Y}, Y)$ are tetragonally related.*

Proof. The lines l_X and l_Y intersect in a generic point D_Z of T. By 6.6, after renaming, $X = X_1$ and $Y = X_2$ are discriminant curves for Z. By 1.5 they are tetragonally related. Q.E.D.

We know the Prym-embedded curves in a Fano variety E_Z associated to a generic double solid Z (5.5). For a generic $x \in A$, we want to describe the Prym-embedded curves in $\Theta . \Theta_x$.

Lemma 6.9. *Suppose that A is generic and that $(\tilde{X}, X)$ is a generic element of $P^{-1}(A)$. Let $(\tilde{Y}, Y), (\tilde{U}, U)$ in $P^{-1}(A)$ be such that the lines l_X, l_Y, l_U are in a plane section of T and suppose that $(\tilde{Y}, Y), (\tilde{U}, U)$ are generic for this property. Then we have: either $((\tilde{X}, X), (\tilde{Y}, Y), (\tilde{U}, U))$ or $((\tilde{X}_\lambda, X_\lambda), (\tilde{Y}_\lambda, Y_\lambda), (\tilde{U}_\lambda, U_\lambda))$ is a tetragonally related triple.*

Proof. By 6.8, $(\tilde{X}, X)$ is tetragonally related to $(\tilde{Y}, Y)$ and also to $(\tilde{U}, U)$. Let g_1 and g_2 be the g^1_4's (on X) relating $(\tilde{X}, X)$ to $(\tilde{Y}, Y)$ and $(\tilde{U}, U)$ respectively. The base locus of the 1-dimensional family of divisors in A corresponding to the hyperplanes in $|2\Theta|_{00}$ containing the plane $\langle l_Y, l_X, l_U \rangle$ spanned by l_X, l_Y, l_U is

$$(\Sigma(X) \cup \Sigma(X_\lambda)) \cap (\Sigma(Y) \cup \Sigma(Y_\lambda)) \cap (\Sigma(U) \cup \Sigma(U_\lambda)).$$

The intersection of this with $\Sigma(X)$ is

$$\Sigma(X) \cap (\Sigma(Y) \cup \Sigma(Y_\lambda)) \cap (\Sigma(U) \cup \Sigma(U_\lambda)). \tag{2}$$

Let

$$S_i := \{[p, q] : h^0(g_i - \pi p - \pi q) > 0\} \subset \Sigma(X)$$

and

$$T_i := \{[p, q] : h^0(K_X - g_i - \pi p - \pi q) > 0\} \subset \Sigma(X).$$

By 5.9, the intersection (2) is equal to $(S_1 \cup T_1) \cap (S_2 \cup T_2)$. This being of dimension 1, we must have either $S_1 = S_2$ or $T_1 = T_2$ or $S_1 = T_2$ or $T_1 = S_2$.

In any of the first two cases, $g_1 = g_2$ and $\{(\tilde{X}, X), (\tilde{Y}, Y), (\tilde{U}, U)\}$ is a tetragonally related triple. In any of the last two cases, $g_1 = K_X - g_2$ and $\{(\tilde{X}, X), (\tilde{Y}, Y), (\tilde{U}_\lambda, U_\lambda)\}$ is a tetragonally related triple. This is equivalent to: $\{(\tilde{X}_\lambda, X_\lambda), (\tilde{Y}_\lambda, Y_\lambda), (\tilde{U}_\lambda, U_\lambda)\}$ is a tetragonally related triple (via the opposite of the g^1_4 on U_λ). Q.E.D.

6.10. Let x be a generic element of A (generic in $\mathcal{A}_4$). Let $\tilde{X}_\lambda$ be a Prym-embedded curve in $\Theta . \Theta_x$, then, by 3.16, $x = [p, q]$ for some $p, q \in \tilde{X}$. If g_i $(i = 1, \ldots, 5)$ are the 5 g^1_4's such that $h^0(g_4^1 - \pi p - \pi q) > 0$ (as in the proof of 6.7) and $((\tilde{Y}_i, Y_i), (\tilde{U}_i, U_i))$ is the pair of Prym-curves tetragonally related to $(\tilde{X}, X)$ via g_i, then, by 2.4 and 3.16, we have Prym-embeddings of $(\tilde{Y}_i)_\lambda, (\tilde{U}_i)_\lambda$ in $\Theta . \Theta_x$: this gives us 11 curves in $\Theta . \Theta_x$ and 11 lines in $H = \tilde{h}(x)$. The curves obtained are nonisomorphic because, by 5.10, each of them determines the g_4^1 through which it is tetragonally related to $(\tilde{X}, X)$. Also, the lines corresponding to them are distinct because, by 4.9, the Prym-curves are not images of each other by λ.

We claim that for x generic, no three lines of the form l_X in H pass through the same point. Otherwise the three lines will have to be in a plane: if they are not, H will be equal to their span which would be tangent to T and this contradicts the genericity of x. By 6.9, the three lines correspond to a tetragonally related triple $\{(\tilde{Y}, Y), (\tilde{Y}', Y'), (\tilde{Y}'', Y'')\}$. Let Z be a double solid such that D_Z is equal to the intersection point of $l_Y, l_{Y'}$ and $l_{Y''}$. Then $\Sigma(Y) \cup \Sigma(Y') \cup \Sigma(Y'') \subset D_Z$. As we must have, for instance, $Y = X_1$, $Y' = X_2$, $Y'' = X_{12}$ with the notation of 1.5, this is not possible generically by 5.2.

Now, considering Y_1 instead of X, one can repeat the same procedure and obtain 8 new (by 6.9 and the fact that no three lines in H have a common intersection point) lines in H and Prym-embedded curves in $\Theta . \Theta_x$. Then we obtain again 8 new lines and Prym-embedded curves with U_1 instead of Y_1.

So we have all the Prym-embedded curves in $\Theta . \Theta_x$ together with their corresponding lines $(11 + 8 + 8)$. Notice that, if we fix $(\tilde{X}, X), (\tilde{Y}, Y) = (\tilde{Y}_1, Y_1)$ and $(\tilde{U}, U) = (\tilde{U}_1, U_1)$, then for any hyperplane H containing l_X, l_Y and l_U, all the other lines in H (associated to Prym-curves) encounter l_X, l_Y or l_U because the corresponding Prym-curves are tetragonally related to $(\tilde{X}, X)$, $(\tilde{Y}, Y)$ or $(\tilde{U}, U)$. Also l_X, l_Y, l_U are the only lines (corresponding to Prym-curves) in their plane (consider, for instance, the hyperplane tangent to T at $l_X \cap l_Y$).

We are now ready to prove

Theorem 6.11. *Suppose that A is generic. Then T is a cubic threefold.*

Proof. Consider a generic tetragonal triple $\{(\tilde{X}, X), (\tilde{Y}, Y), (\tilde{U}, U)\}$. Then the lines l_X, l_Y, l_U are in a plane V because they intersect two by two by 5.9 and they do not all pass through the same point (6.10). We are going to show that $V \cap T$ is the union of these three lines, each occuring with multiplicity 1.

Suppose that there is a point $t \in V \cap T$ which is not on $l_X \cup l_Y \cup l_U$. Let l be a line through t, corresponding to some Prym-curve. Let H be the hyperplane spanned by l and V. Then, by 6.10, all lines (corresponding to Prym-curves) in H encounter one of the lines l_X, l_Y, l_U. Hence l is in V. But V contains only l_X, l_Y, l_U (6.10). Contradiction.

Now take a generic point $t \in l_X$. The tangent space to T at t contains exactly 21 lines (6.4) which are: l_X, five other lines through t (6.5) and the lines incident to any two of them. None of these is in V except l_X (6.10). If $T_t(T)$ contains V, it contains more than 21 lines: this contradicts 6.4. Q.E.D.

In 6.12–6.26 we suppose A to be an arbitrary element of $\mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. We proceed with the study of the properties of T.

Theorem 6.12. *Suppose that $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. Then T is an irreducible cubic threefold.*

Proof. By 6.2, T has pure dimension 3. By continuity from the generic case, the degree of T is less than or equal to 3. Let H be a generic hyperplane in $|2\Theta|_{00}$. By 5.10, H is spanned by lines associated to Prym-curves. So T cannot be a hyperplane because then $T \cap H$ cannot span H. The 27 distinct lines l_X in H (5.12) correspond to (generic) Prym-curves $(\tilde{X}, X)$ in P^s (4.8) and intersect at least as much as when A is generic. For instance, for $(\tilde{X}, X)$ in some dense open subset of P^s, by continuity (6.10, 6.11), H contains 5 planes each containing l_X and two other distinct lines corresponding to a pair of Prym-curves tetragonally related to $(\tilde{X}, X)$ through the same g^1_4. Suppose that T is a union of hyperplanes. By 6.2 and 3.17iii), we can choose $(\tilde{X}, X)$ in such a way that l_X is not contained in the intersection of two of the components of T. This forces the five planes we mentioned above to be all in one component of T. Repeating this argument for other lines l_Y in H, we obtain that all the lines l_Y in H are in the same component of T: since H is generic, this is impossible because then $T \cap H$ does not span H (5.10).

The threefold T is not an irreducible quadric because otherwise any plane containing three lines corresponding to tetragonally related Prym-curves would be contained in T. As there is at most a one-dimensional family of planes in an irreducible quadric each plane would have to contain a one-dimensional family of lines l_X. On the other hand, the intersection of a generic hyperplane H with T would be the union of two planes containing all the lines l_X in H of which there are exactly 27.

If T is the union of a quadric and a hyperplane H_0, out of three lines l_X in a plane in H, one has to be in H_0. Then $H \cap H_0$ would be a plane containing more than 3 lines l_X, so it contains a one-dimensional family of them. But this cannot hold for H generic. Q.E.D.

Theorem 6.13. *Suppose that $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. Then T has at worst isolated double points as singularities. For $(\tilde{X}, X) \in P^{-1}(A)$ such that l_X is well-defined (after a choice of isomorphism $P(\tilde{X}, X) \cong A$ up to translations and $-\mathrm{id}$), the discriminant curve of T for its conic bundle structure obtained from the projection from l_X is the plane quintic Q parametrizing singular quadrics through κX. If we denote by Q_λ the plane quintic parametrizing singular quadrics through κX_λ, then $Q = Q_\lambda$.*

Proof. Suppose that T has a triple point t. Then T is the cone of vertex t over an irreducible cubic surface S. If S has a triple point, i.e., S is a cone over an irreducible cubic plane curve, then T has a triple line l_0 and every line in T is in some plane contained in T and containing l_0. Let l_X be a line corresponding to a Prym-curve $(\tilde{X}, X)$ which is smooth or of type $(1, 1)$. Then, by continuity, 5.1 implies that a generic element of l_X contains $\Sigma(Y) \cup \Sigma(Y_\lambda)$ for some Prym-curve $(\tilde{Y}, Y)$ tetragonally related to $(\tilde{X}, X)$. We can choose $(\tilde{X}, X)$ and $(\tilde{Y}, Y)$ such that $l_Y \cap l_X \neq l_0 \cap l_X$, hence, since l_Y also intersects l_0, l_Y is contained in the plane V spanned by l_X and l_0. In the same way, V contains the lines corresponding to Prym-curves tetragonally related to $(\tilde{Y}, Y)$, etc. This is easily seen to imply that then V contains at least a 2-dimensional family of lines corresponding to Prym-curves. Then all the lines l_X have to be contained in a finite number of planes in T because $P^{-1}(A)$ is two-dimensional (3.18). Then, by definition, T would be the union of these planes. This is impossible by 6.2. Hence S has at worst double points and it contains a finite number of lines. By 5.12, it contains 27 lines, hence it is smooth and the triple point of T is unique.

Suppose now that T has a double curve C. Then every hyperplane section of T is singular. A general hyperplane section of T contains a finite number of lines and as it is singular it contains less than 27 lines. This contradicts 5.12.

So T has isolated singularities.

Suppose that $(\tilde{X}, X)$ is smooth and generic in some irreducible component F_0 of P^s (3.17, 3.3). The surface $\Sigma(X)$ cannot be contained in the Γ_{00}-divisor corresponding to any singular point of T because $(\tilde{X}, X)$ is generic and $\bigcup_{F_0} \Sigma(X') = A$. So l_X does not contain any singular points of T and the projection from l_X exhibits the blow up of T along l_X as a conic bundle over $\mathbb{P}^2$. The discriminant curve in $\mathbb{P}^2$ for this conic bundle is the image of the set of plane sections of T containing l_X that are unions of three (possibly equal) lines. As the lines corresponding to a tetragonal triple are in a plane (see 6.11 for instance) and the plane representation of a plane quintic is unique, the discriminant curve contains Q and Q_λ. As the discriminant curve has degree 5, it is equal to Q and to Q_λ. A singular point of T projects to a singular point of Q of the same multiplicity or more. As Q has only nodes as singularities, T has at worst double points as singularities. Knowing this, the last assertions of the theorem remain valid by continuity for any l_X.

Theorem 6.14. *T is singular if and only if $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$ has a vanishing theta-null. For A generic in θ_{null}, T has only one node.*

Proof. The cubic threefold T is singular if and only if, for all $(\tilde{X}, X)$, Q is singular. Equivalently, as the elements of some dense open subset of $P^{-1}(A)$ are all smooth or of type (1, 1) and, by continuity, for all $(\tilde{X}, X)$, X has a vanishing theta-null (a Prym-curve of type (1, 1) has *even* vanishing theta-nulls by [B1], pp. 182–183). Hence the locus of singular cubic threefolds is contained in the image by P of the locus $\widehat{\theta_{\text{null}}}$ in $\mathscr{P}_5 \backslash P^{-1} \bar{\mathscr{J}}_4$ parametrizing Prym-curves $(\tilde{X}, X)$ such that X has a vanishing theta-null. This has dimension 11.

By [B1], pp. 184–191, $\widehat{\theta_{\text{null}}}$ has 2 irreducible components $\theta_{\text{null}1}$ and $\theta_{\text{null}2}$, the elements of these are Prym-curves with even and odd vanishing theta-nulls respectively.

By [B1], p. 183, $\theta_{\text{null}} = P(\theta_{\text{null}1})$. Also, by [M2], pp. 344–347, for any element A of θ_{null}, the smooth elements of $P^{-1}(A)$ are in $\theta_{\text{null}1}$ so their associated plane quintics are singular. Since the associated plane quintics of Prym-curves of type (1, 1) are always singular, by continuity and by 6.13, all the discriminant curves of T are singular.

By 3.14 and 3.9, $\theta_{\text{null}2}$ maps dominantly to $\mathscr{A}_4$ with one-dimensional fibers everywhere.

It follows that T is singular on θ_{null} exactly.

Generically on $\theta_{\text{null}1}$, the plane quintics Q (corresponding to elements of $\theta_{\text{null}1}$) are generic in the space of plane quintics with nodes. This is easily seen to imply that T is generic in the space of cubic threefolds with isolated double points, therefore it has only one node. Q.E.D.

Recall that $\tilde{h}$ is the lift of h to the blow up $\tilde{A}$ of A at 0. We give a precise description of the generic fibers of $\tilde{h}$. This will permit us to find the degree of $\tilde{h}$ (or h) and to relate $|2\Theta|_{00}$

directly to the space of quadrics through $\kappa X \subset \mathbb{P}^4$. We then show that $\tilde{h}$ is a morphism and determine its ramification and branch loci.

6.15. Choose a smooth $(\tilde{X}, X) \in P^s$. Consider the net N of quadrics containing κX with the plane quintic Q parametrizing the singular ones (1.3). Let $\psi : X^{(2)} \to N^* \cong (\mathbb{P}^2)^*$ be the morphism associating to $\{p, q\}$ the line l_{pq} in N consisting of all quadrics containing the line $\langle p, q \rangle$ in $|K_X|^*$. Choose a *generic* line L in N. Let $(\tilde{Y}, Y)$ and $(\tilde{Y}', Y')$ be tetragonally related to $(\tilde{X}, X)$ through two g^1_4's g_1 and g_2 whose corresponding quadrics are two distinct points of $Q \cap L$ (see 1.3).

6.16. Recall that whenever we mention $\Sigma(X)$ or l_X we are implicitly assuming that we are given an isomorphism $P(\tilde{X}, X) \cong A$ up to translations and $-$ id. Choose $(\tilde{X}, X)$ and L in such a way that l_X, l_Y and l'_Y are well-defined with base loci as in 5.7 and l_Y and $l_{Y'}$ span the hyperplane H in $|2\Theta|_{00}$ which is the inverse image of L by the projection from l_X. For any Prym-curve $(\tilde{U}, U)$, let $\tilde{\Sigma}(U)$ be the strict transform of $\Sigma(U)$ in $\tilde{A}$ and let $\tilde{l}_U$ be the pencil of divisors in $\tilde{A}$ obtained as the strict transforms of the elements of l_U. Considering H as a point of $(|2\Theta|_{00})^*$, since l_X and l_Y span H, $\tilde{h}^{-1}(H)$ is the base locus of $\tilde{l}_Y \cup \tilde{l}_{Y'}$, i.e.,

$$\tilde{h}^{-1}(H) = (\tilde{\Sigma}(Y) \cup \tilde{\Sigma}(Y_\lambda)) \cap (\tilde{\Sigma}(Y') \cup \tilde{\Sigma}(Y'_\lambda)) \subset \tilde{\Sigma}(X) \cup \tilde{\Sigma}(X_\lambda).$$

Suppose that $\tilde{h}([p, q]) = H$ with $[p, q] \in \tilde{\Sigma}(X)$, then, by 5.9, for $i = 1$ and 2: either $h^0(g_i - \pi p - \pi q) > 0$ or $h^0(K_X - g_i - \pi p - \pi q) > 0$.

That is, in the canonical space of X, the line $\langle \pi p, \pi q \rangle$ is in the intersection S of the singular quadrics q_i corresponding to g_i for $i = 1, 2$.

Since L is generic, there are 16 lines in S that are all secant to κX (see [DS], pp. 86–87 for instance) and none of these lines is tangent to κX. Hence we have $16.4 = 64 = 2^6$ distinct points in $\tilde{\Sigma}(X)$ which project to these and do not map to 0 in A. Counting the points in $\tilde{\Sigma}(X) \cup \tilde{\Sigma}(X_\lambda)$ we obtain $2.2^6 = 2^7$ distinct preimages for H which do not map to base points of $|2\Theta|_{00}$ by 5.13.

Corollary 6.17. *Suppose that $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. Then the degree of $\tilde{h}$ (or h) is 2^7.*

Definition 6.18. We define the multiplicity of $|2\Theta|_{00}$ at 0 in the following way: For any generic subsystem H of codimension 1 of $|2\Theta|_{00}$ the base locus of H is a finite set. This is the union of 0 and a set of distinct points, distinct from 0. The point 0 occurs with a certain multiplicity in the intersection of the elements of H: more precisely, this is the length of the maximal Artinian subscheme of A with underlying set 0 and contained in every element of H. This length is an upper-semicontinuous function on $(|2\Theta|_{00})^*$ and is constant on a nonempty open subset of $(|2\Theta|_{00})^*$. The multiplicity at 0 for H in this nonempty open subset is the multiplicity of $|2\Theta|_{00}$ at 0.

Remark 6.19. It is clear from the above definition that the sum of the multiplicity at 0 of $|2\Theta|_{00}$ and the number of points distinct from 0 in the base locus of a generic H is equal to the number of points in the base locus of a generic subsystem of (projective) dimension 3 of $|2\Theta|$. Hence 6.17 implies that the multiplicity at 0 of $|2\Theta|_{00}$ is 4^4 because $(2[\Theta])^4 = 2^7.3 = 2^7 + 4^4$.

Let E be the exceptional divisor of $\tilde{A}$. For each Γ_{00}-divisor D, let $B(D)$ be the projectivized tangent cone to D at 0. Recall that $\tau|2\Theta|_{00}$ is the linear system of the quartic tangent cones $B(D)$. We can now prove theorem 4ii) of the introduction.

Corollary 6.20. *Suppose that $A \in \mathcal{A}_4 \backslash \bar{\mathcal{J}}_4$. Then the base locus of $\tau|2\Theta|_{00}$ is empty and $\tilde{h}$ is a morphism.*

Proof. Consider four generic Γ_{00}-divisors $D_1, \ldots, D_4$. Let $b : \tilde{A} \to A$ be the blow up map. Let $\tilde{D}_i$ be the strict transform of D_i in $\tilde{A}$.

We have to show that $\tilde{D}_1 \cap \ldots \cap \tilde{D}_4 \cap E$ is empty and that the D_i's all have multiplicity 4 at 0. The D_i's being generic, the first part is a consequence of 6.16. The second part is a consequence of 6.19. Q.E.D.

Together with a result of [BD2] (pp. 32–33, see also [Iz1]), this gives:

Let A be in $\mathcal{A}_4 \backslash \mathcal{A}_{4\mathrm{dec}}$, then A is a jacobian if and only if the base locus $V_{\mathrm{inf}}(\Gamma_{00})$ of $\tau|2\Theta|_{00}$ is nonempty. In that case, if the unique quadric containing the canonical curve is smooth, $V_{\mathrm{inf}}(\Gamma_{00})$ is the canonical curve or, if the quadric containing the canonical curve is not smooth, $V_{\mathrm{inf}}(\Gamma_{00})$ is the union of the canonical curve and the vertex of the quadric containing it.

Corollary 6.21. *Suppose that $A \in \mathcal{A}_4 \backslash \bar{\mathcal{J}}_4$. The linear system $\tau|2\Theta|_{00}$ has projective dimension greater than or equal to 3. Equivalently, the kernel of $\tau : \Gamma_{00} \to S^4$ (see 2.1.1) has dimension (as a vector space) at most 1.*

Proof. If the linear system is a net (or has smaller dimension), its base locus is nonempty.

Suppose that $A \in \mathcal{A}_4 \backslash \bar{\mathcal{J}}_4$. Let $(\tilde{X}, X) \in P^{-1}(A)$ and $D \in |2\Theta|_{00}$ be both generic (see the beginning of section 1). Fix $\phi : P(\tilde{X}, X) \xrightarrow{\cong} A$. Let $q \in N$ be the quadric in $|K_X|^*$ which is the image of D by the projection from l_X, where N is, as before, the net of quadrics containing κX. Let $\tilde{D}$ be the strict transform of D in $\tilde{A}$. We would like to note the following geometric fact which we will use in 6.23 to extend the definition of l_X to all of $P^{-1}(A)$.

Proposition 6.22. *The intersection $\tilde{D} \cap \tilde{\Sigma}(X)$ is the set of $[s, t]$ such that $\langle \pi s, \pi t \rangle$ lies in q.*

Proof. The divisor D corresponds to a hyperplane H_D in $(|2\Theta|_{00})^*$ and

$$\tilde{D} \cap \tilde{\Sigma}(X) = \tilde{h}^{-1}(H_D \cap \tilde{h}(\tilde{\Sigma}(X))) \cap \tilde{\Sigma}(X).$$

Now, $\tilde{\Sigma}(X) \cup \tilde{\Sigma}(X_\lambda)$ is the base locus of the pencil of divisors obtained as the strict transforms of the elements of l_X. Hence, $\tilde{h}(\tilde{\Sigma}(X))$ which is its image, is the base locus of a pencil of hyperplanes, i.e., a plane, in $(|2\Theta|_{00})^*$. This plane is the set of hyperplanes in $|2\Theta|_{00}$ which contain l_X, so, via the projection from $l_X : |2\Theta|_{00} \to N$, we can, and will, canonically identify it with N^*. Let $l^* = H_D \cap N^*$. An element l of $l^* \subset N^*$ corresponds to a line L in N. By 6.16, for l generic, $\tilde{h}^{-1}(l) \cap \tilde{\Sigma}(X)$ is the set of elements $[s, t]$ such that $\langle \pi s, \pi t \rangle$ is contained in the intersection of the quadrics of L. As $l^* = \{l : L \ni q\}$ and $\tilde{h}^{-1}(l^*) \cap \tilde{\Sigma}(X)$ is the closure of the union of the $\tilde{h}^{-1}(l) \cap \tilde{\Sigma}(X)$ for l generic in l^*, 6.22 is proved.

Recall that we showed in 5.7, 5.11 that the pencil l_X was well-defined over P^λ (with base locus $\Sigma(X)$) and a dense open subset of P^s (with base locus $\Sigma(X) \cup \Sigma(X_\lambda)$) (see 3.17 for the notation).

Definition and Corollary 6.23. *Suppose that $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. The pencil l_X is well-defined for all $(\tilde{X}, X) \in P^s$ (with any given choice of $\phi : P(\tilde{X}, X) \xrightarrow{\cong} A$ up to translations and* $-$ id) *and its base locus contains $\Sigma(X) \cup \Sigma(X_\lambda)$.*

Proof. It is enough to show that the span of $\tilde{h}(\Sigma(X))$ is a plane. It is contained in a plane because it is so generically. Also, it is not a line, otherwise $|2\Theta|_{00}$ induces a pencil on $\Sigma(X)$: by continuity, we deduce from 6.22 that the base locus of this pencil contains the set of pairs which project to elements of $X^{(2)}$ whose corresponding secants are in a pencil of quadrics in $|K_X|^*$: such a set contains points other than 0. This contradicts the fact that the base locus of $|2\Theta|_{00}$ is 0 (5.13). The last assertion is clear. Q.E.D.

So l_X is now defined for all $(\tilde{X}, X) \in P^{-1}(A)$ and $\phi : P(\tilde{X}, X) \xrightarrow{\cong} A$ (determined up to translations and $-$ id) *for $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$.*

Before we determine the branch locus and the ramification locus of $\tilde{h}$, we note the following

Lemma 6.24. *Suppose that $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. Then $\tilde{h}$ has no 2-dimensional fibers.*

Proof. If there is a surface S contracted to a point by $\tilde{h}$, then the trace on S of the strict transform of every Γ_{00}-divisor would be a fixed curve. This contradicts 5.13 or 6.20.

Recall that E is the exceptional divisor in $\tilde{A}$. We define $T^* \subset (|2\Theta|_{00})^*$ to be the set of hyperplanes in $|2\Theta|_{00}$ which are tangent to T at some point.

Theorem 6.25. *Suppose that $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. Then the branch locus of $\tilde{h}$ is the union of $R_0 = \tilde{h}(E)$ and T^*. The variety $\tilde{h}^{-1}(R_0)$ is equal to $E \cup \Delta$ where Δ is the union of the diagonals $\Delta(X)$ of the surfaces $\Sigma(X)$ ($(\tilde{X}, X) \in P^{-1}(A)$). The variety $\tilde{h}^{-1}(T^*)$ is equal to $\mathscr{E}_A \cup R'$. Here R' is the closure of the codimension* 1 *subvariety of elements x of A such that "$\Theta . \Theta_x$ contains at most* 21 *Prym-embedded curves which are not images of each other by a translation or* $-$ id, (*at least*) 6 *of which count twice" and $\mathscr{E}_A$ is the set of elements $x \in A$ such that $\Theta . \Theta_x$ contains a Prym-embedded curve and its image by λ. If A is generic, this definition of $\mathscr{E}_A$ agrees with the one given in* 5.5.

The ramification locus of $\tilde{h}$ is $\Delta \cup E \cup \mathscr{E}_A$. If A is generic, the ramification is generically simple on each component.

Proof. We first suppose that A is generic. Let H be a hyperplane in $|2\Theta|_{00}$ and let $(\tilde{X}, X)$ be an element of $P^{-1}(A)$ such that $l_X \subset H$. Suppose that, by the projection from l_X, H projects to a line L such that $L \cap Q$ contains at least two distinct points. Let $(\tilde{Y}, Y)$, $(\tilde{Y}', Y')$, g_1 and g_2 be as in 6.16. Then, since the base loci of $\tilde{l}_Y$ and $\tilde{l}_{Y'}$ are $\tilde{\Sigma}(Y) \cup \tilde{\Sigma}(Y_\lambda)$ and $\tilde{\Sigma}(Y') \cup \tilde{\Sigma}(Y'_\lambda)$ respectively (5.6), we have

$$\tilde{h}^*(H) \subset (\tilde{\Sigma}(Y) \cup \tilde{\Sigma}(Y_\lambda)) \cap (\tilde{\Sigma}(Y') \cup \tilde{\Sigma}(Y'_\lambda))$$

where $\tilde{h}^*(H)$ is the scheme-theoretic fiber of $\tilde{h}$ at H. The right hand side is contained in $\tilde{\Sigma}(X) \cup \tilde{\Sigma}(X_\lambda)$ hence is finite by 5.9. So $\tilde{h}^{-1}(H)$ is also finite. The number of points, counted with multiplicities, is the same on both sides by 6.16, hence the two sides are equal. As the family of plane quintics which have contact of order 5 at some point with a line is of dimension 10 and A is generic, the fibers of $\tilde{h}$ are finite except, possibly, for a finite number of them.

Now, describing the fibers of $\tilde{h}$ as in 6.16, one can easily deduce that, in codimension 1, $\tilde{h}$ can ramify in 3 ways at H:

i) one of the inverse images x of H lies in $\tilde{\Sigma}(X) \cap \tilde{\Sigma}(X_\lambda) \setminus E$,

ii) the surface S ($= q_1 \cap q_2$ in 6.16) intersection of the quadrics of L contains less than 16 lines,

iii) one of the inverse images of H is in $\Delta(X)$ (and one in E).

In case i) $\tilde{X}$ and $\tilde{X}_\lambda$ are Prym-embedded in $\Theta . \Theta_x$. As T is smooth (6.14), the family of lines in T is smooth and irreducible [CG], hence every line in T corresponds to some Prym-curve. Prym-embedded curves in $\Theta . \Theta_x$ correspond to lines in H and $(\tilde{X}, X)$ and $(\tilde{X}_\lambda, X_\lambda)$ correspond to the same line, so the number of lines in $T_H = T \cap H$ is less than 27, T_H is singular and H is tangent to T.

In case ii) also H is tangent to T: The quartic Del Pezzo surface S contains 16 distinct lines unless it is not smooth, in which case L is tangent to Q. As we are only concerned with points in some dense open subset of the ramification divisor of $\tilde{h}$, we can suppose that Q is nonsingular and that $L \cap Q$ has multiplicity 2 at a unique quadric q_1 and multiplicity 1 elsewhere (if Q has a node, for x in some dense open subset of the divisor $\bigcup_{Q \text{ nodal}} \Sigma(X)$, L is transverse to Q). Since Q is nonsingular, q_1 is of rank 4 and it is easily seen that S has one node, say o, 4 of its lines contain o and "count twice". Take $\langle \pi p, \pi q \rangle$ to be one of the lines through o, then it is easily seen that

$$h^0(g_1 - \pi p - \pi q) > 0 \quad \text{and} \quad h^0(K_X - g_1 - \pi p - \pi q) > 0$$

where g_1 is cut on X by one ruling of q. So, if $(\tilde{Y}, Y)$ is tetragonally related to $(\tilde{X}, X)$ via g_1, $\Theta . \Theta_x$ contains Prym-embeddings of $\tilde{Y}$ and $\tilde{Y}_\lambda$: as in i) H is tangent to T.

In cases i) and ii), let $D \in T$ be the image of H by the birational map $T^* \to T$.

If we suppose that l_X does not contain the node of T_H, then, (in a way similar to 6.16), $\tilde{h}^{-1}(H) \cap \tilde{\Sigma}(X)$ has (generically in codimension 1) 48 distinct elements: 16 of these occur with multiplicity 2 in the scheme-theoretic fiber $\tilde{h}^*(H)$ and are the elements $[s, t]$ of $\tilde{\Sigma}(X)$ corresponding to lines $\langle \pi s, \pi t \rangle \subset S$ containing o. The 32 others correspond to lines in S which do not contain o and occur with multiplicity 1 in $\tilde{h}^*(H)$. Hence, in total, $\tilde{h}^{-1}(H)$ has 96 elements: 32 ramification points of $\tilde{h}$ and 64 points which are not ramification points.

If x is one of the 32 ramification points, then, as we saw, $\Theta . \Theta_x$ contains Prym-embeddings of some curve and its image by λ. The Fano varieties of the double solids with associated Γ_{00}-divisor D contain Prym-embeddings of 6 curves and their images by λ and give us 32 distinct elements in $\tilde{h}^{-1}(H)$ (5.4, 6.4). Hence $\mathscr{E}_A$ is the only component of the ramification locus with image T^*. Write $\tilde{h}^{-1}(T^*) = \mathscr{E}_A \cup R'$. Then, for x in some dense open subset of R', $\Theta.\Theta_x$ contains 21 nonisomorphic Prym-embedded curves.

In case iii), in a way similar to 6.16 again, one sees easily that $\tilde{h}^{-1}(H)$ has (generically) one element in E, which has multiplicity 2 in $\tilde{h}^*(H)$, and one element in Δ also occurring with multiplicity 2 in $\tilde{h}^*(H)$. This completes the proof for A generic.

Let $\tilde{\mathbb{A}}$ be the blow up of the universal family $\phi : \mathbb{A} \to \mathscr{H}$ (see 6.3) along its zero section. Let $\tilde{\mathbb{A}}'$ be its restriction to the locus $\mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. By 6.20, the maps $\tilde{h}$ define a morphism

$$\tilde{\mathbb{H}} : \tilde{\mathbb{A}}' \to \mathbb{P}^*|_{\mathscr{A}_4 \backslash \bar{\mathscr{J}}_4}$$

generically finite on each fiber A (5.10). Here $\mathbb{P}$ is the bundle $\mathbb{P}\phi_* \mathscr{O}_{\mathbb{A}}(2\Theta)$ for some choice of a universal theta divisor on $\mathbb{A}$. The ramification locus of $\tilde{\mathbb{H}}$ is a divisor and intersects each fiber A in a divisor because the degree of $\tilde{h}$ is always the same (6.17). So the ramification locus of $\tilde{\mathbb{H}}$ is the closure of the ramification locus of its restriction to any dense open subset of $\tilde{\mathbb{A}}'$. Hence, from the result in the generic case, we deduce that it has three components: the exceptional locus $\mathbb{E}$ in $\tilde{\mathbb{A}}'$, $\mathscr{E} = \bigcup \mathscr{E}_A$ and the union $\mathbb{D}$ of the diagonals $\Delta(X)$ of $\Sigma(X)$ for all Prym-curves $(\tilde{X}, X)$. The components $\mathbb{E}$ and $\mathbb{D}$ have the same image by $\tilde{\mathbb{H}}$ whereas the image of $\mathscr{E}_A$ is the set of hyperplanes in $|2\Theta|_{00}$ which either contain less than 27 lines corresponding to Prym-curves or contain a 1-dimensional family of them, i.e., T^*. The statement about R' is clear. Q.E.D.

We now have a precise description of the possible singularities of T (these are all isolated double points by 6.13).

Proposition 6.26. *Suppose that* $A \in \mathscr{A}_4 \backslash \bar{\mathscr{J}}_4$. *The double points of* T *are the divisors* $2\Theta_\upsilon$ *where* υ *is a vanishing theta-null on* A.

Proof. Let D be a Γ_{00}-divisor which is a double point of T. Then the set of hyperplanes of $|2\Theta|_{00}$ containing D is a component H_D of T^* which is a hyperplane in $(|2\Theta|_{00})^*$. By 6.25, $\tilde{h}$ is ramified on H_D, hence (the scheme-theoretic inverse image) $\tilde{h}^*(H_D) = nD'$ for some divisor $D' \subset A$ and $n > 1$. However, $\tilde{h}^*(H_D) = D$, $D \equiv 2\Theta$ and $[\Theta]$ is a minimal homology class, hence $n = 2$ and D' is a symmetric theta divisor on A. So $D' = \Theta_\upsilon$ for some $\upsilon \in A$. As $2D' \in |2\Theta|_{00}$, it follows that υ is a vanishing theta-null on A.

Conversely, it is easily seen that every vanishing theta-null υ on A gives rise to a double point of T via $\upsilon \mapsto 2\Theta_\upsilon$.

From now on in this section we suppose *A generic*. We wish to say a few words about points of order 2. Recall that F is the Fano variety of lines in T. We first observe

Lemma 6.27. *The morphism* $l : P^{-1}(A) \to F$, *which to each* $(\tilde{X}, X)$ *associates its line* l_X, *is an unramified double cover.*

Proof. The map $l : P^{-1}(A) \to F$ is a quasi-finite morphism (5.6). As F is an irreducible projective surface ([CG], p. 312) and $P^{-1}(A)$ is complete ([B1], p. 179), this is a finite morphism and its degree is 2. As λ is fixed point free by 3.17, the cover l is unramified. Q.E.D.

Fix $(\tilde{X}, X) \in P^{-1}(A)$ with X smooth. Let $Q \subset N$ be as before (1.3, 6.15) and suppose Q smooth. Recall (6.13) that the plane quintic Q also parametrizes singular quadrics through κX_λ. Therefore, on JQ we are given three points of order 2:

– α with associated double cover $\tilde{Q} \to Q$ and $P(Q, \alpha) = JT$: $\tilde{Q}$ parametrizes the family of lines in T which are incident to l_X. The morphism $\tilde{Q} \to Q$ sends a line l incident to l_X to its projection from l_X which lies on $Q \subset \mathbb{P}^2$ because Q is the discriminant curve of the projection of T from l_X (6.13). It is well-known [CG] that $P(\tilde{Q}, Q) = JT$.

– α' with associated double cover $\tilde{Q}' \to Q$ and $P(Q, \alpha') = JX$ (1.3).

– α'' with associated double cover $\tilde{Q}'' \to Q$ and $P(Q, \alpha'') = JX_\lambda$ (1.3).

The Weil pairing on $(JQ)_2$ (the group of points of order 2 in JQ) is given by ([M1], p. 182)

$$(\beta, \beta') \equiv h^0(g_5^2) + h^0(g_5^2 \otimes \beta) + h^0(g_5^2 \otimes \beta') + h^0(g_5^2 \otimes \beta \otimes \beta') \mod 2 .$$

Proposition 6.28. *The points $\alpha, \alpha', \alpha''$ are the nonzero elements of an $\mathbb{F}_2$-vector-subspace V_2 of $(JQ)_2$ which is totally isotropic with respect to $(,)$.*

Proof. Let $\hat{Q}$ be the inverse image of $\tilde{Q}$ in $P^{-1}(A)$. The involution of $\hat{Q}$ for the cover $\hat{Q} \to \tilde{Q}$ is λ and, by 6.8, $\hat{Q}$ parametrizes Prym-curves tetragonally related to $(\tilde{X}, X)$. Let q be an element of Q. Let g_1 and h_1 $(\in \tilde{Q}')$ be the two opposite g_4^1's of X above q (1.3). Let $(\tilde{Y}, Y)$ and $(\tilde{U}, U)$ be tetragonally related to $(\tilde{X}, X)$ via g_1, then (see 3.13) $(\tilde{Y}_\lambda, Y_\lambda)$ and $(\tilde{U}_\lambda, U_\lambda)$ are tetragonally related to $(\tilde{X}, X)$ via h_1. Sending $(\tilde{Y}, Y)$ and $(\tilde{U}, U)$ to g_1, we see that $\hat{Q} \to Q$ factors through $\tilde{Q}'$. Analogously, $\hat{Q} \to Q$ factors through $\tilde{Q}''$. Let $\sigma_{\alpha'}$ and $\sigma_{\alpha''}$ be the involutions of $\hat{Q}$ for the covers $\hat{Q} \to \tilde{Q}'$ and $\hat{Q} \to \tilde{Q}''$. Then $\sigma_{\alpha'}(\tilde{Y}, Y) = (\tilde{U}, U)$ and $\sigma_{\alpha'}(\tilde{Y}_\lambda, Y_\lambda) = (\tilde{U}_\lambda, U_\lambda)$. Similarly $\sigma_{\alpha''}(\tilde{Y}, Y) = (\tilde{U}_\lambda, U_\lambda)$ and $\sigma_{\alpha''}(\tilde{Y}_\lambda, Y_\lambda) = (\tilde{U}, U)$. Therefore, the three involutions commute and the product of two of them is equal to the third one. Thus they are the nonzero elements of a group of automorphisms of $\hat{Q}$ isomorphic to $(\mathbb{F}_2)^2$. This proves the first assertion, the second assertion is a simple computation.

The variety JT has a canonical theta divisor Θ_T with a unique singular point (of multiplicity 3) at 0 ([B4], p. 190). Hence there is a canonical $\mathbb{F}_2$-valued quadratic form on $(JT)_2$ defined by sending a point β of order 2 to the multiplicity of Θ_T at β modulo 2. Let $r: \tilde{Q} \to Q$ be the double cover given by α. It follows from [V], section 3, that the canonical $\mathbb{F}_2$-valued quadratic form on $(JT)_2$ coincides with $\beta \mapsto h^0(\tilde{Q}, \beta(r^* g_5^2)) \mod 2$. We say that $\beta \in (JT)_2$ is even if $h^0(\tilde{Q}, \beta(r^* g_5^2)) \equiv 0 \mod 2$.

Proposition 6.29. *The point of order 2 in JT associated to the double cover $l: P^{-1}(A) \to F$ (via the identification of JT with the albanese variety of F* ([CG], p. 334)) *is even.*

Proof. In the exact sequence

$$(3) \qquad 0 \to \{\alpha\} \to \{\alpha\}^\perp \to (JT)_2 \to 0$$

associated to our Prym construction ([M2], p. 332, here "$\perp$" means "orthogonal complement with respect to $(,)$") V_2 projects to a subgroup of order 2 generated by a point μ. The point μ gives (by restriction to F) an unramified double cover $l': \tilde{F} \to F$ whose restriction to $\tilde{Q}$ is $\hat{Q} \to \tilde{Q}$. The embeddings $\hat{Q} \hookrightarrow \tilde{F}$ fit together to give us a morphism

$m: P^{-1}(A) \to \tilde{F}$ such that $l = l'm$. The morphism m is bijective and $\tilde{F}$ is normal, hence m is an isomorphism. As $\mu = r^*\alpha' = r^*\alpha''$ ($\mu \in (JT)_2 \subset (J\tilde{Q})_2$), by the projection formula

$$h^0(\tilde{Q}, \mu(r^*g_5^2)) = h^0(Q, \alpha'(g_5^2)) + h^0(Q, \alpha''(g_5^2)) = 0 + 0$$

(see [ACGH], p. 300 for instance).

Remark 6.30. As Donagi observes, one can recover A from (T, μ): choose a discriminant curve Q for T and obtain V_2 as the inverse image of $\{0, \mu\}$ in the sequence (3) above. Then one obtains JX and JX_λ as the Prym varieties of Q for the nonzero points α', α'' in V_2 distinct from α ($P(Q, \alpha) = JT$). In the exact sequences

$$0 \to \{\alpha'\} \to \{\alpha'\}^\perp \to (JX)_2 \to 0,$$
$$0 \to \{\alpha''\} \to \{\alpha''\}^\perp \to (JX_\lambda)_2 \to 0$$

analogous to (3), V_2 projects to two subgroups of order 2 generated by η and η_λ respectively and one recovers A as $P(X, \eta) = P(X_\lambda, \eta_\lambda)$.

6.31. Here we show how to obtain Donagi's construction [Do4] from ours. This will show that for a generic ppav our cubic threefold is isomorphic to Donagi's and our involution λ coincides with his.

Consider a plane section of T, union of three lines l_X, l_Y, l_U. By 6.9 we can suppose that $\{(\tilde{X}, X), (\tilde{Y}, Y), (\tilde{U}, U)\}$ is a tetragonally related triple. The three plane quintics Q, Q_1, Q_2 associated to X, Y, U with their double covers $\tilde{Q}, \tilde{Q}_1, \tilde{Q}_2$ form a tetragonal triple:

Let $t \in Q$ be the common image of l_Y and l_U by the projection from l_X. For each line $l \subset T$ incident to l_Y there are 4 lines in T containing $l \cap l_Y$ besides l and l_Y: these give a lifting in $\tilde{Q}$ of a divisor of $|g_5^2 - t|$ on Q; similarly for lines incident to l_U.

Let $(\tilde{D}', D')$ and $(\tilde{D}'', D'')$ be tetragonally related to $(\tilde{Q}', Q)$ through $|g_5^2 - t|$, then D' and D'' are trigonal because the common Prym variety of the tetragonal triple is JX: by [M2], p. 344, and a dimension count, the only smooth curves with Prym varieties generic jacobians of dimension 5 are trigonal curves and plane quintics (there is only one $(\tilde{Q}', Q) \in \mathscr{P}_6$ verifying $P(\tilde{Q}', Q) = JX$ with Q a plane quintic). The same is true for the tetragonal triple obtained from $(\tilde{Q}'', Q)$, and also for those obtained with Q_1 and Q_2.

Let W be the curve of genus 7 trigonally related to the triple (Q, Q_1, Q_2) (1.4). The curve W comes with the three points β, β_1, β_2 of order 2 with respective Prym varieties JQ, JQ_1, JQ_2 and with a totally isotropic (with respect to the antisymmetric form $(\,,)$ introduced just before 6.28) $\mathbb{F}_2$-vector space V_3 of dimension 3 which is the common inverse image of the totally isotropic vector spaces in $(JQ)_2$ (called V_2 in 6.28), $(JQ_1)_2$, $(JQ_2)_2$ by the exact sequences analogous to (3).

So we have 7 curves of genus 6 whose jacobians are Pryms of this trigonal curve for the elements of $\mathbb{P} V_3$. Three of these are the plane quintics and the other 4 are the trigonal curves D', D'' and their analogues. The three points β, β_1, β_2 are on a line in $\mathbb{P} V_3$. The same is true for every set of three points associated to a tetragonal triple. Hence each line is associated to a ppav of dimension 5 which is the Prym variety of the tetragonal triple associated to that line; six of these are jacobians and the last one is JT; they all come with a point of order 2 image of V_3 (these are μ and η's with our previous notations). The common Prym variety for the ppav's of dimension 5 in the configuration (in the more general sense of Mumford in [M2], section 2, "Prym varieties" can be defined for every abelian variety with certain data) is A. The above was done by Donagi in reverse order (start with a trigonal W and V_3): this is how he introduced (T, μ) for a generic ppav.

We represent $\mathbb{P}V_3$ by a "triangle" (figure 1).

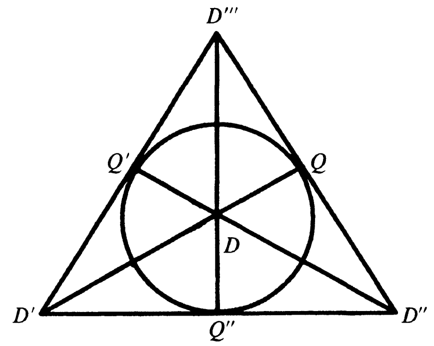


Figure 1. The triangle

We represent V_3 by a "cube". One of the vertices of the "cube" is the origin (figure 2).

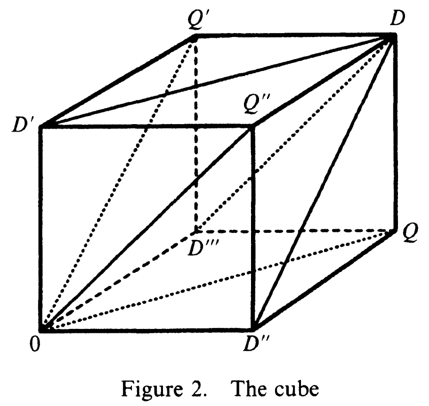


Figure 2. The cube

Now dualize V_3: we represent this by another "cube" with vertices: the origin, $T, X, Y, U, X_\lambda, Y_\lambda, U_\lambda$ (figure 3).

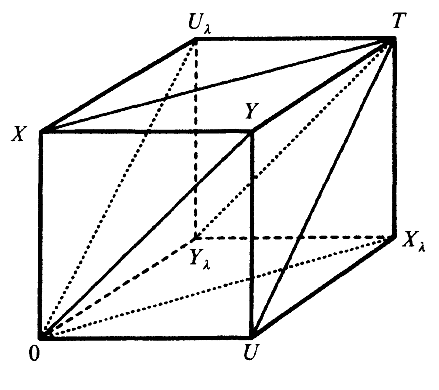


Figure 3. The dual of the cube

The projectivization of the "cube" is another "triangle": $\mathbb{P}(V_3)^*$ (figure 4).

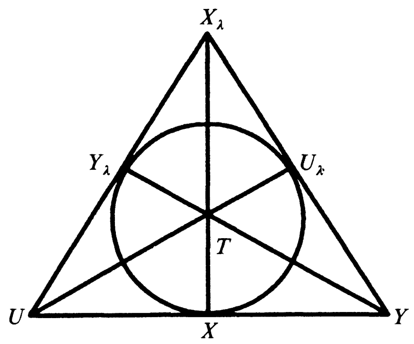


Figure 4. The dual of the triangle

Embed $\mathbb{P}(V_3)^*$ in $\mathbb{P}^5(F_2)$ by the Veronese map. Then project from T: we can represent the image of $\mathbb{P}(V_3)^*$ by an "octahedron" with vertices the six "straight lines" in figure 1 or the six "planes" (in the euclidian sense) through 0 in figure 2. Each vertex corresponds to one of the curves X, Y, U, X_λ, Y_λ, U_λ (see also the proof of 5.8).

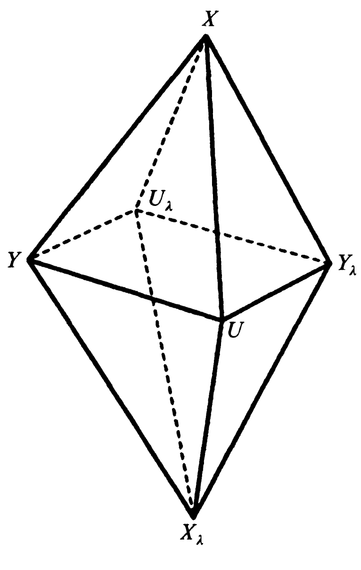


Figure 5. The octahedron

7. (Singular) cubic threefolds over $\mathcal{J}_4 \backslash \mathcal{J}_{\text{hyp}}$

We wish to define the cubic threefold for the jacobian $A = JC$ of a smooth non-hyperelliptic curve of genus 4 and prove the assertions of theorem 2 of the introduction in this case.

As we saw in 4.2 and 4.3, $P^{-1}(A)$ has two components, each isomorphic to the quotient of $C^{(2)}$ by the automorphism group of C and interchanged by λ. In particular, $P^{-1}(A)$ is two-dimensional.

By [W2], the base locus $V(\Gamma_{00})$ of $|2\Theta|_{00}$ is set-theoretically equal to $C - C \cup \{\pm(K_C - 2g_3^1)\}$. (By [Iz1] the equality is scheme-theoretic outside 0.)

Using 4.3 and with the same notation, it is easily seen that

$$\Sigma(C_{pq}) = C - C \cup W_2 - p - q \cup p + q - W_2 .$$

Also, for a trigonal $(\tilde{X}, X) \in P^{-1}(A)$, associated to g_4^1 on C:

$$\Sigma(X) = \{s + t - s' - t' : h^0(g_4^1 - s - t) > 0 \text{ and } h^0(g_4^1 - s' - t') > 0\} .$$

For $(\tilde{X}, X)$ smooth, we can define the pencil $l_X = l_{X_\lambda}$ as in 5.7, the base locus of each pencil will be $\Sigma(X) \cup \Sigma(X_\lambda)$. So we can define T to be the reduced closed subvariety of $|2\Theta|_{00}$ containing the lines corresponding to Prym-curves. The variety T is irreducible because $P^{-1}(A)$ has two irreducible components exchanged by λ.

We still have the projection from $l_X : |2\Theta|_{00} \to N$. Under this projection, the lines corresponding to Prym-curves tetragonally related to X go to points on the plane quintic Q parametrizing quadrics through κX which are singular at a point of κX (see 1.3).

So T cannot be a plane nor a hyperplane, T is a threefold and by continuity the degree of T is less than or equal to 3.

Recall that in 4.5 we computed the number of Prym-embedded curves in a generic intersection of translates of Θ for a jacobian. Hence a generic hyperplane contains exactly 27 distinct lines. In a way analogous to 6.12 and 6.13, one proves

Proposition 7.1. *Suppose that* $A \in \mathscr{J}_4 \backslash \mathscr{J}_{\text{hyp}}$. *Then* T *is a cubic threefold with at worst a finite number of double points.*

As before it follows that T has conic-bundle structures over the nets N with discriminant curves Q. If $(\tilde{X}, X)$ is smooth and trigonal, Q is obtained from X by identifying two points. We deduce

Proposition 7.2. *Suppose that* $A \in \mathscr{J}_4 \backslash \mathscr{J}_{\text{hyp}}$. *Then* T *has exactly one double point.*

Let ξ be one of the two (possibly equal) g_3^1's on C and let $\xi' = |K_C - \xi|$. The two components of $P^{-1}(JC)$ intersect in the set of singular curves C_{pq} such that $h^0(\xi - p - q) > 0$ or $h^0(\xi' - p - q) > 0$. One sees immediately that

$$\Sigma(C_{pq}) \cup \Sigma((C_{pq})_\lambda) \subset (W_3 - \xi) \cup (W_3 - \xi')$$

which is a Γ_{00}-divisor, it follows easily that $l_{C_{pq}}$ is well-defined with base locus $\Sigma(C_{pq}) \cup \Sigma((C_{pq})_\lambda)$.

Suppose, for instance, that $h^0(\xi - p - q) > 0$. Then ξ induces a g_3^1 on C_{pq}. Hence, by continuity (from the smooth trigonal case, see 1.3), the plane quintic Q_{pq} associated to C_{pq} is obtained from C_{pq} by identifying two points. But Q_{pq} is also obtained from the curve C'

parametrizing the set of lines through the double point of T by identifying two pairs of points. Hence $C' \cong C$:

Lemma 7.3. *C parametrizes the set of lines through the double point of T.*

Corollary 7.4. *Suppose that* $A \in \mathcal{J}_4 \backslash \mathcal{J}_{\text{hyp}}$. *Then* C' *is nondegenerate and the quadric tangent cone to T at its double point has rank* $\geqq 3$.

Proof. The curve C' is irreducible and nondegenerate because, for all p, q such that $h^0(\xi - p - q) > 0$, Q_{pq} is its projection from one of its points, namely $s = \xi - p - q$. The quadric tangent cone to T at its double point contains C' hence it is of rank $\geqq 3$.

By [CG], when T has an ordinary double point, $C' = \kappa C$. Hence, for C generic, $C' = \kappa C$. By continuity, C' is always a space sextic of degree 6. Since we know that C' is nondegenerate and isomorphic to C, it has to be equal to κC because a smooth non-hyperelliptic curve of genus 4 has only one embedding as a nondegenerate space sextic. This completes the proof of theorem 2 of the introduction.

Notations

$\mathcal{A}_4$	Introduction	$\mathcal{P}_5$	Introduction
$(\tilde{X}, X)$	Introduction	P	Introduction
JT, JX	Introduction	A	Introduction
F	Introduction	μ	Introduction
T, T_A	Introduction	λ	Introduction
$(\tilde{X}_\lambda, X_\lambda)$	Introduction		
$\mathcal{J}_4$	Introduction	$\bar{\mathcal{J}}_4$	Introduction
$\mathcal{J}_{\text{hyp}}$	Introduction	$\bar{\mathcal{J}}_{\text{hyp}}$	Introduction
$\mathcal{A}_{4\text{dec}}$	Introduction	Θ	Introduction
Γ_{00}-divisor	Introduction	$\|2\Theta\|_{00}$	Introduction
$\Sigma(X)$	Introduction and section 1.2		
l_X	Introduction and 5.6	θ_{null}	Introduction
$\tau\|2\Theta\|_{00}$	Introduction	h	Introduction
$\tilde{h}$	Introduction	$\tilde{A}$	Introduction
T^*	Introduction	l	Introduction
Θ_t, Θ_x	Introduction and section 2.3	Γ_{00}	section 1.1
π	section 1.2	η	section 1.2
σ	section 1.2	A^-	section 1.2
ω_X	section 1.2	ν	section 1.2
L_p	section 1.2	χX	section 1.2
$\tilde{X}^{(2)}$	section 1.2	Σ_A	section 1.2
$[p, q]$	section 1.2	$\text{Pic}^d X$	section 1.3
W_d^r	section 1.3	g_d^r	section 1.3
$\text{Div}^d X$	section 1.3	K_X	section 1.3
N	section 1.3	κX	section 1.3
Q	section 1.3	Θ'	section 1.3
$\tilde{Q}'$	section 1.3	h^0	section 1.4

Z	section 1.5	B	section 1.5
$p_1, \ldots, p_6$	section 1.5	$\tilde{Z}$	section 1.5
JZ	section 1.5	$\mathcal{X}$	section 1.5
J	section 1.5	$\tilde{p}$	section 1.5
$\mathcal{Q}_i$	section 1.5	AJ	section 1.5
$(\tilde{X}_i, X_i)$	section 1.5	$(\tilde{X}'_i, X'_i)$	section 1.5
$(\tilde{X}_{ij}, X_{ij})$	section 1.5	$(\tilde{X}'_{ij}, X'_{ij})$	section 1.5
ϱ	section 1.5	$\imath$	section 1.5
$\mathbb{P}T_0A$	section 1.5	$\mathcal{F}$	section 1.5
$\imath_{\mathcal{F}}$	section 1.5	$\mathcal{G}$	section 1.5
$\imath Z, \imath\mathcal{F}Z$	section 1.5	$Z_i, \imath Z_i$	section 1.5
τ	2.1.1	Θ_x, θ_ξ	2.1.1
T_0A	section 2.1	D_Z	section 2.1
$\mathcal{X}_A$	section 2.1	E_Z	section 2.1
t_x	section 2.3	W_p	section 2.4
$\langle , \rangle$	section 2.5	type$(0, 2)$	3.2
type $(1, 1)$	3.3	$\mathcal{P}_{1,1}$	3.3
$\mathcal{A}_{1,1}$	3.3	$\mathbf{R}$	3.4
$\mathbf{A}$	3.4	$\mathbf{P}(\tilde{X}, X)$	3.17
P^s	3.17	$C_{pq}, \tilde{C}_{pq}$	4.2
$\mathcal{E}_A$	5.5 and 6.25	$\mathcal{E}'$	6.3
$\mathbb{A}$	6.3	$\mathcal{H}$	6.3

Bibliography

[ACGH] *E. Arbarello, M. Cornalba, P. Griffiths, J. Harris*, Geometry of Algebraic curves, vol. I, Springer-Verlag, Berlin–Heidelberg–New York 1985.

[BC] *C. Barton, H. Clemens*, A result on the Chow ring of a generic principally polarized abelian variety of dimension four, Comp. Math. **34** (1977), 49–67.

[B1] *A. Beauville*, Prym varieties and the Schottky problem, Invent. Math. **41** (1977), 149–196.

[B2] *A. Beauville*, Variétés de Prym et jacobiennes intermédiaires, Ann. Sci. E. N. S. (4) **10** (1977), 309–391.

[B3] *A. Beauville*, Sous-variétés spéciales des variétés de Prym, Comp. Math. **45** (1982), 357–383.

[B4] *A. Beauville*, Les singularités du diviseur Θ de la jacobienne intermédiaire de l'hypersurface cubique dans $\mathbb{P}^4$, Lect. Notes Math. **947** (1982), 190–208.

[BD1] *A. Beauville, O. Debarre*, Sur le Problème de Schottky pour les variétés de Prym, Ann. Sc. Norm. Sup. Pisa (4) **14** (1987), 613–623.

[BD2] *A. Beauville, O. Debarre*, Sur les fonctions thêta du second ordre, Arithmetic of Complex manifolds, Lect. Notes Math. **1399** (1989), 27–39.

[C1] *H. Clemens*, Double Solids, Adv. Math. **47** (1983), 107–230.

[C2] *H. Clemens*, The fiber of the Prym map and the period map for double solids, as given by Ron Donagi, University of Utah, unpublished.

[CG] *H. Clemens, P. Griffiths*, The intermediate jacobian of the cubic threefold, Ann. Math. **95** (1972), 281–356.

[De1] *O. Debarre*, Sur les variétés abéliennes dont le diviseur thêta est singulier en codimension 3, Duke Math. J. **57** no 1 (1988), 221–273.

[De2] *O. Debarre*, Intersection of theta divisors in a 4-dimensional Prym, Note, University of Iowa, unpublished.

[Dm] *M. Demazure*, Séminaire sur les singularités des surfaces, Lect. Notes Math. **777** (1980), 23–69.

[DO] *I. Dolgachev, D. Ortland*, Point sets in projective spaces and theta functions, Astérisque **165** (1988).

[Do1] *R. Donagi*, The tetragonal construction, Bull. A. M. S. **4** (1981), 181–185.

[Do2] *R. Donagi*, The fiber of the period map for double solids of genus 4, Notes, University of Pennsylvania, unpublished.

[Do3] *R. Donagi*, The Schottky problem, Theory of moduli, Lect. Notes Math. **1337** (1988), 84–137.

[Do4] *R. Donagi*, The fibers of the Prym map, Proceedings of the 1990 Schottky Problem conference, Contemp. Math.

[DS] *R. Donagi, R. Smith*, The structure of the Prym map, Acta Math. **146** (1981), 25–102.

[GG] *B. van Geemen, G. van der Geer*, Kummer varieties and the moduli spaces of abelian varieties, Am. J. Math. **108** (1986), 615–642.

[GH] *P. Griffiths, J. Harris*, Principles of algebraic geometry, Wiley, New York 1978.

[Ig] *J.I. Igusa*, Theta Functions, Springer-Verlag, New York–Heidelberg–Berlin 1972.

[Iz1] *E. Izadi*, Fonctions thêta du second ordre sur la jacobienne d'une courbe lisse, Math. Ann. **289** (1991), 189–202.

[Iz2] *E. Izadi*, On the moduli space of 4-dimensional principally polarized abelian varieties, Thesis, University of Utah 1991.

[M1] *D. Mumford*, Theta characteristics of an algebraic curve, Ann. Sci. Éc. Norm. Sup. **4** (1971), 181–192.

[M2] *D. Mumford*, Prym varieties I, Contributions to Analysis (L.V. Ahlfors, I. Kra, B. Maskit, L. Niremberg, eds.), Academic Press, New York (1974), 325–350.

[M3] *D. Mumford*, Varieties defined by quadratic equations (with an appendix by G. Kempf), Questions on algebraic varieties, ed. Cremonese, Roma (1970), 29–100.

[R] *S. Recillas*, Jacobians of curves with a g_4^1 are Prym varieties of trigonal curves, Bol. Soc. Math. Mexicana **19** (1974), 9–13.

[S] *I.R. Shafarevich*, Algebraic Surfaces, Proc. Steklov Inst. Math. **75** (1965), A.M.S. Providence, R.I., 1967.

[V] *R. Varley*, Weddle's surfaces, Humbert curves, and a certain 4-dimensional abelian variety, Amer. J. Math. **108** (1986), 931–952.

[W1] *G.E. Welters*, Abel-Jacobi isogenies for certain types of Fano threefolds, Math. Centre Tracts **141**, Math. Centrum, Amsterdam 1981.

[W2] *G.E. Welters*, The surface $C - C$ on Jacobi varieties and second order theta functions, Acta Math. **157** (1986), 1–22.

Harvard University, Department of Mathematics, 1 Oxford Street, Cambridge, MA 02138, USA

e-mail: izadi@math.harvard.edu

Eingegangen 6. April 1993, in revidierter Fassung 17. November 1994