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COUNTER-EXAMPLES OF HIGH CLIFFORD INDEX TO PRYM-TORELLI

E. IZADI AND H. LANGE

Abstract

We give examples of curves of arbitrarily high Clifford index such that the Prym map is not injective at any of their étale double covers.

Introduction

To any (non-trivial) étale double covering $\kappa : \widetilde{X} \to X$ of a smooth projective curve X of genus $g \geq 2$ one can associate a principally polarized abelian variety $P(\kappa)$ of dimension $g-1$ in a canonical way, the *Prym variety* of κ. This induces a morphism

$$pr_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$$

from the moduli space $\mathcal{R}_g$ of (non-trivial) étale double coverings of curves of genus g to the moduli space $\mathcal{A}_{g-1}$ of principally polarized abelian varieties of dimension $g-1$, called the *Prym map*. It was shown independently by Friedman-Smith [10], Kanev [13], Welters [20] and Debarre [6], that pr_g is generically injective for $g \geq 7$. On the other hand, Mumford showed in [16, p. 346], that the Prym variety of an étale double cover of a hyperelliptic curve is the product of the Jacobians of two other hyperelliptic curves (one of which can have genus 0). Via a dimension count, this implies that the Prym map has positive dimensional fibers on the locus of hyperelliptic Jacobians. Then Beauville remarked in [2, Contre-exemple 7.13, p. 388] that pr_g is not injective for $3 \leq g \leq 10$ at some non-hyperelliptic curves. In [7], Donagi gave a construction showing that pr_g is not injective at any étale double cover

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of a curve X admitting a map $X \to \mathbb{P}^1$ of degree 4 under some generality assumptions. Moreover, he conjectured (see [7, Conjecture 4.1] or [18, p. 253]) that pr_g is injective at any $\kappa : \widetilde{X} \to X$, whenever X does not admit a g^1_4.

Verra showed in [19] that pr_{10} is not injective at any étale double cover of a general plane sextic. In [17], Naranjo found examples of admissible double covers of nodal curves, where the Prym map fails to be injective, which were not obtained from Donagi's construction. However, the curves which either admit a g^1_4 or are plane sextics (more precisely, have a g^2_6) are exactly the curves of Clifford index ≤ 2. Furthermore, Naranjo's examples are double covers of specializations of bielliptic curves, hence specializations of curves of Clifford index ≤ 2. So, in [14], Lange and Sernesi ask whether pr_g is injective at $\kappa : \widetilde{X} \to X$ whenever X is of Clifford index ≥ 3. It is the aim of this paper to show that this is not the case. Our main result is the following theorem.

Theorem 0.1. *For any integer N there is a curve X of Clifford index at least N such that the Prym map is not injective at any étale double cover of X.*

If $\rho_4 : X \to Y$ is a ramified 4-fold cover of smooth projective curves, the tetragonal construction generalizes immediately to associate to any étale double cover $\kappa : \widetilde{X} \to X$ two other étale double covers $\tau_i : \widetilde{C}_i \to C_i$ where C_i is a 4-fold cover of Y of the same genus as X. It was shown in [12, Paragraph 6.5] that the corresponding Prym varieties $P = P(\kappa)$ and $P_i = P(\tau_i)$ are isogenous. In the special case $Y \cong \mathbb{P}^1$ Donagi showed that they are isomorphic. We show that, under small generality assumptions, they are also isomorphic for an arbitrary curve Y, which leads to Theorem 0.1.

We know two existing proofs of Donagi's theorem, one via degeneration which is given in [8] and the other using the cohomology class of the Abel-Prym curve (see [4]). Neither existing proof seems to generalize to arbitrary base curves Y. We give a proof of this using an explicit correspondence that induces the isogeny $P \to P_i$. We show that this isogeny has degree $2^{2g_X - 2}$ and factors through multiplication by 2, which gives the isomorphism. This proof works in particular in the case $Y = \mathbb{P}^1$, thus giving a third proof of Donagi's theorem.

In a different direction, we consider the question of the existence of irreducible curves representing multiples of the minimal class in Prym varieties. The minimal cohomology class for curves in an abelian variety A of dimension g with principal polarization Θ is

$$\frac{[\Theta]^{g-1}}{(g-1)!}.$$

Our construction above of the curves $\widetilde{C}_i$ can also be done if we replace 4-fold covers ρ_4 by n-fold covers ρ_n, for any $n \geq 3$. We prove the following (see Theorem 6.6 below).

Theorem 0.2. *For any unramified double cover $\kappa : \widetilde{X} \to X$ and any simply ramified n-sheeted cover $\rho_n : X \to Y$ as in Section 1, the class of the image of $\widetilde{C}_i$ in the Prym variety P of κ is*

$$[\widetilde{C}_i] = 2^{n-1} \frac{[\Theta_P]^{g_X - 2}}{(g_X - 2)!}$$

where Θ_P is the principal polarization of P as the Prym variety of the double cover κ.

Our original motivation for wanting to produce examples of curves with highly split Jacobians (see [12]: the curves $\widetilde{C}_i$ have this property) was to find examples of integral curves representing multiples of the minimal class in principally polarized abelian varieties.

We work over the field of complex numbers.

1. Summary of previous results

1.1. Let us recall the setup and some of the results of [12]. Consider the following maps of smooth projective curves

$$\widetilde{X} \xrightarrow{\kappa} X \xrightarrow{\rho_n} Y$$

where Y is of genus g_Y, ρ_n is a *simply* ramified cover of degree $n \geq 3$, X has genus g_X and $\widetilde{X}$ is an étale double cover of X which is NOT obtained by base change from a double cover of Y.

Then Y embeds into the symmetric power $X^{(n)}$ via the map sending a point y of Y to the divisor obtained as the sum of its preimages in X. In other words, if $\overline{x}_1, \ldots, \overline{x}_n$ are the pre-images of y in X, with one point repeated if it is a ramification point of ρ_n, then y is sent to the point

$$\overline{z} := \overline{x}_1 + \ldots + \overline{x}_n \in X^{(n)}.$$

Let $\widetilde{C} \subset \widetilde{X}^{(n)}$ be the curve defined by the fiber product diagram

$$\begin{array}{ccc} \widetilde{C} & \longrightarrow & \widetilde{X}^{(n)} \\ \downarrow & & \downarrow \kappa^{(n)} \\ Y & \longrightarrow & X^{(n)}. \end{array} \tag{1.1}$$

In other words, the curve $\widetilde{C}$ parametrizes the liftings of points of Y to $\widetilde{X}$. In [12, Lemma 1.1] we proved that $\widetilde{C}$ is smooth.

Recall that the Norm map $Nm : Pic^n\widetilde{X} \to Pic^n X$ can be defined as $\mathcal{O}_{\widetilde{X}}(D) \mapsto \mathcal{O}_X(\kappa_* D)$ and that its kernel has two connected components that are translates of the Prym variety P of the double cover $\kappa : \widetilde{X} \to X$. Therefore the fibers of the induced map

$$Nm|_Y : Nm^{-1}(Y) \longrightarrow Y \hookrightarrow X^{(n)} \longrightarrow Pic^n X$$

are disjoint unions of two translates of P. Let $\widetilde{Y} \to Y$ be the étale double cover parametrizing the components of the fibers of $Nm|_Y$. Then, by the definition of $\widetilde{C}$, the composite map

$$\widetilde{C} \hookrightarrow \widetilde{X}^{(n)} \longrightarrow Pic^n \widetilde{X}$$

induces a map $\widetilde{C} \to \widetilde{Y}$ whose composition with $\widetilde{Y} \to Y$ is the natural map $\widetilde{C} \to Y$ from (1.1). We proved in [12, pp. 187] that the curves $\widetilde{C}$ and $\widetilde{Y}$ have the same number of connected components. We shall assume, as in [12], that the double cover $\widetilde{Y} \to Y$ is trivial, i.e., the curve $\widetilde{C}$ has two connected components which we denote $\widetilde{C}_1$ and $\widetilde{C}_2$. As explained in [12, pp. 187–188], a sufficient condition for this is for the monodromy of the cover $\widetilde{X} \to Y$ to factor through the Weyl group of D_n. This is always the case if $Y \cong \mathbb{P}^1$.

The curve $\widetilde{C}$ has an involution σ defined as follows. For each i, let x_i and x_i' be the two preimages of $\overline{x}_i$ in $\widetilde{X}$. Then

$$z := x_1 + \ldots + x_n$$

is a point of $\widetilde{C}$ and

$$\sigma(z) = x_1' + \ldots + x_n'.$$

Let C be the quotient of $\widetilde{C}$ by σ.

The degrees of the maps $\widetilde{C} \to Y$ and $C \to Y$ are 2^n and 2^{n-1}, respectively. It is easily seen that σ is fixed-point-free if $n \geq 3$. Also, we can see that for each ramification point $\overline{x}_1 = \overline{x}_2$ of ρ_n there are 2^{n-2} ramification points in a fiber of $\widetilde{C} \to Y$ obtained as $x_1 + x_1' + D_{n-2}$ where D_{n-2} is one of the 2^{n-2} divisors on $\widetilde{X}$ lifting $\overline{x}_3 + \ldots + \overline{x}_n$.

Two liftings of $\overline{z}$ are in the same connected component of $\widetilde{C}$ if and only if they differ by an even number of points of $\widetilde{X}$. Half of the divisors $x_1 + x_1' + D_{n-2}$ lie in $\widetilde{C}_1$ and the other half lies in $\widetilde{C}_2$. So the degree of the map $\widetilde{C}_i \to Y$ is 2^{n-1} for $i = 1$ and 2, and $\widetilde{C}_1$ and $\widetilde{C}_2$ have the same genus.

Writing the degree of the ramification divisor of ρ_n as

$$\deg(R_{X/Y}) = 2g_X - 2 - n(2g_Y - 2),$$

the genus of $\widetilde{C}_1$ and $\widetilde{C}_2$ is

$$g_{\widetilde{C}_i} = 2^{n-3}\left(g_X - 1 - (n-4)(g_Y - 1)\right) + 1. \tag{1.2}$$

If n is odd, the involution σ exchanges the two components of $\widetilde{C}$, hence induces isomorphisms

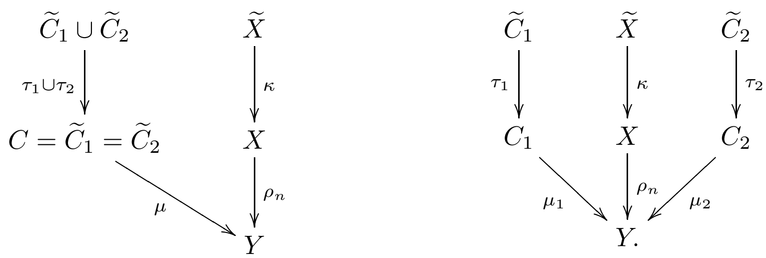
$$\widetilde{C}_1 \cong \widetilde{C}_2 \cong C.$$

If n is even, the involution σ acts on each component of $\widetilde{C}$; hence, C also has two connected components, say C_1 and C_2. For $n \geq 4$, since σ is fixed-point-free, we compute the genus of C_1 and C_2 to be

$$g_{C_i} = 2^{n-4}\left(g_X - 1 - (n-4)(g_Y - 1)\right) + 1. \tag{1.3}$$

So we have the following diagrams:

n odd *n even*



In the sequel we work with one of the components $\widetilde{C}_i$, say $\widetilde{C}_1$.

2. The correspondences S and S^t

For our constructions we shall need the following correspondences. We define S to be the reduced curve

$$S = \{(z = x_1 + \cdots + x_n, x) \in \widetilde{C}_1 \times \widetilde{X} \mid x = x_i \text{ for some } i\}$$

and S^t to be its transpose, i.e.

$$S^t = \{(x, z = x_1 + \cdots + x_n) \in \widetilde{X} \times \widetilde{C}_1 \mid x = x_i \text{ for some } i\}.$$

As maps of curves to divisors, S and S^t are given by

$$S : \begin{cases} \widetilde{C}_1 & \to & \operatorname{Div}^n(\widetilde{X}) \\ z = x_1 + \cdots + x_n & \mapsto & x_1 + \cdots + x_n \end{cases}$$

and

$$S^t : \begin{cases} \widetilde{X} & \to & \operatorname{Div}^{2^{n-2}}(\widetilde{C}_1) \\ x & \mapsto & \sum x + x_2^{\epsilon_2} + \cdots + x_n^{\epsilon_n} \end{cases}$$

where $\kappa(x_i) = \overline{x}_i$, $\kappa(x) = \overline{x}$, $\rho_n^{-1}\rho_n(\overline{x}) = \{\overline{x}, \overline{x}_2, \cdots, \overline{x}_n\}$ and the sum is to be taken over all $\epsilon_i = 1$ or -1 with $\sum \epsilon_i \equiv 0[2]$ and $x_i^1 = x_i$, $x_i^{-1} = x_i'$ for $i =$

$2, \cdots, n$. In other words, the sum is taken over all divisors of $\tau_1^{-1}\mu_1^{-1}\rho_n\kappa(x)$ in $\widetilde{C}_1$ containing x.

As in [12], for each $k \in \{1, \dots, n\}$ and $z = x_1 + \dots + x_n \in \widetilde{C}$, we denote by

$$[k + (n-k)'](z)$$

the sum of all the points of $\widetilde{C}$ where k of the x_i are added to $(n-k)$ of the x_i', the indices i being all distinct. When k is even, this induces a map, also denoted $[k + (n-k)']$, from $\widetilde{C}_1$ to $Div(\widetilde{C}_1)$.

We need the following.

Lemma 2.1. *We have*

$$S^t S = \sum_{i=0}^{[\frac{n-1}{2}]} (n-2i)[(2i)' + (n-2i)].$$

Proof. Choose a point $z \in \widetilde{C}_1$. Then $z = x_1 + \dots + x_n \in \widetilde{X}^{(n)}$, i.e., the image of z in $Div^n \widetilde{X}$ is $x_1 + \dots + x_n$. The map S^t will send this to

$$\sum_{i=1}^{n} (x_i + \widetilde{X}^{(n-1)}) \cap \widetilde{C}_1.$$

It is easy to see that this is equal to the expression in the statement of the lemma. □

Lemma 2.2. *For all $x \in \widetilde{X}$, we have*

$$S(S^t(x)) = 2^{n-2}x + 2^{n-3}\sum_{i=2}^{n}(x_i + x_i').$$

Proof. Immediate from the definitions. □

The correspondences S and S^t induce homomorphisms of the corresponding Jacobians in the usual way. We denote these homomorphisms by

$$s : J\widetilde{C}_1 \longrightarrow J\widetilde{X} \qquad \text{and} \qquad s^t : J\widetilde{X} \longrightarrow J\widetilde{C}_1.$$

Assume that n is even. It follows from the definition of the involution on $\widetilde{C}_1$ that, on the Jacobians,

$$s\sigma = {}'s \qquad \text{and} \qquad s^{t\prime} = \sigma s^t.$$

Therefore the homomorphisms s and s^t induce homomorphisms of the Prym varieties

$$P := \mathrm{Prym}(\kappa) \quad \text{and} \quad P_1 := \mathrm{Prym}(\tau_1) \tag{2.1}$$

of the involutions σ and $'$, which we again denote by s and s^t:

$$s : P_1 \longrightarrow P, \qquad s^t : P \longrightarrow P_1.$$

Note that, for k even, the endomorphism induced by $[k' + (n-k)]$ on $J\widetilde{C}_1$ induces an endomorphism on P_1. An immediate consequence of Lemmas 2.1 and 2.2 is the following.

Corollary 2.3. *Suppose n is even. The homomorphisms s and s^t between the Prym varieties P_1 and P satisfy the identities*

$$s^t s = \sum_{i=0}^{[\frac{n-1}{2}]} (n-2i)[(2i)' + (n-2i)]$$

and

$$ss^t = 2^{n-2} \cdot 1_P$$

where $[(2i)' + (n-2i)]$ denotes the endomorphism induced on P_1 by $[(2i)' + (n-2i)]$ on $J\widetilde{C}_1$.

In particular, s^t is an isogeny from P to an abelian subvariety of P_1.

3. Comparison of $\mathrm{Prym}(\tau_1)$ and $\mathrm{Prym}(\kappa)$ for $n = 4$

When $n = 4$, from (1.2) and (1.3) we see that

$$\dim P_1 = \dim P = g_X - 1.$$

Hence, when $n = 4$, Corollary 2.3 becomes

Corollary 3.1. *Suppose $n = 4$. The homomorphisms s and s^t between the Prym varieties P_1 and P are isogenies satisfying the identities*

$$s^t s = 4 \cdot 1_{P_1}$$

and

$$ss^t = 4 \cdot 1_P.$$

Proof. For $n = 4$, the first identity in Corollary 2.3 is

$$s^t s = 4 \cdot 1_{P_1} + [2' + 2].$$

Since P_1 is the image of the endomorphism $1_{J\widetilde{C}_1} - \sigma$ of $J\widetilde{C}_1$, and $[2' + 2]\sigma = [2 + 2'] = [2' + 2]$, we see that $[2' + 2]$ induces the zero endomorphism 0_{P_1} on P_1.

The second identity is immediate. □

Corollary 3.2.

$$\deg(s : P_1 \to P) = \deg(s^t : P \to P_1) = 2^{2 \dim P_1} = \deg(2_{P_1}) = \deg(2_P).$$

Proof. The fact that S^t is the transposed correspondence of S implies that $s^t : P \to P_1$ is the transposed endomorphism of $s : P_1 \to P$ with respect to the canonical principal polarizations. In particular, $\deg s^t = \deg s$. So Corollary 2.3 implies the assertion. □

Proposition 3.3. *The isogeny $s : P_1 \to P$ factors via the multiplication by 2 endomorphism $2_{P_1} : P_1 \to P_1$.*

Proof. Recall that $J\widetilde{C}_1 = H^0(\omega_{\widetilde{C}_1})^*/H_1(\widetilde{C}_1,\mathbb{Z})$ and $J\widetilde{X} = H^0(\omega_{\widetilde{X}})^*/H_1(\widetilde{X},\mathbb{Z})$. If, by a slight abuse of notation, $s : H^0(\omega_{\widetilde{C}_1})^* \to H^0(\omega_{\widetilde{X}})^*$ and $s^t : H^0(\omega_{\widetilde{X}})^* \to H^0(\omega_{\widetilde{C}_1})^*$ also denote the liftings of the homomorphisms s and s^t, it is a standard fact that they satisfy the relations

$$(\alpha, s(\beta))_{\widetilde{X}} = (s^t(\alpha), \beta)_{\widetilde{C}_1}$$

for all $\alpha \in H_1(\widetilde{X},\mathbb{Z})$ and $\beta \in H_1(\widetilde{C}_1,\mathbb{Z})$, where (,) denote the intersection products. Moreover, the canonical principal polarizations $E_{J\widetilde{X}}$ and $E_{J\widetilde{C}_1}$ are given by

$$E_{J\widetilde{X}}(v,w) = -(v,w)_{\widetilde{X}}$$

for all $v, w \in H_1(\widetilde{X},\mathbb{Z})$ and

$$E_{J\widetilde{C}_1}(v_1,w_1) = -(v_1,w_1)_{\widetilde{C}_1}$$

for all $v_1, w_1 \in H_1(\widetilde{C}_1,\mathbb{Z})$. This implies

$$E_{J\widetilde{X}}(v, s(w_1)) = E_{J\widetilde{C}_1}(s^t(v), w_1)$$

for all $v \in H_1(\widetilde{X},\mathbb{Z}),\ w_1 \in H_1(\widetilde{C}_1,\mathbb{Z})$.

Now let

$$P = V^-/\Lambda^- \quad \text{and} \quad P_1 = V_1^-/\Lambda_1^-$$

where as usual V^- and V_1^- are the anti-invariant vector subspaces of $H^0(\omega_{\widetilde{X}})^*$ and $H^0(\omega_{\widetilde{C}_1})^*$ with respect to the liftings of the involutions $'$ and σ. The canonical principal polarizations of $J\widetilde{X}$ and $J\widetilde{C}_1$ restrict to twice the principal polarizations Ξ on P and Ξ_1 and P_1. Hence, $E_{J\widetilde{X}} = 2E_\Xi$ and $E_{J\widetilde{C}_1} = 2E_{\Xi_1}$ and we get

$$E_\Xi(v, s(w_1)) = E_{\Xi_1}(s^t(v), w_1)$$

for all $v \in \Lambda^-$ and $w_1 \in \Lambda_1^-$. Using Corollary 2.3 this implies for all $v_1, w_1 \in \Lambda_1^-$,

$$\begin{aligned} s^*E_\Xi(v_1,w_1) = E_\Xi(s(v_1), s(w_1)) &= E_{\Xi_1}(s^t s(v_1), w_1) \\ &= E_{\Xi_1}(4v_1, w_1) \\ &= E_{\Xi_1}(2v_1, 2w_1) = 2^*E_{\Xi_1}(v_1,w_1). \end{aligned}$$

So $s^*E_\Xi = 2^*E_{\Xi_1}$ and s^*E_Ξ takes integer values on the lattice $\frac{1}{2}\Lambda_1^-$. Hence, the points of order 2 map to zero by s because Ξ is a principal polarization and s is an isogeny; in particular, by e.g. [4, Corollary 2.4.4, page 36], the isogeny s factors via the 2-multiplication 2_{P_1}. □

As an immediate consequence we get the main result of this section.

Theorem 3.4. *Let $\rho_4 : X \to Y$ be a simply ramified cover of degree 4 of smooth projective curves and $\kappa : \widetilde{X} \to X$ be an étale double cover which is not obtained by base change from a double cover of Y. Assume that the curve $\widetilde{C}$ obtained by diagram (1.1) has 2 connected components, one of which is $\widetilde{C}_1$. If P and P_1 denote the associated Prym varieties defined in (2.1), then we have:*

The homomorphism $s : J\widetilde{C}_1 \to J\widetilde{X}$ induces an isomorphism of principally polarized abelian varieties

$$(P_1, \Xi_1) \simeq (P, \Xi).$$

Proof. According to Proposition 3.3 the isogeny $s : P_1 \to P$ factors as follows

$$\begin{array}{ccc} P_1 & \xrightarrow{\quad s \quad} & P \\ & {\scriptstyle 2_{P_1}} \searrow \quad \nearrow {\scriptstyle \psi} & \\ & P_1 & \end{array}.$$

According to Corollary 3.2, $\deg(s : P_1 \to P) = \deg 2_{P_1}$. Hence, $\psi : P_1 \to P$ is an isomorphism. By construction it respects the polarizations. □

4. Non-isomorphy of the coverings

In order to show that Theorem 3.4 gives examples of the non-injectivity of the Prym map, we have to show that for general coverings ρ_4, the coverings κ and τ_1 are non-isomorphic and that the Clifford index of the curve X will be at least 3. First, we show more generally:

Lemma 4.1. *Suppose that the cover $X \xrightarrow{\rho_n} Y$ has exactly two ramification points of index 1 in one fiber and is otherwise simply ramified, then, for any choice of étale double cover $\widetilde{X} \xrightarrow{\kappa} X$, the curve $\widetilde{C}$ has 2^{n-4} singular points that are described as follows.*

Let y be the point of Y such that there are two ramification points of index 1, say $\overline{x}_1$ and $\overline{x}_2$ in $\rho_n^{-1}(y)$. Let $\overline{x}_3, \dots, \overline{x}_{n-2}$ be the other (distinct) points in $\rho_n^{-1}(y)$. Then the singular points of $\widetilde{C}$ are the points $x_1 + x_1' + x_2 + x_2' + x_3 + \cdots + x_{n-2}$ where $\kappa^{-1}(\overline{x}_1) = \{x_1, x_1'\}$, $\kappa^{-1}(\overline{x}_2) = \{x_2, x_2'\}$ and, for $i \geq 3$, x_i is a point of $\widetilde{X}$ lying above $\overline{x}_i$.

Proof. As in the proof of Lemma 1.1 on page 186 of [12], we can see that C_1 and C_2 are smooth above all non-branch points of Y and all simple branch points of Y. For all i, let x_i and x_i' be the points of $\widetilde{X}$ above $\overline{x}_i$. We need to analyze the local structure of $\widetilde{C}$ at the points $2x_1 + 2x_2 + x_3 + \cdots + x_{n-2}$, $x_1 + x_1' + 2x_2 + x_3 + \cdots + x_{n-2}$ and $x_1 + x_1' + x_2 + x_2' + x_3 + \cdots + x_{n-2}$, the cases of the other points of $\widetilde{C}$ being similar either to these or to cases we considered in Lemma 1.1 of loc. cit.. All these points lie above the point $2\overline{x}_1 + 2\overline{x}_2 + \overline{x}_3 + \ldots + \overline{x}_{n-2}$ of $X^{(n)}$ which, with our conventions, we identify with the point y of Y. As in that proof, since $\widetilde{C}$ is defined by the fiber product diagram (1.1), its tangent space is the pull-back of the tangent space of Y. The tangent space to Y is a subspace of the tangent space of $X^{(n)}$ which, at the point $2\overline{x}_1 + 2\overline{x}_2 + \overline{x}_3 + \cdots + \overline{x}_{n-2}$, can be canonically identified with

$$\mathcal{O}_{2\overline{x}_1}(2\overline{x}_1) \oplus \mathcal{O}_{2\overline{x}_2}(2\overline{x}_2) \bigoplus_{i=3}^{n-2} \mathcal{O}_{\overline{x}_i}(\overline{x}_i).$$

At the point $2x_1 + 2x_2 + x_3 + \cdots + x_{n-2}$, the tangent space to $\widetilde{X}^{(n)}$ can be canonically identified with

$$\mathcal{O}_{2x_1}(2x_1) \oplus \mathcal{O}_{2x_2}(2x_2) \bigoplus_{i=3}^{n-2} \mathcal{O}_{x_i}(x_i).$$

The differential of $\kappa^{(n)}$ sends $\mathcal{O}_{x_i}(x_i)$ isomorphically to $\mathcal{O}_{\overline{x}_i}(\overline{x}_i)$ and sends $\mathcal{O}_{2x_1}(2x_1)$ and $\mathcal{O}_{2x_2}(2x_2)$ isomorphically to $\mathcal{O}_{2\overline{x}_1}(2\overline{x}_1)$ and $\mathcal{O}_{2\overline{x}_2}(2\overline{x}_2)$ respectively. Hence, the differential of $\kappa^{(n)}$ is an isomorphism and $\widetilde{C}$ is smooth at $2x_1 + 2x_2 + x_3 + \cdots + x_{n-2}$.

The tangent space to $\widetilde{X}^{(n)}$ at $x_1 + x_1' + 2x_2 + x_3 + \ldots x_{n-2} \in \widetilde{X}^{(n)}$ can be canonically identified with

$$\mathcal{O}_{x_1}(x_1) \oplus \mathcal{O}_{x_1'}(x_1') \oplus \mathcal{O}_{2x_2}(2x_2) \bigoplus_{i=3}^{n-2} \mathcal{O}_{x_i}(x_i).$$

The differential of $\kappa^{(n)}$ sends $\mathcal{O}_{x_i}(x_i)$ and $\mathcal{O}_{2x_2}(2x_2)$ isomorphically to $\mathcal{O}_{\overline{x}_i}(\overline{x}_i)$ and $\mathcal{O}_{2\overline{x}_2}(2\overline{x}_2)$ respectively, and sends $\mathcal{O}_x(x)$ and $\mathcal{O}_{x'}(x')$ both isomorphically to the subspace $\mathcal{O}_{\overline{x}}(\overline{x})$ of $\mathcal{O}_{2\overline{x}}(2\overline{x})$. This case is therefore entirely similar to the case of a simple ramification point considered in the proof of Lemma 1.1 of [12] and $\widetilde{C}$ is smooth at such a point.

Finally, the tangent space to $\widetilde{X}^{(n)}$ at $x_1+x_1'+x_2+x_2'+x_3+\ldots+x_{n-2} \in \widetilde{X}^{(n)}$ can be canonically identified with

$$\mathcal{O}_{x_1}(x_1) \oplus \mathcal{O}_{x_1'}(x_1') \oplus \mathcal{O}_{x_2}(x_2) \oplus \mathcal{O}_{x_2'}(x_2') \bigoplus_{i=3}^{n-2} \mathcal{O}_{x_i}(x_i).$$

For all i, the differential of $\kappa^{(n)}$ sends $\mathcal{O}_{x_i}(x_i)$ isomorphically to $\mathcal{O}_{\overline{x}_i}(\overline{x}_i)$ and, for $i = 1$ or 2, sends $\mathcal{O}_{x_i}(x_i)$ and $\mathcal{O}_{x_i'}(x_i')$ both isomorphically to the subspace $\mathcal{O}_{\overline{x}_i}(\overline{x}_i)$ of $\mathcal{O}_{2\overline{x}_i}(2\overline{x}_i)$. Its kernel is therefore two-dimensional and it follows that $\widetilde{C}$ is singular at $x_1 + x_1' + x_2 + x_2' + x_3 + \ldots + x_{n-2}$. □

Corollary 4.2. *Suppose that $n = 4$ and*

$$X \xrightarrow{\rho_4} Y$$

is a generic member of a family of covers of degree 4 which has a special member as in the lemma. Then, for any double cover

$$\widetilde{X} \xrightarrow{\kappa} X,$$

the curves C_1, C_2 and the curves $\widetilde{C}_1$, $\widetilde{C}_2$ are non-isomorphic and non-isomorphic to X, respectively, $\widetilde{X}$.

Proof. In the situation of the lemma, the curve $\widetilde{C}$ has one singular point. Hence, one of $\widetilde{C}_1$ or $\widetilde{C}_2$, say $\widetilde{C}_1$, is singular and the other is smooth. Since, by the description in the lemma, the singular point is fixed by the involution σ, the curve C_1 is singular and C_2 is smooth. In particular, they are non-isomorphic and C_1, respectively $\widetilde{C}_1$ is non-isomorphic to X, respectively $\widetilde{X}$. Hence, the three curves and their double covers are also non-isomorphic for a generic choice of ρ_4. □

Corollary 4.3. *Suppose that $n = 4$. For each $e \geq 4$, there exists an (irreducible) family of simply ramified covers of degree 4 ramified at e points, with monodromy group $\mathfrak{S}_4$, such that for its generic member*

$$X \xrightarrow{\rho_4} Y$$

and, for any double cover

$$\widetilde{X} \xrightarrow{\kappa} X,$$

the curves C_1, C_2 and the curves $\widetilde{C}_1$, $\widetilde{C}_2$ are non-isomorphic and non-isomorphic to X, respectively, $\widetilde{X}$.

Proof. Given a cover $\rho_4 : X \to Y$ and a point $y \in Y$ which is not a branch point, choose a labeling $\phi : \{1, \dots, 4\} \to \rho_4^{-1}(y)$. For each branch point $b \in Y$ of ρ_4, choose a loop in Y based at y and going around b once. Lifting this loop to X and using the labeling ϕ, we obtain a permutation $\tau_b \in \mathfrak{S}_4$. Choose an odd integer $d \geq 3$. Given a curve Y, we can choose a small disc B in Y and d distinct points $b_1, \dots, b_d$ in B to be the branch points of ρ_4. We can then topologically construct a 4-sheeted cover $\rho_4 : X \to Y$ ramified at the d points in such a way that b_1 is a branch point with two simple ramification points above it and, for $2 \leq i \leq d$, the point b_i is a simple branch point of ρ_4. We can furthermore construct the cover in such a way that its monodromy group is the full symmetric group $\mathfrak{S}_4$. (Note that a cover with monodromy group $\mathfrak{S}_4$ has at least 4 ramification points.) This cover will deform to simply ramified covers with monodromy group $\mathfrak{S}_4$ and we can apply Corollary 4.2. □

Note that by, e.g., [11], *the space of simply ramified covers of degree n with monodromy group $\mathfrak{S}_n$ and at least $2n$ ramification points is irreducible.*

Remark 4.4. *In the case* $Y = \mathbb{P}^1$*, Donagi notes in* [8, p. 77] *that any two of the towers* $\widetilde{X} \to X \to \mathbb{P}^1$*,* $\widetilde{C}_1 \to C_1 \to \mathbb{P}_1$ *and* $\widetilde{C}_2 \to C_2 \to \mathbb{P}_1$ *differ by an outer automorphism of order* 3 *of the Weyl group* $W(D_4)$*. This also applies to arbitrary* Y*. By standard covering theory, two such towers on* Y *are isomorphic if and only if they differ by an inner automorphism of* $W(D_4)$*. After working out the details, this would give a second proof of the non-isomorphy of the covers* $\widetilde{X} \to X$*,* $\widetilde{C}_1 \to C_1$ *and* $\widetilde{C}_2 \to C_2$*.*

5. The Clifford indices of the counterexamples

In order to give an estimate for the Clifford index of the curve X we use the following well-known inequality of Castelnuovo (see [5]): Suppose X is a smooth projective curve admitting two maps $f_1 : X \to Y_1$, $f_2 : X \to Y_2$ of degrees $n_1, n_2 \geq 2$ which do not both factor via a map $X \to Z$ of degree ≥ 2. Then

$$g_X \leq (n_1 - 1)(n_2 - 1) + n_1 g_{Y_1} + n_2 g_{Y_2}. \tag{5.1}$$

For a modern proof of this inequality, note that the assumption implies that the map $(f_1, f_2) : X \to Y_1 \times Y_2$ is birational onto its image and apply the adjunction formula.

Moreover, if $\operatorname{cliff} X$ and $\operatorname{gon} X$ denote the Clifford index and the gonality of X, recall the following inequality, which is valid for any smooth projective curve (see [9]),

$$\operatorname{cliff} X + 2 \leq \operatorname{gon} X \leq \operatorname{cliff} X + 3. \tag{5.2}$$

Finally, recall that a covering is called *simple* if it cannot be written as a composition of 2 coverings of degree ≥ 2.

Lemma 5.1. *Let* $f : X \to Y$ *be a simple covering of degree* n *of smooth projective curves with ramification divisor of degree* δ*. If* $\delta \geq 2(n-1)n \operatorname{gon} Y$*, then*

$$\operatorname{gon} X = n \operatorname{gon} Y.$$

Proof. Certainly we have $\operatorname{gon} X \leq n \operatorname{gon} Y$. Suppose that $g : X \to \mathbb{P}^1$ is a map of degree $m < n \operatorname{gon} Y$. Since f is a simple covering, we may apply inequlity (5.1) to f and g, which gives, using the Hurwitz formula,

$$n(g_Y - 1) + 1 + \frac{\delta}{2} = g_X \leq (n-1)(m-1) + n g_Y.$$

This implies

$$\delta \leq 2(n-1)m < 2(n-1)n \operatorname{gon} Y,$$

which contradicts the assumption. □

Corollary 5.2. *If $n = 4$ and $\rho_4 : X \to Y$ is a simple covering of a general curve Y of genus g_Y with ramification divisor of degree $\delta \geq 24 \operatorname{gon} Y$. Then*

$$\operatorname{cliff} X \geq 2g_Y - 1.$$

Proof. For a general curve Y of genus g_Y we have $\operatorname{gon} Y = [\frac{g+3}{2}]$. Now Lemma 5.1 and inequality (5.2) give

$$\operatorname{cliff} X \geq \operatorname{gon} X - 3 = 4 \operatorname{gon} Y - 3 \geq 2g_Y - 1.$$

□

Certainly this estimate is not the best possible. Moreover, one can use the same method to give a more precise result for any curve Y with given gonality or Clifford index.

6. The class of $\widetilde{C}_i$ in the Prym variety P

6.1. Let the notation again be as in Section 1. In particular, $\rho_n : X \to Y$ is a simply ramified cover of degree $n \geq 3$.

In this section we compute the class of the image of $\widetilde{C}_i$ in the Prym variety $P = \operatorname{Prym}(\kappa)$.

6.2. We map $\widetilde{C}_i$ into the Prym variety P of $\kappa : \widetilde{X} \longrightarrow X$ via the composition

$$\widetilde{C}_i \longrightarrow J\widetilde{X} \longrightarrow J\widetilde{X}/\kappa^* JX = P.$$

We shall compute the cohomology class $[\widetilde{C}_i]$ of the image of $\widetilde{C}_i$ by this map via degeneration to the case where Y is an irreducible rational curve of arithmetic genus g_Y. First we note the following.

Proposition 6.1. *Assume $n \geq 3$. The curves $\widetilde{C}_1$ and $\widetilde{C}_2$ have the same cohomology class in P.*

Proof. The proof of [3, Proposition 1, page 360] goes through without change. □

In the case where $Y \simeq \mathbb{P}^1$, the cohomology class $[\widetilde{C}_i]$ in P was computed by Beauville in the proof of [3, Proposition 2, p. 363] to be

$$[\widetilde{C}_i] = 2^{n-1} \frac{[\Theta_P]^{(g_X - 2)}}{(g_X - 2)!}$$

where Θ_P is the principal polarization of P as the Prym variety of κ. Here we generalize this formula to the case where Y is not isomorphic to $\mathbb{P}^1$.

Remark 6.2. In [3, Proposition 2, p. 363], Beauville could further divide his class by 4 because his map from $\widetilde{C}$ into P factored through 2-multiplication $2_P : P \to P$. Except when $n = 4$, this is not necessarily the case in our situation and we cannot divide the class by 4.

We first generalize the formula to the case where $Y \cong \mathbb{P}^1$, but the double cover $\widetilde{X} \to X$ is ramified. Note that in the ramified case, the curve $\widetilde{C}$ does not necessarily split into the union of the two curves $\widetilde{C}_1$ and $\widetilde{C}_2$. It is however defined in the same way as a subscheme of $\widetilde{X}^{(n)}$ by the Cartesian diagram

$$\begin{array}{ccc} \widetilde{C} & \longrightarrow & \widetilde{X}^{(n)} \\ \downarrow & & \downarrow \kappa^{(n)} \\ Y & \longrightarrow & X^{(n)}. \end{array} \tag{6.1}$$

It has therefore a well-defined cohomology class in $\widetilde{X}^{(n)}$. We push this class forward to the Jacobian $J\widetilde{X}$ and then to the Prym variety $P = J\widetilde{X}/JX$. We denote this push-forward class by $[\widetilde{C}]$: it is a well-defined element of the cohomology ring of P.

Proposition 6.3. *Consider coverings*

$$\widetilde{X} \xrightarrow{\kappa} X \xrightarrow{\rho_n} Y \cong \mathbb{P}^1$$

where κ is a ramified double cover and ρ_n is a ramified n-sheeted cover. Define the curve $\widetilde{C}$ as before and map $\widetilde{C}$ to the Prym variety P of κ. Then the push-forward class $[\widetilde{C}]$ is

$$[\widetilde{C}] = 2^{n-g_X+1} \frac{[\Theta_P]^{(g_{\widetilde{X}}-g_X-1)}}{(g_{\widetilde{X}}-g_X-1)!}.$$

Proof. Let $\varphi_{X,n} : X^{(n)} \to JX$ and $\varphi_{\widetilde{X},n} : \widetilde{X}^{(n)} \to J\widetilde{X}$ denote Abel maps such that the following diagram commutes

$$\begin{array}{ccc} \widetilde{X}^{(n)} & \xrightarrow{\varphi_{\widetilde{X},n}} & J\widetilde{X} \\ \kappa^{(n)} \downarrow & & N \downarrow \\ X^{(n)} & \xrightarrow{\varphi_{X,n}} & JX \end{array}$$

where N denotes the norm map. Let $\eta_{X,n}$ and $\eta_{\widetilde{X},n}$ be the respective cohomology classes of $X^{(n-1)} + p \subset X^{(n)}$ and $\widetilde{X}^{(n-1)} + \widetilde{p} \subset \widetilde{X}^{(n)}$ where p and $\widetilde{p}$ are points of X and $\widetilde{X}$. Hence

$$\varphi_{\widetilde{X},n_*}\eta^k_{\widetilde{X},n} = \frac{[\Theta_{\widetilde{X}}]^{k+g_{\widetilde{X}}-n}}{(k+g_{\widetilde{X}}-n)!} \tag{6.2}$$

where $\Theta_{\widetilde{X}}$ is the principal polarization of $\widetilde{X}$ as the Jacobian of $\widetilde{X}$. As in [3, p. 363] we rewrite Macdonald's formula [15, p. 337] for the class of the image of Y in $X^{(n)}$ as

$$[Y] = \sum_{\alpha=0}^{n-1} \binom{n-1-g_X}{\alpha} \eta^\alpha_{X,n} \frac{\varphi^*_{X,n}[\Theta_X]^{n-1-\alpha}}{(n-1-\alpha)!}.$$

Therefore, in $\widetilde{X}^{(n)}$, the cohomolgy class of $\widetilde{C}$ is

$$[\widetilde{C}]_{\widetilde{X}^{(n)}} = \sum_{\alpha=0}^{n-1} \binom{n-1-g_X}{\alpha} 2^\alpha \eta_{\widetilde{X},n}^\alpha \frac{\varphi_{\widetilde{X},n}^* N^*[\Theta_X]^{n-1-\alpha}}{(n-1-\alpha)!}.$$

Using (6.2) and the projection formula, the push-forward of this to $J\widetilde{X}$ is

$$[\widetilde{C}]_{J\widetilde{X}} = \sum_{\alpha=0}^{n-1} \binom{n-1-g_X}{\alpha} 2^\alpha \frac{[\Theta_{\widetilde{X}}]^{\alpha+g_{\widetilde{X}}-n}}{(\alpha+g_{\widetilde{X}}-n)!} \frac{N^*[\Theta_X]^{n-1-\alpha}}{(n-1-\alpha)!}.$$

It is easy to see that it follows from [16, pp. 329–330] that

$$4[\Theta_{\widetilde{X}}] = 2N^*[\Theta_X] + q_n^*[\Theta_P]$$

where $q_n : J\widetilde{X} \to P$ is the projection. Inserting, developping and using the fact that

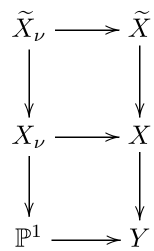
$$\left\{q_{n*}N^* : H^k(JX,\mathbb{Q}) \longrightarrow H^{k-2g_X}(P,\mathbb{Q})\right\} = \begin{cases} 0 & \text{if } k \neq 2g_X \\ 2^{2(g_{\widetilde{X}}-g_X)} & \text{if } k = 2g_X, \end{cases}$$

we obtain the class $[\widetilde{C}]$ in P:

$$\begin{aligned}[\widetilde{C}] &= 2^{n-g_X+1} \sum_{\alpha=0}^{n-1} \binom{n-1-g_X}{\alpha} \binom{g_X}{n-1-\alpha} \frac{[\Theta_P]^{g_{\widetilde{X}}-g_X-1}}{(g_{\widetilde{X}}-g_X-1)!} \\ &= 2^{n-g_X+1} \frac{[\Theta_P]^{g_{\widetilde{X}}-g_X-1}}{(g_{\widetilde{X}}-g_X-1)!}\end{aligned}$$

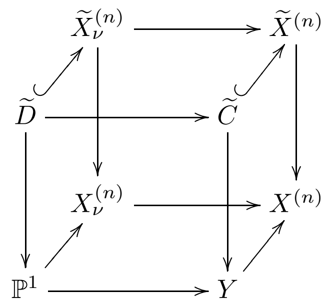
because $\sum_{\alpha=0}^{n-1} \binom{n-1-g_X}{\alpha}\binom{g_X}{n-1-\alpha} = 1$ (see [3, p. 364]). □

6.3. Now we generalize the construction to the case where Y is an irreducible rational nodal curve of arithmetic genus g_Y, $\rho_n : X \to Y$ is a simply ramified n-sheeted cover with branch locus disjoint from the nodes of Y, $\kappa : \widetilde{X} \to X$ is a 2-sheeted cover, ramified at the nodes of X, such that near each ramification point, the covering involution does not exchange the two branches of $\widetilde{X}$. Note that the latter is a type of Beauville admissible double cover. Consider the Cartesian diagram of normalizations:

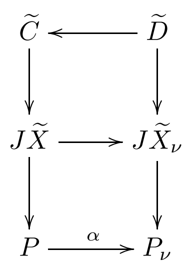


which induces the Cartesian diagram (the curves $\widetilde{C}$ and $\widetilde{D}$ below are defined by the diagram)

(6.3)



and hence the diagram



where P and P_ν are, respectively, the generalized Prym varieties of the covers $\widetilde{X} \to X$ and $\widetilde{X}_\nu \to X_\nu$.

Let $\beta : P_\nu \to P$ be the pseudo-inverse of α, i.e., β is the unique isogeny such that $\alpha \circ \beta = 2_{P_\nu}$ and $\beta \circ \alpha = 2_P$. Note that the degree of α is $2^{ng_Y - 1}$. Hence, the degree of β is $2^{2g_X - ng_Y - 1}$.

Proposition 6.4. *We have*

$$\beta_* \frac{[\Theta_{P_\nu}]^{g_X - 2}}{(g_X - 2)!} = 2^{g_X - ng_Y + 1} \frac{[\Theta_P]^{g_X - 2}}{(g_X - 2)!},$$

where Θ_{P_ν} is the principal polarization of P_ν as the Prym variety of the double cover $\widetilde{X}_\nu \to X_\nu$.

Proof. By [1, p. 159] the map α is an isogeny with kernel an isotropic subgroup with respect to $\Theta_{\widetilde{X}}|_P$ of points of order 2. This implies that $\alpha^*[\Theta_{P_\nu}] = [\Theta_{\widetilde{X}}|_P]$, equivalently, $\alpha^*[\Theta_{P_\nu}] = 2[\Theta_P]$. The latter implies $\beta^*[\Theta_P] = 2[\Theta_{P_\nu}]$. Therefore

$$\beta^* \frac{[\Theta_P]^{g_X-2}}{(g_X-2)!} = 2^{g_X-2} \frac{[\Theta_{P_\nu}]^{g_X-2}}{(g_X-2)!}.$$

Poincaré duality gives the following commutative diagram

$$\begin{array}{ccc} H^{2g_X-4}(P,\mathbb{Q}) & \xrightarrow{\beta^*} & H^{2g_X-4}(P_\nu,\mathbb{Q}) \\ d\downarrow & & \downarrow \\ H_2(P,\mathbb{Q}) & \xleftarrow{\beta_*} & H_2(P_\nu,\mathbb{Q}) \end{array}$$

where the right hand vertical map is Poincaré duality and the left hand vertical map is Poincaré duality multiplied by the degree $d = 2^{2g_X - ng_Y - 1}$ of β. Now the proposition follows. □

Corollary 6.5. *The class* $[\widetilde{C}]$ *in* P *is*

$$[\widetilde{C}] = 2^n \frac{[\Theta_P]^{g_X-2}}{(g_X-2)!}.$$

Proof. Using the previous two propositions and the fact that $\beta \circ \alpha = 2_P$, we compute

$$[\widetilde{C}] = \frac{1}{4}\beta_*[\widetilde{D}] = \frac{1}{4}2^{n-g_{X_\nu}+1}\beta_* \frac{[\Theta_{P_\nu}]^{g_X-2}}{(g_X-2)!} = 2^{n-g_{X_\nu}-1+g_X-ng_Y+1} \frac{[\Theta_P]^{g_X-2}}{(g_X-2)!}.$$

□

Theorem 6.6. *For any unramified double cover* $\kappa : \widetilde{X} \to X$ *and any simply ramified* n*-sheeted cover* $\rho_n : X \to Y$ *as in Section* 1 *the class of the image of* $\widetilde{C}_i$ *in the Prym variety* P *of* κ *is*

$$[\widetilde{C}_i] = 2^{n-1} \frac{[\Theta_P]^{g_X-2}}{(g_X-2)!}.$$

Proof. Any tower of smooth curves $\widetilde{X} \to X \to Y$ as usual is a member of a connected family

$$\widetilde{\mathcal{X}} \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow M$$

of coverings of curves where all fibers are smooth except at one point $0 \in M$ where the fiber is a tower $\widetilde{X}_0 \to X_0 \to Y_0$ with Y_0 irreducible rational nodal, $X_0 \to Y_0$ simply ramified of degree n with branch locus disjoint from the nodes of Y_0 and $\widetilde{X}_0 \to X_0$ admissible 2-sheeted and ramified at the nodes of X_0. The family of Prym varieties of the fibers of the double covers $\widetilde{X} \to X$ is then a family of principally polarized abelian varieties on M. The construction of the curves $\widetilde{C}$ on each fiber can be done globally on the family to produce a family of curves $\widetilde{\mathcal{C}} \to M$. The fibers of $\widetilde{\mathcal{C}}$ map into the family of Prym varieties and their classes live in a local system of free abelian groups and vary continuously on M. Therefore the classes are constant and given by the same formula as in Corollary 6.5. We then use Proposition 6.1 to obtain the classes $[\widetilde{C}_i]$. □

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Department of Mathematics, Boyd Graduate Studies Research Center, University of Georgia, Athens, Georgia 30602-7403

E-mail address: izadi@math.uga.edu

Department Mathematik, Universität Erlangen-Nürnberg, Bismarckstrasse 1 1/2, D-91054 Erlangen, Germany

E-mail address: lange@mi.uni-erlangen.de