Geometriae Dedicata (2005) 115: 33–63
DOI 10.1007/s10711-005-7152-2

Deforming Curves in Jacobians to Non-Jacobians II: Curves in $C^{(e)}$, $3 \leqslant e \leqslant g-3^{\star}$

E. IZADI
Department of Mathematics, Boyd Graduate Studies Research Center, University of Georgia, Athens, GA 30602-7403, U.S.A. e-mail: izadi@math.uga.edu

(Received: 22 April 2004; accepted in final form: 11 May 2005)

Abstract. We introduce deformation theoretic methods for determining when a curve X in a nonhyperelliptic Jacobian JC will deform with JC to a non-Jacobian. We apply these methods to a particular class of curves in symmetric powers $C^{(e)}$ of C where $3 \leqslant e \leqslant g-3$. More precisely, given a pencil g^1_d of degree d on C, let X be the curve parametrizing divisors of degree e in divisors of g^1_d (see the paper for the precise scheme-theoretical definition). Under certain genericity assumptions on the pair (C, g^1_d), we prove that if X deforms infinitesimally out of the Jacobian locus with JC then either $d=2e$, $\dim H^0(g^1_d)=e$ or $d=2e+1$, $\dim H^0(g^1_d)=e+1$. The analogous result in the case $e=2$ without genericity assumptions was proved earlier.

Mathematics Subject Classification (2000). Primary 14K12, 14C25; Secondary 14B10, 14H40.

Key words. Abelian variety, curve, deformation, Jacobian, Prym, moduli space of Abelian varieties.

Introduction

This is a second paper where we introduce deformation theory methods which can be applied to finding curves in families of principally polarized Abelian varieties (ppav) containing jacobians. One of our motivations for finding interesting and computationally tractable curves in ppav is to solve the Hodge conjecture for the primitive cohomology of the theta divisor which we explain below. For other motivations, we refer to the prequel [5] to this paper.

Let (A, Θ) be a ppav over $\mathbb{C}$ of dimension $g \geqslant 4$ such that Θ is smooth. Since any Abelian variety (over $\mathbb{C}$) is isogenous to such an Abelian variety, the Hodge conjectures for arbitrary Abelian varieties are equivalent to the Hodge conjectures for principally polarized Abelian varieties with smooth theta divisors.

* This material is based upon work partially supported by the National Science Foundation under Grant No. DMS-0071795. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

The primitive part $K(\Theta,\mathbb{Q})$ of the cohomology of Θ can be defined as the kernel of the map $H^{g-1}(\Theta,\mathbb{Q})\longrightarrow H^{g+1}(A,\mathbb{Q})$ obtained by Poincaré Duality from push-forward on homology. The space $K(\Theta,\mathbb{Q})$ defines a Hodge substructure of the cohomology of Θ of level $g-3$ (see p. 562 of [8]; the proof there works also for $g>4$). The generalized Hodge conjecture then predicts that there is a family of curves in Θ such that $K(\Theta,\mathbb{Q})$ is contained in the image of its Abel–Jacobi map. The Abel–Jacobi map for a family of curves can be defined as follows.

Let $\mathcal{C}\to S$ be a family of curves with S smooth, complete and irreducible of dimension d such that there is a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{q} & \Theta \\ p\downarrow & & \\ S & & . \end{array}$$

The Abel–Jacobi map for this family of curves is by definition $AJ:=q_*p^*:H^{2d-(g-3)}(S,\mathbb{Q})\to H^{g-1}(\Theta,\mathbb{Q})$. The image of the Abel–Jacobi map defines a Hodge substructure of level $\leqslant g-3$ of the cohomology of Θ.

For Abelian fourfolds one interesting family of curves is the family of Prym-embedded curves in Θ and it is proved in [8] that it does give a solution to the Hodge conjecture for $K(\Theta,\mathbb{Q})$. In dimension $\geqslant 6$ there are no known families of interesting curves in the theta divisor of a general ppav.

Let us briefly explain our methods, similar to [5]. We always assume C to be a smooth nonhyperelliptic curve of genus $g\geqslant 4$. After identifying $JC=\mathrm{Pic}^0C$ with $A:=\mathrm{Pic}^{g-1}C$ by tensoring with a fixed invertible sheaf of degree $g-1$, Riemann's theta divisor is

$$\Theta:=\{\mathcal{L}\in\mathrm{Pic}^{g-1}C:h^0(\mathcal{L})>0\}.$$

Consider a subvariety X of A contained in 'many' translates Θ_a of Θ. As in [5], for each such translate Θ_a, we have a map

$$\nu_a:H^1(T_A)\longrightarrow H^1(\mathcal{I}_{Z_{g-1}}(\Theta_a)|_X),$$

obtained from Green's exact sequence ([4], see Section 3 below) which factors through the first order obstruction map

$$\nu:H^1(T_A)\longrightarrow H^1(N_{X/JC}),$$

where $N_{X/A}$ is the normal sheaf to X in A (see Section 2). Hence, if $\nu_a(\eta)$ is not zero for some a, so is $\nu(\eta)$.

The main difference between the method used here and that of [5] is in Sections 5 and 6 below which are more difficult for $e>2$ and are where we need some assumptions of genericity on X.

We apply the above to families of curves in Jacobians which are natural generalizations of Prym-embedded curves in tetragonal Jacobians. More precisely, let C be a nonhyperelliptic curve of genus g with a g^1_d (a pencil of degree d). In Section 1 below we define a curve $X_e(g^1_d)$ whose reduced support is

$$X_e(g^1_d)_{\mathrm{red}} := \{D_e : \exists D \in C^{(d-e)} \text{ such that } D_e + D \in g^1_d\} \subset C^{(e)}$$

where $2 \leqslant e \leqslant d$ and $C^{(e)}$ is the eth symmetric power of C. We map $X := X_e(g^1_d)$ and $C^{(e)}$ to $C^{(g-1)}$ and then to A by adding a fixed divisor $q := \sum_{i=1}^{g-1-e} q_i$. If $d \geqslant e+1$, the map is nonconstant on X. We call $W_e + q$ the image of $C^{(e)}$ in A via this map. Given a one-parameter infinitesimal deformation of the Jacobian of C normal to the Jacobian locus $\mathcal{J}_g$ we ask when the curve X deforms with it. Let $Z_{g-1} \subset C^{(g-1)}$ be the locus where the map

$$\begin{aligned} \psi_0 : C^{(g-1)} &\longrightarrow \Theta \subset A \\ D &\longmapsto \mathcal{O}_C(D) \end{aligned}$$

fails to be an isomorphism and let $Z_q \subset C^{(e)}$ be the locus defined by the exactness of the sequence

$$N_{W_{e+q/A}}|_{C^{(e)}} \longrightarrow N_{\Theta/A}|_{C^{(e)}} = \mathcal{O}_{C^{(e)}}(\Theta) \longrightarrow \mathcal{O}_{Z_{(q)}}(\Theta) \longrightarrow 0.$$

We prove the following

THEOREM 1. *Assume* $3 \leqslant e \leqslant g-3$ (*so* $g \geqslant 6$), *for all* $e' \leqslant e$ *the curve* $X_{e'}(g^1_d)$ *is irreducible and reduced and the set of* q *for which* $X \cap Z_{g-1} \neq X \cap Z_q$ *has dimension at most* $g-e-3$. *Further suppose that the curve C is nontrigonal and nonbielliptic of genus* $\geqslant 7$ *or nontrigonal of genus 6. If* $X_e(g^1_d)$ *deforms out of* $\mathcal{J}_g$ *then*

- *either* $h^0(g^1_d) = e$ *and* $d = 2e$
- *or* $h^0(g^1_d) = e+1$ *and* $d = 2e+1$.

The irreducibility assumption on the curves $X_{e'}(g^1_d)$ is used in Section 5. The assumptions on C are only used in Section 6 where we need to control the dimension of W^1_{g-1-e}.

By Appendix 8.3 below, for (C, L) in a nonempty open subset of the (irreducible) Hurwitz scheme of smooth curves with maps of degree d to $\mathbb{P}^1$ (with simple ramification) the hypotheses of the theorem are satisfied. In case $e=2$, we proved this result in [5] without the assumptions of genericity. We expect that for $e>2$ the result will still hold for reducible curves but nonreduced curves might deform in directions which are contained in the intersection of the spans, in $S^2H^1(\mathcal{O}_A) \subset H^1(\mathcal{O}_A)^{\otimes 2} \cong H^1(T_A)$ of the divisors parametrized by the curve X (this intersection is empty for reduced curves but could be nonempty for nonreduced curves).

In the case $e=2$, $d=4$, the curve X is a Prym-embedded curve (see [13]), hence deforms out of $\mathcal{J}_g$ into the locus of Prym varieties.

As explained in Section 7, Theorem 1 shows that when $3 \leqslant e \leqslant g-3$ the most interesting cases in which X will possibly deform out of $\mathcal{J}_g$ are those in which

C is bielliptic, e any integer between 3 and $g-3$ or C is any curve of genus g between 6 and 10, $e=3$ and $L\subset g^2_6$. In both these cases, it is likely that the curve X will deform out of $\mathcal{J}_g$. Although Jacobians of bielliptic curves form a subvariety of dimension $2g-2$ of $\mathcal{J}_g$, the curves obtained from bielliptic curves will likely deform to large families of ppav: such a situation is analogous to the case of tetragonal Jacobians of dimension $\geqslant 7$ where the curve $X_2(g^1_4)$ deforms to a general Prym but does NOT deform to a general Jacobian.

So we have some families of curves (including any X with $e=g-2$) which could possibly deform to non-Jacobians. We need a different approach to prove that higher order obstructions to deformations vanish: this will be presented in detail in the forthcoming paper [6] and the idea behind it is the following. For each Θ_a containing X, one has the map of cohomology groups of normal sheaves

$$H^1(N_{X/JC})\longrightarrow H^1(N_{\Theta_a/JC|X})=H^1(\mathcal{O}_X(\Theta_a))$$

whose kernel contains all the obstructions to the deformations of X since we will only consider algebraizable deformations of JC for which the obstructions to deforming Θ_a vanish. If one can prove that the intersection of these kernels is the image of the first order algebraizable deformations of JC, i.e., the image of $S^2H^1(\mathcal{O}_C)\subset H^1(T_{JC})$, it will follow that the only obstructions to deforming X with JC are the first order obstructions.

Finally, we would like to mention that from curves one can obtain higher-dimensional subvarieties of an Abelian variety. For a discussion of this we refer the reader to [7].

Plan of the paper: In Section 1, we define the curves $X_e(L)$ via their ideals for which we write down a concrete workable resolution. We compute their genus, define their maps to A and prove a useful lemma about divisors parametrized by $X_{e+1}(L)$ and $X_{e+2}(L)$. In Section 2, we define the first order obstruction map $\nu_e\colon S^2H^1(\mathcal{O}_C)\to H^1(N_{W_e/A}|_X)$ which we use to prove Theorem 1. In Section 3, we compute the translates Θ_a of Θ containing X and show how we can 'replace' ν_e by the collection of maps $S^2H^1(\mathcal{O}_C)\to H^1(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta))$. Our method will consist in finding when these maps can have nontrivial kernels. In Section 4, we decompose these maps into compositions of $S^2H^1(\mathcal{O}_C)\to H^0(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta))$ and $H^0(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta))\to H^1(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta))$ which we then analyze separately in Sections 5 and 6, respectively. In Section 5, we prove that for any $\eta\in S^2H^1(\mathcal{O}_C)\backslash H^1(T_C)$, there exists a translate Θ_a of Θ containing X such that η is *not* in the kernel of $S^2H^1(\mathcal{O}_C)\to H^0(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta))$. In Section 6 we prove that for 'almost all' Θ_a containing X, the coboundary map $H^0(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta))\to H^1(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta))$ is injective unless $d=2e$ and $h^0(L)=e$ or $d=2e+1$ and $h^0(L)=e+1$ which proves Theorem 1. In Section 7, we describe the consequences of Theorem 1. Finally, we gather some useful technical results in the Appendix.

NOTATION AND CONVENTIONS

We will denote linear equivalence of divisors by $\sim$.

For any divisor or coherent sheaf D on a scheme X, denote by $h^i(D)$ the dimension of the cohomology group $H^i(D) = H^i(X, D)$. For any subscheme Y of X, we will denote by $\mathcal{I}_{Y/X}$ the ideal sheaf of Y in X and by $N_{Y/X}$ the normal sheaf of Y in X. When there is no ambiguity, we drop the subscript $_X$ from $\mathcal{I}_{Y/X}$ or $N_{Y/X}$. The tangent sheaf of X will be denoted by $T_X := \mathcal{H}\mathrm{om}(\Omega^1_X, \mathcal{O}_X)$ and the dualizing sheaf of X by ω_X. By the genericity of any property on X, we mean genericity on every irreducible component.

We let C be a smooth nonhyperelliptic curve of genus g over the field $\mathbb{C}$ of complex numbers. For any positive integer n, denote by $C^{(n)}$ the nth symmetric power of C. Note that $C^{(n)}$ parametrizes the effective divisors of degree n on C.

We denote by K an arbitrary canonical divisor on C. Since C is not hyperelliptic, its canonical map $C \to |K|^*$ is an embedding and throughout this paper we identify C with its canonical image. For a divisor D on C, we denote by $V(D)$ and $\langle D \rangle$ its respective spans in $H^0(K)^* = H^0(\omega_C)^* = H^1(\mathcal{O}_C)$ and $|K|^* = \mathbb{P}H^0(\omega_C)^* = \mathbb{P}H^1(\mathcal{O}_C)$. Meaning

$$V(D) := H^0(\omega_C(-D))^\perp \subset H^0(\omega_C)^*, \quad \langle D \rangle := H^0(\omega_C(-D))^\perp \subset |K|^*$$

so that for instance, for any $t \in C$, $\langle nt \rangle$ is the $(n-1)$st osculating space to $C \subset |K|^*$, the first osculating space being the projective tangent line to C.

Since we will mostly work with the Picard group $\mathrm{Pic}^{g-1}C$ of invertible sheaves of degree $g-1$ on C, we put $A := \mathrm{Pic}^{g-1}C$. Let Θ denote the natural theta divisor of A, i.e.,

$$\Theta := \{\mathcal{L} \in A : h^0(\mathcal{L}) > 0\}.$$

The multiplicity of Θ at $\mathcal{L} \in \Theta$ is $h^0(\mathcal{L})$ ([2] Chapter VI p. 226). So the singular locus of Θ is

$$\mathrm{Sing}(\Theta) := \{\mathcal{L} \in A : h^0(\mathcal{L}) \geqslant 2\}.$$

There is a map

$$\begin{aligned} \mathrm{Sing}_2(\Theta) &\longrightarrow |I_2(C)| \\ \mathcal{L} &\longmapsto Q(\mathcal{L}) := \bigcup_{D \in |\mathcal{L}|} \langle D \rangle, \end{aligned}$$

where $\mathrm{Sing}_2(\Theta)$ is the locus of points of order 2 on Θ and $|I_2(C)|$ is the linear system of quadrics containing the canonical curve C. This map is equal to the map sending $\mathcal{L}$ to the (quadric) tangent cone to Θ at $\mathcal{L}$ and its image $\mathcal{Q}$ generates $|I_2(C)|$ (see [4] and [14]). Any $Q(\mathcal{L}) \in \mathcal{Q}$ has rank $\leqslant 4$. The singular locus of $Q(\mathcal{L})$ cuts C in the sum of the base divisors of $|\mathcal{L}|$ and $|\omega_C \otimes \mathcal{L}^{-1}|$. The rulings of Q cut the divisors of the moving parts of $|\mathcal{L}|$ and $|\omega_C \otimes \mathcal{L}^{-1}|$ on C (see [1]).

For any divisor or invertible sheaf a of degree 0 and any subscheme Y of A, we let Y_a or $Y+a$ denote the translate of Y by a. By a g^r_d we mean a (not necessarily complete) linear system of degree d and dimension r. We call W^r_d the subscheme of $\mathrm{Pic}^d C$ parametrizing invertible sheaves $\mathcal{L}$ with $h^0(\mathcal{L}) > r$ with its determinantal scheme structure (see [2, Chapter IV]).

For any effective divisor E of degree e on C and any positive integer $n \geqslant e$, let $C^{(n-e)}_E \subset C^{(n)}$ be the image of $C^{(n-e)}$ in $C^{(n)}$ by the morphism $D \longmapsto D+E$. For any divisor $E = \sum_{i=1}^r n_i t_i$ on C, let C^{div}_E denote the divisor $\sum_{i=1}^r n_i C^{(n-1)}_{t_i}$ on $C^{(n)}$. For a linear system L on C, we denote by C^{div}_L any divisor $C^{\mathrm{div}}_E \subset C^{(n)}$ with $E \in L$.

By infinitesimal deformation we always mean *flat* first-order infinitesimal deformation.

1. The curve $X := X_e(g^1_d)$ and its Useful Properties

Suppose $2 \leqslant e \leqslant g-1$ and let L be a pencil of degree $d \geqslant e+2$ on C. We would like to define a curve X whose underlying set will be

$$\{D_e : \exists D \in C^{(d-e)} \text{ such that } D_e + D \in L\}.$$

If L contains reduced divisors, then X is reduced and can be defined by the above set. If L does not contain reduced divisors then we need to define a scheme structure on X. Although we suppose X integral in this paper, we will define it in general since the definition is simple in the general case. Furthermore, we define the curve by its ideal sheaf whose description we will use later on. We do this in such a way that our nonreduced curves will be flat limits of the reduced ones. Note that the restriction $d \geqslant e+2$ avoids trivial cases where either the maps $X \to A$ are constant or the cohomology class of the image $\overline{X}$ of X is equal to the minimal class in which case we know that $\overline{X}$ does not deform out of the Jacobian locus [11].

Let $W(L) \subset H^0(L)$ be the vector subspace whose projectivization is $L \subset |L|$. The underlying set of X is the subset of $C^{(e)}$ where the elements of $W(L)$ are dependent. A scheme structure can be defined on this set in the following way. Let $D^e \subset C^{(e)} \times C$ be the universal divisor and let q_e and p_e be the first and second projections from $C^{(e)} \times C$ onto $C^{(e)}$ and C, respectively. Then the global evaluation of sections of $\mathcal{O}_C(L)$ on divisors of degree e is the map

$$H^0(L) \otimes \mathcal{O}_{C^{(e)}} \longrightarrow V^e_L := q_{e*}(p_e^* \mathcal{O}_C(L)|_{D^e}) \tag{1.1}$$

obtained by push-forward via q_e from the evaluation map

$$p_e^* \mathcal{O}_C(L) \longrightarrow p_e^* \mathcal{O}_C(L)|_{D^e}.$$

So X is the locus where the evaluation map $W(L) \otimes \mathcal{O}_{C^{(e)}} \to V^e_L$ has rank $\leqslant 1$. Therefore, since X is of (the expected) pure dimension 1, by Eagon and Northcott

[3, Theorem 2, p. 201], there is an exact sequence

$$0 \to \Lambda^e V_L^{e*} \otimes S^{e-2} W(L) \to \cdots \to \Lambda^4 V_L^{e*} \otimes S^2 W(L) \to \Lambda^3 V_L^{e*} \otimes W(L) \to \Lambda^2 V_L^{e*} \to \mathcal{I}_{X/C^{(e)}} \to 0. \tag{1.2}$$

Since our construction can be done globally in families, we see that our non-reduced curves X are indeed flat limits of reduced curves X.

The natural morphism $C \to \mathbb{P}^1$ obtained from L gives a morphism $X \to \mathbb{P}^1$ and, using the Hurwitz formula, one sees immediately that X has arithmetic genus

$$g_X = -\binom{d}{e} + (g-1+d)\binom{d-2}{e-1} + 1.$$

This works at least when X is smooth. When X is not smooth, we obtain the arithmetic genus by specialization from the smooth case.

1.1 Having defined X in $C^{(e)}$, we define the curve $\overline{X}$ that we are really interested in as its image in A up to translation. For this we first choose $g-e-1$ general points $p_1, \ldots, p_{g-e-1}$ in C and map $C^{(e)}$ to $C^{(g-1)}$ and A by the respective morphisms

$$\begin{aligned} \phi_p : C^{(e)} &\longrightarrow C^{(g-1)} & \psi_p : C^{(e)} &\longrightarrow A \\ D_e &\longmapsto D_e + \sum_{i=1}^{g-e-1} p_i & D_e &\longmapsto \mathcal{O}_C\Big(D_e + \sum_{i=1}^{g-e-1} p_i\Big). \end{aligned}$$

The first map is an embedding and the second map is a rational resolution of its image which is a determinantal variety. The fibers of ψ_p are the complete linear systems in $C^{(e)}$. Therefore, in particular, if we let W_e be the image of $C^{(e)}$ in A, then $\psi_{p_*}(\mathcal{O}_{C^{(e)}}) = \mathcal{O}_{W_e}$. We define $\overline{X}$ to be the curve whose ideal is $\psi_{p_*}(\mathcal{I}_{X/C^{(e)}})$. It immediately follows that if X is integral, then so is $\overline{X}$. Replacing X by $X_{d-e}(L)$ if necessary, we will assume that $d \geqslant 2e$. We have the following.

LEMMA 1.1. *Suppose $e \leqslant g-2$ and L contains reduced divisors. Then*

(1) *there are divisors $D \in X_{e+1}(L)$ such that $h^0(D) = 1$,*
(2) *assume $e \leqslant g-3$ and*

  (a) *either $d \geqslant 2e+2$,*
  (b) *or $d = 2e+1$, $h^0(L) \leqslant e$,*
  (c) *or $d = 2e$, $h^0(L) \leqslant e-1$;*

  then there are divisors $D \in X_{e+2}(L)$ such that $h^0(D) = 1$.

Proof. If $d \geqslant 2g-2$, then a general divisor of L is reduced and spans at least a hyperplane in the canonical space of C. So we can choose a subdivisor of degree $e+1$ (resp. $e+2$ if $e \leqslant g-3$) of it which spans a linear subspace of dimension e (resp. $e+1$) of $|K|^*$ and hence satisfies the lemma. If $d \leqslant 2g-3$, then, by Clifford's

Theorem, since C is not hyperelliptic, we have $2(h^0(L)-1)<d$, hence $h^1(L)<g-d/2\leqslant g-e$. So $h^0(K-L)=h^1(L)\leqslant g-e-1$ and a general divisor of L is reduced and spans a linear subspace of $|K|^*$ of dimension at least e. Therefore, it has a subdivisor of degree $e+1$ which spans a linear space of dimension e and hence satisfies the first part of the lemma. For the second part, the assumptions in conjunction with Clifford's Theorem imply that $h^1(L)\leqslant g-e-2$ and an analogous reasoning proves the second part. □

2. The First-Order Obstruction Map

2.1. From now on in the rest of the paper, we shall always assume that X (hence $\overline{X}$) is integral, i.e., reduced and irreducible. It is immediate that the irreducibility of X implies that L has no base points. Note that the converse to this is not true as is easily seen by assuming that C maps nontrivially to a curve of positive genus and taking L to be the inverse image of a pencil on the curve of lower positive genus.

Recall that we also assume $d\geqslant 2e$, and, by Lemma 1.1, a general $D_e\in X$ satisfies $h^0(D_e)=1$ so that the map $X\to\overline{X}$ is birational.

2.2. Since $\overline{X}$ is reduced, the obstructions to deformations of $\overline{X}$ with A live in $Ext^1_{\overline{X}}(\mathcal{I}_{\overline{X}/A}/\mathcal{I}^2_{\overline{X}/A},\mathcal{O}_{\overline{X}})$ (see [9, Lemma 2.13, p. 33 and Proposition 2.14, p. 34]). We have the usual map

$$\mathcal{I}_{\overline{X}/A}/\mathcal{I}^2_{\overline{X}/A}\longrightarrow\Omega^1_A|_{\overline{X}} \tag{2.1}$$

from which we obtain the map of exterior groups

$$H^1(T_A|_{\overline{X}})\longrightarrow Ext^1(\mathcal{I}_{\overline{X}/A}/\mathcal{I}^2_{\overline{X}/A},\mathcal{O}_{\overline{X}}). \tag{2.2}$$

Composing this with restriction

$$H^1(T_A)\longrightarrow H^1(T_A|_{\overline{X}}),$$

we obtain the first order obstruction map

$$\nu\colon\ H^1(T_A)\to Ext^1(\mathcal{I}_{\overline{X}/A}/\mathcal{I}^2_{\overline{X}/A},\mathcal{O}_{\overline{X}}).$$

Given an infinitesimal deformation $\eta\in H^1(T_A)$, the curve $\overline{X}$ deforms with A in the direction of η if and only if $\nu(\eta)=0$.

2.3. The local to global spectral sequence for the exterior sheaves of $\mathcal{I}_{\overline{X}/A}/\mathcal{I}^2_{\overline{X}/A}$ provides the exact sequence

$$0\longrightarrow H^1(N_{\overline{X}/A})\longrightarrow Ext^1(\mathcal{I}_{\overline{X}/A}/\mathcal{I}^2_{\overline{X}/A},\mathcal{O}_{\overline{X}})\longrightarrow H^0(\mathcal{E}xt^1(\mathcal{I}_{\overline{X}/A}/\mathcal{I}^2_{\overline{X}/A},\mathcal{O}_{\overline{X}})).$$

The composition

$$H^1(T_A) \longrightarrow Ext^1(\mathcal{I}_{\overline{X}/A}/\mathcal{I}^2_{\overline{X}/A}, \mathcal{O}_{\overline{X}}) \longrightarrow H^0(\mathcal{E}xt^1(\mathcal{I}_{\overline{X}/A}/\mathcal{I}^2_{\overline{X}/A}, \mathcal{O}_{\overline{X}}))$$

sends η to the obstruction to deform $\overline{X}$ with it locally. Since A is smooth, every deformation of A is locally trivial and locally $\overline{X}$ deforms with it trivially. Therefore, the image of η by the above composition is zero and the obstruction map ν factors through $H^1(N_{\overline{X}/A})$:

$$\nu\colon\ H^1(T_A) \longrightarrow H^1(N_{\overline{X}/A}).$$

Alternatively, the dual of the map (2.1) gives us the map of cohomology groups

$$H^1(T_A|_{\overline{X}}) \longrightarrow H^1(N_{\overline{X}/A}).$$

whose composition with the inclusion $H^1(N_{\overline{X}/A}) \longrightarrow \mathrm{Ext}^1(\mathcal{I}_{\overline{X}/A}/\mathcal{I}^2_{\overline{X}/A}, \mathcal{O}_{\overline{X}})$ is (2.2).

2.4. From the inclusion $\overline{X} \subset W_e$, we obtain the map

$$N_{\overline{X}/A} \longrightarrow N_{W_e/A}|_{\overline{X}}$$

which gives us the map of cohomology groups

$$H^1(N_{\overline{X}/A}) \longrightarrow H^1(N_{W_e/A}|_{\overline{X}}).$$

We call ν_e the composition of this with ν and the pull-back $H^1(N_{W_e/A}|_{\overline{X}}) \to H^1(N_{W_e/A}|_X)$ obtained from the surjective morphism $X \to \overline{X}$:

$$\nu_e\colon\ H^1(T_A) \longrightarrow H^1(N_{W_e/A}|_X).$$

If $\nu_e(\eta) \neq 0$, then, a fortiori, $\nu(\eta) \neq 0$.

2.5. The choice of the polarization Θ provides an isomorphism $H^1(T_A) \cong H^1(\mathcal{O}_C)^{\otimes 2}$ via which the algebraic (i.e. globally unobstructed) infinitesimal deformations with which Θ deforms are identified with the elements of the subspace $S^2H^1(\mathcal{O}_C) \subset H^1(\mathcal{O}_C)^{\otimes 2} \cong H^1(T_A)$. Via this identification, the space of infinitesimal deformations of (A, Θ) as a Jacobian is naturally identified with $H^1(T_C) \subset S^2H^1(\mathcal{O}_C)$. The Serre dual of this last map is multiplication of sections

$$S^2H^0(K) \longrightarrow H^\circ(2K)$$

whose kernel is the space $I_2(C)$ of degree 2-forms vanishing on the canonical image of C. To say that we consider an infinitesimal deformation of (A, Θ) out of the Jacobian locus, means that we consider $\eta \in S^2H^1(\mathcal{O}_C) \backslash H^1(T_C)$ which is therefore equivalent to say that we consider $\eta \in S^2H^1(\mathcal{O}_C)$ such that there is $Q \in I_2(C)$ with $(Q, \eta) \neq 0$. Here we denote by

$$(,)\colon\ S^2H^0(K) \otimes S^2H^1(\mathcal{O}_C) \longrightarrow S^2H^1(K) \cong \mathbb{C}$$

the pairing obtained from Serre Duality.

3. Translates of Θ Containing W_e and the Obstruction Map

To prove our main theorem we use translates of Θ which contain W_e. We choose $g-1-e$ general points $p_1, \dots, p_{g-1-e}$ in C that we fix from now on and define ϕ_p, ψ_p and W_e as in Section 1.1. We first only suppose $1 \leqslant e \leqslant g-1$, specializing later to the case $3 \leqslant e \leqslant g-3$ which is of interest to us.

LEMMA 3.1. *Suppose* $1 \leqslant e \leqslant g-1$. *The subvariety* W_e *is contained in a translate* Θ_a *of* Θ *if and only if there exists* $\sum q_i \in C^{(g-e-1)}$ *such that* $a = \sum p_i - \sum q_i$.

Proof. For any points $q_1, \dots, q_{g-e-1}$ of C, the image of $C^{(e)}$ in A by the corresponding map ψ_q is contained in the divisor $\Theta_{\sum p_i - \sum q_i}$. Conversely, if W_e is contained in a translate Θ_a of Θ, then we have $h^0(D_e + \sum p_i - a) > 0$, for all $D_e \in C^{(e)}$. Equivalently, for all $D_e \in C^{(e)}$, we have $h^0(K + a - \sum p_i - D_e) > 0$, i.e., $h^0(K + a - \sum p_i) \geqslant e+1$ and $-a + \sum p_i$ is effective. □

3.1. Choose $a \in \mathrm{Pic}^0 C$ such that $W_e \subset \Theta_a$ (i.e., $a = \sum p_i - \sum q_i$ as above). Equivalently $W_e - a \subset \Theta$. Let ϕ_q and ψ_q denote the analogues of ϕ_p and ψ_p for $\sum q_i$ instead of $\sum p_i$ so that $W_e - a$ is the image of ψ_q in A.

LEMMA 3.2. *Suppose* $1 \leqslant e \leqslant g-1$. *The differential of the map* ψ_q:

$$\psi_{q*}\colon\ T_{C^{(e)}} \to \psi_q^* T_A$$

is injective.

Proof. Since ψ_q is generically injective, this map is also generically injective. If it were not injective, then its kernel would be a subsheaf of $T_{C^{(e)}}$ supported on a proper subvariety of $C^{(e)}$. This is not possible since $T_{C^{(e)}}$ is a locally free sheaf. □

DEFINITION 3.3. *For* $1 \leqslant e \leqslant g-1$ *and any effective divisor* $\sum q_i$ *of degree* $g-1-e$, *define the sheaf* $N'_{\sum q_i} = N'_q$ *on* $C^{(e)}$ *as the quotient of* $\psi_q^* T_A$ *by the image of* $T_{C^{(e)}}$, *i.e., by the exactness of the sequence*

$$0 \longrightarrow T_{C^{(e)}} \xrightarrow{\psi_{q*}} \psi_q^* T_A \longrightarrow N'_q \longrightarrow 0. \tag{3.1}$$

In case $e = g-1$, then the divisor $\sum q_i$ is 0 and we have only one morphism $\psi_0\colon C^{(g-1)} \to \Theta$ which is the natural morphism. Then, by [4, (1.20), p. 89],

$$N'_0 = \mathcal{I}_{Z_{g-1}}(\Theta)$$

where, as in the Introduction, the scheme Z_{g-1} is the locus where the map ψ_0 fails to be an isomorphism. In other words we have the exact sequence

$$0 \longrightarrow T_{C^{(g-1)}} \longrightarrow \psi_0^* T_A \longrightarrow \mathcal{I}_{Z_{g-1}}(\Theta) \longrightarrow 0. \tag{3.2}$$

For the convenience of the reader, we mention that the scheme Z_{g-1} is a determinantal scheme of codimension 2. If $g \geqslant 5$ or if $g=4$ and C has two distinct g^1_3's, the scheme Z_{g-1} is reduced and is the scheme-theoretical inverse image of the singular locus of Θ.

3.2. In general, it is immediate that outside of the subscheme Z_e where ψ_q fails to be an embedding, the sheaf N'_q is isomorphic to (the pull-back of) the normal sheaf of $W_e - a$ in A:

$$N'_q|_{C^{(e)}_{\sum q_i}\setminus Z_e} \xrightarrow{\cong} N_{W_e-a/A}|_{C^{(e)}_{\sum q_i}\setminus Z_e}.$$

The locus Z_e is locally defined, in the same way as Z_{g-1}, by the maximal minors of the differential $\psi_{q*}\colon T_{C^{(e)}} \to \psi_q^* T_A$. Note that Z_e is independent of the choice of $\sum q_i$ and is the scheme-theoretic inverse image of W^1_e. The support of Z_e is the subset of $C^{(e)}$ parametrizing divisors D_e with $h^0(D_e) \geqslant 2$.

3.3. Combining sequences (3.1) and (3.2) with the tangent bundles sequence for $C^{(e)}_{\sum q_i} \subset C^{(g-1)}$, we obtain the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
0 & & 0 & & & & \\
\downarrow & & \downarrow & & & & \\
T_{C^{(e)}} & = & T_{C^{(e)}} & & & & \\
\downarrow & & \downarrow & & & & \\
T_{C^{(g-1)}}|_{C^{(e)}_{\sum q_i}} & \to & T_A|_{C^{(e)}_{\sum q_i}} & \to & \mathcal{I}_{Z_{g-1}}(\Theta)|_{C^{(e)}_{\sum q_i}} & \to & 0 \\
\downarrow & & \downarrow & & \| & & \\
N_{C^{(e)}_{\sum q_i}/C^{(g-1)}} & \to & N'_{\sum q_i} & \to & \mathcal{I}_{Z_{g-1}}(\Theta)|_{C^{(e)}_{\sum q_i}} & \to & 0, \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

where the leftmost horizontal maps are injective if and only if $h^0(\sum q_i) = 1$ (see Lemma 5.1 below) and the maps of sheaves

$$N_{C^{(e)}_{\sum q_i}/C^{(g-1)}} \longrightarrow N'_{\sum q_i} \longrightarrow \mathcal{I}_{Z_{g-1}}(\Theta)|_{C^{(e)}_{\sum q_i}}$$

are defined by the commutativity of the diagram. From the definition of N'_q it is clear that the horizontal maps in the commutative diagram of natural maps of normal sheaves

$$\begin{array}{ccc}
T_A|_{W_e-a} & \longrightarrow & N_{\Theta/A}|_{W_e-a} = \mathcal{O}_\Theta(\Theta)|_{W_e-a} \\
\| & & \uparrow \\
T_A|_{W_e-a} & \longrightarrow & N_{W_e-a/A}
\end{array}$$

factor as follows after pull-back via ψ_q

$$\begin{array}{ccccccc}
T_A|_{C^{(e)}_{\sum q_i}} & \twoheadrightarrow & N'_0|_{C^{(e)}_{\sum q_i}} = \mathcal{I}_{Z_{g-1}}(\Theta)|_{C^{(e)}_{\sum q_i}} & \twoheadrightarrow & \mathcal{I}_{Z_{g-1}\cap C^{(e)}_{\sum q_i}}(\Theta) & \longrightarrow & \mathcal{O}_{C^{(e)}_{\sum q_i}}(\Theta) \\
\| & & \uparrow & & & & \uparrow \\
T_A|_{C^{(e)}_{\sum q_i}} & \twoheadrightarrow & N'_q = N'_{\sum q_i} & & \longrightarrow & & N_{W_e-a/A}|_{C^{(e)}_{\sum q_i}}
\end{array}$$

3.4. The image of the map of normal sheaves

$$N_{W_e-a/A}\longrightarrow \mathcal{O}_{W_e-a}(\Theta)=N_{\Theta/A}|_{W_e-a}$$

is the twist of a sheaf of ideals by $\mathcal{O}_{W_e-a}(\Theta)$. As in the Introduction, we let Z_q be the subscheme of $C^{(e)}$ defined by the pull-back of this sheaf of ideals. So by definition

$$N_{W_e-a/A}|_{C^{(e)}_{\sum q_i}}\twoheadrightarrow \mathcal{I}_{Z_q}(\Theta)\hookrightarrow \mathcal{O}_{C^{(e)}_{\sum q_i}}(\Theta).$$

By the above $\mathcal{I}_{Z_q}/C^{(e)}$ contains $\mathcal{I}_{Z_{g-1}\cap C^{(e)}_{\sum q_i}}$, hence the subscheme Z_q is contained in $Z_{g-1}\cap C^{(e)}_{\sum q_i}$. Furthermore, because $C^{(e)}_{\sum q_i}\to W_e-a$ is an isomorphism outside Z_e, we have $Z_q\backslash Z_e=Z_{g-1}\cap C^{(e)}_{\sum q_i}\backslash Z_e$ as schemes.

Restricting the diagram in Section 3.3 to X_{-a} after composing the maps into N'_q with the map

$$N'_q\longrightarrow N_{W_e-a/A}|_{C^{(e)}_{\sum q_i}},$$

we obtain the commutative diagram with exact middle row and left column

$$\begin{array}{ccccccc}
T_{C^{(e)}_{\sum q_i}}|_{X_{-a}} & = & T_{C^{(e)}_{\sum q_i}}|_{X_{-a}} & & & & \\
\downarrow & & \downarrow & & & & \\
T_{C^{g-1}}|_{X_{-a}} & \to & T_A|_{X_{-a}} & \to & \mathcal{I}_{Z_{g-1}}(\Theta)|_{X_{-a}} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
N_{C^{(e)}_{\sum q_i}/C^{(g-1)}}|_{X_{-a}} & \to & N_{W_e-a/A}|_{X_{-a}} & \to & \mathcal{I}_{Z_q}(\Theta)|_{X_{-a}} & \to & 0. \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}$$

So we have the commutative diagram

$$\begin{array}{ccccc}
S^2H^1(\mathcal{O}_C)\subset H^1(T_A) & \longrightarrow & H^1(T_A|_{X_{-a}}) & \longrightarrow & H^1(\mathcal{I}_{Z_{g-1}}(\Theta)|_{X_{-a}}) \\
\| & & \downarrow & & \downarrow \\
S^2H^1(\mathcal{O}_C)\subset H^1(T_A) & \longrightarrow & H^1(N_{W_e-a/A}|_{X_{-a}}) & \longrightarrow & H^1(\mathcal{I}_{Z_{g-1}}(\Theta)|_{X_{-a}}).
\end{array}$$

Translation by a induces the identity on $H^1(T_A)$ and isomorphisms

$$H^1(T_A|_{X_{-a}})\cong H^1(T_A|_X),$$

$$H^1(N_{W_e-a/A}|_{X_{-a}})\cong H^1(N_{W_e/A}|_X),$$

so that the kernel of

$$\nu_e\colon S^2H^1(\mathcal{O}_C)\longrightarrow H^1(N_{W_e/A}|_X)$$

is equal to the kernel of the map

$$S^2H^1(\mathcal{O}_C) \longrightarrow H^1(N_{W_e-a/A}|_{X_{-a}})$$

obtained from ν_e by translation. Therefore, the previous diagram proves the following theorem.

THEOREM 3.4. *The kernel of the map*

$$\nu_e\colon S^2H^1(\mathcal{O}_C) \longrightarrow H^1(N_{W_e/A}|_X)$$

is contained in the kernel of the map obtained from the above

$$S^2H^1(\mathcal{O}_C) \longrightarrow H^1(\mathcal{I}_{Z_q}(\Theta)|_{X_{-a}})$$

for all a such that Θ_a contains W_e.

A fortiori, the kernel of ν_e is contained in the kernel of the composition

$$S^2H^1(\mathcal{O}_C) \longrightarrow H^1(\mathcal{I}_{Z_q}(\Theta)|_{X_{-a}}) \longrightarrow H^1(\mathcal{I}_{Z_q\cap X_{-a}}(\Theta)).$$

3.5. *Hypothesis.* From now on we will assume that the set of q for which $Z_q \cap X_{-a} \neq Z_{g-1} \cap X_{-a}$ as schemes has dimension at most $g-e-3$ and we have chosen q outside of this set, i.e., in such a way that

$$Z_q \cap X_{-a} = Z_{g-1} \cap X_{-a}.$$

By Appendix 8.3, this will be the case for all $q \in C^{(g-1-e)}$ on some open subset of the space of pairs (C, L).

Under the assumptions of Theorem 1, we shall prove that for any $\eta \in S^2H^1(\mathcal{O}_C)\backslash H^1(T_C)$, there exists a such that Θ_a contains W_e and the image of η in $H^1(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta))$ is nonzero unless

- either $d = 2e$ and $h^0(L) = e$,
- or $d = 2e+1$ and $h^0(L) = e+1$.

4. The Kernel of the Map $S^2H^1(\mathcal{O}_C) \longrightarrow H^1(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta))$

4.1. From now on we shall use divisors $\sum q_i \in C^{(g-1-e)}$ such that $h^0(\sum q_i) = 1$ and, for $a = \sum p_i - \sum q_i$, we have $X_{-a} \not\subset Z_{g-1}$ and $X_{-a} \cap Z_{g-1} \neq \emptyset$. Such divisors exist by Lemmas 6.1 and 6.2 below.

4.2. The above map is equal to the composition

$$S^2H^1(\mathcal{O}_C) \longrightarrow H^1(\mathcal{I}_{Z_{g-1}}(\Theta)) \longrightarrow H^1(\mathcal{I}_{Z_{g-1}}(\Theta)|_{X_{-a}}) \longrightarrow H^1(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta)). \tag{4.1}$$

From the usual exact sequence

$$0 \longrightarrow \mathcal{I}_{Z_{g-1}}(\Theta) \longrightarrow \mathcal{O}_{C^{(g-1)}}(\Theta) \longrightarrow \mathcal{O}_{Z_{g-1}}(\Theta) \longrightarrow 0$$

we obtain the embedding

$$H^0(\mathcal{O}_{Z_{g-1}}(\Theta)) \hookrightarrow H^1(\mathcal{I}_{Z_{g-1}}(\Theta)).$$

By [4, p. 95], the image of $S^2H^1(\mathcal{O}_C)$ in $H^1(\mathcal{I}_{Z_{g-1}}(\Theta))$ is contained in $H^0(\mathcal{O}_{Z_{g-1}}(\Theta))$. Using the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{I}_{Z_{g-1}}(\Theta) & \longrightarrow & \mathcal{O}_{C^{g-1}}(\Theta) & \longrightarrow & \mathcal{O}_{Z_{g-1}}(\Theta) & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & \mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta) & \longrightarrow & \mathcal{O}_{X_{-a}}(\Theta) & \longrightarrow & \mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta) & \longrightarrow & 0, \end{array}$$

Composition (4.1) is also equal to the composition

$$S^2H^1(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_{Z_{g-1}}(\Theta)) \longrightarrow H^0\left(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta)\right) \longrightarrow$$
$$\longrightarrow H^1\left(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta)\right).$$

By [4, p. 95] the first map is the following

$$S^2H^1(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_{Z_{g-1}}(\Theta))$$
$$\sum a_{ij}\frac{\partial^2}{\partial z_i \partial z_j} \mapsto \sum a_{ij}\frac{\partial^2 \sigma}{\partial z_i \partial z_j}|_{Z_{g-1}},$$

where $\{z_i\}$ is a system of coordinates on A and σ is a theta function with divisor of zeros equal to Θ. So we have the following description

$$S^2H^1(\mathcal{O}_C) \to H^0(\mathcal{O}_{Z_{g-1}}(\Theta)) \qquad H^0\left(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta)\right) \overset{\text{coboundary}}{\to} H^1\left(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta)\right)$$
$$\sum a_{ij}\frac{\partial^2}{\partial z_i \partial z_j} \mapsto \sum a_{ij}\frac{\partial^2 \sigma}{\partial z_i \partial z_j}|_{Z_{g-1}} \mapsto \sum a_{ij}\frac{\partial^2 \sigma}{\partial z_i \partial z_j}|_{Z_{g-1}\cap X_{-a}} \mapsto ?$$

4.3. We will investigate the kernel of the composition of the first two maps $S^2H^1(\mathcal{O}_C) \to H^0\left(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta)\right)$ and that of the coboundary map $H^0\left(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta)\right) \to H^1\left(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta)\right)$ separately. The kernel of $S^2H^1(\mathcal{O}_C) \to H^0\left(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta)\right)$ is contained in (with equality if and only if $Z_{g-1} \cap X_{-a}$ is reduced) the annihilator of the quadrics of rank $\leqslant 4$ which are the tangent cones to Θ at the points of $Z_{g-1} \cap X_{-a}$.

5. The Kernel of the Map $S^2H^1(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta))$

5.1. An effective divisor $\sum_{i=1}^{g-1-e} q_i \in C^{(g-1-e)}$ gives the embedding of X in $C^{(g-1)}$ defined by $D_e \mapsto D_e + \sum_{i=1}^{g-1-e} q_i$. The union of the images of these maps is a scheme, denoted $X + C^{(g-1-e)} \subset C^{(g-1)}$ whose intersection with Z_{g-1} is the union of the schemes $Z_{g-1}\cap X_{-a}$ as $a=\sum p_i-\sum q_i$ varies. To say that η is in the kernel of the composition

$$S^2H^1(\mathcal{O}_C)\longrightarrow H^0(\mathcal{O}_{Z_{g-1}}(\Theta))\longrightarrow H^0(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta))$$

for every a, means that η is in the kernel of the map

$$S^2H^1(\mathcal{O}_C)\longrightarrow H^0(\mathcal{O}_{Z_{g-1}}(\Theta))\longrightarrow H^0(\mathcal{O}_{Z_{g-1}\cap X+C^{(g-1-e)}}(\Theta)).$$

Since X is reduced, $X+C^{(g-1-e)}$ is also the union of all $C_E^{(g-1-e)}$ with $E\in X$. So we see that the above is also equivalent to η being in the kernel of all the maps

$$S^2H^1(\mathcal{O}_C)\longrightarrow H^0(\mathcal{O}_{Z_{g-1}}(\Theta))\longrightarrow H^0(\mathcal{O}_{Z_{g-1}\cap C_E^{(g-1-e)}}(\Theta)).$$

for all points $E\in X$.

5.2. We compute the kernel of the map

$$S^2H^1(\mathcal{O}_C)\longrightarrow H^0(\mathcal{O}_{Z_{g-1}}(\Theta))\longrightarrow H^0(\mathcal{O}_{Z_{g-1}\cap C_D^{(f)}}(\Theta)).$$

for $1\leqslant f\leqslant g-1$ and D a *reduced* divisor of degree $g-1-f$ such that $h^0(D)=1$. We shall later assume $D\in X$. Recall that a general $D\in X$ is reduced and, by Lemma 1.1, we have $h^0(D)=1$.

LEMMA 5.1. *The sequences*

$$0\longrightarrow T_{C^{g-1}}|_{C_D^{(f)}}\longrightarrow T_A|_{C_D^{(f)}}\longrightarrow \mathcal{I}_{Z_{g-1}}(\Theta)|_{C_D^{(f)}}\longrightarrow 0$$

and

$$0\longrightarrow N_{C_D^{(f)}}\longrightarrow N'_D\longrightarrow \mathcal{I}_{Z_{g-1}}(\Theta)|_{C_D^{(f)}}\longrightarrow 0$$

are exact.

Proof. The first sequence is obtained from sequence (3.2) by restriction to $C_D^{(f)}$, so we only need to prove that the map

$$T_{C^{g-1}}|_{C_D^{(f)}}\longrightarrow T_A|_{C_D^{(f)}}$$

is injective. Since $h^0(D)=1$, this map is generically injective. If it were not injective, then its kernel would be a subsheaf of $T_{C^{(g-1)}}|_{C_D^{(f)}}$ supported on a proper subvariety of $C_D^{(f)}$. This is not possible since $T_{C^{(g-1)}}|_{C_D^{(f)}}$ is a locally free sheaf.

The exactness of the second sequence follows from the first and the diagram in Section 3.3 with $\sum q_i$ replaced with D. □

From the lemma it follows that the kernel of the map

$$H^1(T_A|_{C^{(f)}}) = H^1(T_A) = H^1(\mathcal{O}_C)^{\otimes 2} \longrightarrow H^1(\mathcal{I}_{Z_{g-1}}(\Theta)|_{C_D^{(f)}})$$

is the image of $H^1(T_{C^{(g-1)}}|_{C_D^{(f)}})$. We first need

LEMMA 5.2. *We have*

$$\mathcal{I}_{Z_{g-1}}(\Theta)|_{C_D^{(f)}} \cong \mathcal{I}_{Z_{g-1}\cap C_D^{(f)}}(\Theta)$$

if and only if $Z_{g-1} \cap C_D^{(f)}$ *has codimension* 2 *in* $C_D^{(f)}$.

Proof. The natural exact sequence

$$0 \longrightarrow \mathcal{I}_{Z_{g-1}} \longrightarrow \mathcal{O}_{C^{(g-1)}} \longrightarrow \mathcal{O}_{Z_{g-1}} \longrightarrow 0$$

gives the exact sequence of Tor groups

$$0 \longrightarrow \mathrm{Tor}_1^{C^{(g-1)}}(\mathcal{O}_{Z_{g-1}}, \mathcal{O}_{C_D^{(f)}}) \longrightarrow \mathcal{I}_{Z_{g-1}}|_{C_D^{(f)}} \longrightarrow \mathcal{O}_{C_D^{(f)}} \longrightarrow \mathcal{O}_{Z_{g-1}\cap C_D^{(f)}} \longrightarrow 0.$$

We will first prove that if $Z_{g-1} \cap C_D^{(f)}$ has codimension 2 in $C_D^{(f)}$ then

$$\mathrm{Tor}_1^{C^{(g-1)}}(\mathcal{O}_{Z_{g-1}}, \mathcal{O}_{C_D^{(f)}}) = 0.$$

Write $D = \sum t_i$, then $C_D^{(f)}$ is the complete intersection of the divisors $C_{t_i}^{(g-2)}$ in $C^{(g-1)}$. So we have the Kozsul resolution

$$0 \longrightarrow \mathcal{O}_{C^{(g-1)}}\left(-\sum C_{t_i}^{(g-2)}\right) \longrightarrow \bigoplus_j \mathcal{O}_{C^{(g-1)}}\left(-\sum_{i\neq j} C_{t_i}^{(g-2)}\right) \longrightarrow \cdots$$
$$\cdots \longrightarrow \bigoplus_j \mathcal{O}_{C^{(g-1)}}(-C_{t_j}^{(g-2)}) \longrightarrow \mathcal{O}_{C^{(g-1)}} \longrightarrow \mathcal{O}_{C_D^{(f)}} \longrightarrow 0.$$

If $Z_{g-1} \cap C_D^{(f)}$ has codimension 2 in $C_D^{(f)}$, then $Z_{g-1} \cap C_D^{(f)}$ has codimension f in Z_{g-1}, hence it is the complete intersection of the divisors $C_{t_i}^{(g-2)} \cap Z_{g-1}$ in Z_{g-1}. It follows that the restriction of the above Kozsul resolution to Z_{g-1} remains exact and $\mathrm{Tor}_1^{C^{(g-1)}}(\mathcal{O}_{Z_{g-1}}, \mathcal{O}_{C_D^{(f)}}) = 0$.

Conversely, if

$$\mathcal{I}_{Z_{g-1}}|_{C_D^{(f)}} \cong \mathcal{I}_{Z_{g-1}\cap C_D^{(f)}}$$

then the ideal $\mathcal{I}_{Z_{g-1}\cap C_D^{(f)}}$ has at least two independent local generators everywhere and $Z_{g-1} \cap C_D^{(f)}$ has codimension 2 in $C_D^{(f)}$. □

In such a case the kernel of

$$S^2 H^1(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_{Z_{g-1}}(\Theta)) \longrightarrow H^0(\mathcal{O}_{Z_{g-1}\cap C_D^{(f)}}(\Theta))$$

is the intersection of the image of $H^1(T_{C^{g-1}}|_{C_D^{(f)}})$ with $S^2H^1(\mathcal{O}_C)$.

Recall that $V(D)$ is the vector subspace of $H^1(\mathcal{O}_C)=H^0(\omega_C)^*$ whose projectivization is $\langle D\rangle$.

THEOREM 5.5. *The image of* $H^1(T_{C^{(g-1)}}|_{C_D^{(f)}})$ *in* $H^1(T_A|_{C_D^{(f)}})$ *is the span of* $H^1(T_C)\subset S^2H^1(\mathcal{O}_C)$ *and* $V(D)\otimes H^1(\mathcal{O}_C)$. *The intersection of this span with* $S^2H^1(\mathcal{O}_C)$ *is the span of* $H^1(T_C)$ *and* $S^2V(D)$.

Proof. We use the commutative diagram with exact rows and columns obtained as in Section 3.3

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & T_{C^{(f)}} & = & T_{C^{(f)}} & & \\
 & & \downarrow & & \downarrow & & \\
0\longrightarrow & & T_{C^{(g-1)}}|_{C_D^{(f)}} & \longrightarrow & T_A|_{C_D^{(f)}} & \longrightarrow \mathcal{I}_{Z_{g-1}}(\theta)|_{C_D^{(f)}} & \longrightarrow 0\\
 & & \downarrow & & \downarrow & \| & \\
0\longrightarrow & & N_{C_D^{(f)}/C^{(g-1)}} & \longrightarrow & N'_D & \longrightarrow \mathcal{I}_{Z_{g-1}}(\theta)|_{C_D^{(f)}} & \longrightarrow 0\\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where the injectivity of the leftmost horizontal maps follows from Lemma 5.1. The map $H^1(C^{(f)},\ T_{C^{(f)}})\to H^1(C^{(f)},T_A|_{C_D^{(f)}})$ is injective, hence so is the map $H^1(C^{(f)},T_{C^{(f)}})\to H^1(C^{(f)},\ T_{C^{(g-1)}}|_{C_D^{(f)}})$. Therefore $H^0(C^{(f)},\ T_{C^{(g-1)}}|_{C_D^{(f)}})\to H^0(C^{(f)},\ N_{C_D^{(f)}/C^{(g-1)}})$ is an isomorphism.

Consider now the composition

$$H^0\left(T_{C^{(g-1)}}|_{C_D^{(f)}}\right)\otimes\mathcal{O}_{C^{(f)}}\longrightarrow T_{C^{(g-1)}}|_{C_D^{(f)}}\longrightarrow N_{C_D^{(f)}/C(g-1)}$$

where the first map is evaluation. Then, via the isomorphism

$$H^0\left(C^{(f)},T_{C^{(g-1)}}|_{C_D^{(f)}}\right)\xrightarrow{\cong}H^0(C^{(f)},N_{C_D^{(f)}/C^{(g-1)}}),$$

this composition can be identified with evaluation of global sections

$$H^0(C^{(f)},N_{C_D^{(f)}/C^{(g-1)}})\otimes\mathcal{O}_{C^{(f)}}\longrightarrow(N_{C_D^{(f)}/C^{(g-1)}}).$$

From this we obtain the map

$$H^0(C^{(f)},N_{C_D^{(f)}/C^{(g-1)}})\otimes H^1(\mathcal{O}_{C^{(f)}})\longrightarrow H^1(N_{C_D^{(f)}/C^{(g-1)}}).$$

Write $D=\sum_{i=1}^{g-1-f}t_i$. Since $C_D^{(f)}$ is the complete intersection of the divisors $C_{t_i}^{(g-2)}$ in $C^{(g-1)}$, its normal sheaf is isomorphic to $\oplus_{i=1}^{g-1-f}\mathcal{O}_{C^{(f)}}(C_{t_i}^{(f-1)})$. Therefore, using Appendix 6.1 in [5], the above map can be identified with

$$\bigoplus_{i=1}^{g-1-f}S^{f-1}H^0(t_i)\otimes H^1(\mathcal{O}_C)\longrightarrow\bigoplus_{i=1}^{g-1-f}S^{f-1}H^0(t_i)\otimes H^1(t_i)$$

which is onto because each of the maps $H^1(\mathcal{O}_C) \to H^1(q_i)$ is linear projection which is onto. Therefore, the composition

$$H^0(T_{C^{(g-1)}}|_{C_D^{(f)}}) \otimes H^1(\mathcal{O}_{C^{(f)}}) \longrightarrow H^1(T_{C^{(g-1)}}|_{C_D^{(f)}}) \longrightarrow H^1(N_{C_D^{(f)}/C^{(g-1)}})$$

is onto and hence so is

$$H^1(T_{C^{(g-1)}}|_{C_D^{(f)}}) \longrightarrow H^1(N_{C_D^{(f)}/C^{(g-1)}}).$$

In conclusion we have the exact sequence

$$0 \longrightarrow H^1(T_{C^{(f)}} \longrightarrow H^1(T_{C^{(g-1)}}|_{C_D^{(f)}}) \longrightarrow H^1(N_{C_D^{(f)}/C^{(g-1)}}) \longrightarrow 0$$

and the image of $H^1(T_{C^{(g-1)}}|_{C_D^{(f)}})$ in $H^1(T_A|_{C_D^{(f)}})$ is the span of the images of $H^1(T_C) = H^1(T_{C^{(f)}})$ and $\bigoplus_{i=1}^{g-1-f} S^{f-1} H^0(t_i) \otimes H^1(\mathcal{O}_C)$.

Now, the image of $\bigoplus_{i=1}^{g-1-f} S^f H^0(t_i)$ in $H^0(T_A) = H^1(\mathcal{O}_C)$ is $V(D)$. Indeed, as we saw above, $H^0(T_{C^{(g-1)}}|_{C_D^{(f)}}) \cong H^0(N_{C_D^{(f)}/C_{(g-1)}}) \cong \bigoplus_{i=1}^{g-1-f} H^0(\mathcal{O}_{C^{(f)}}(C_{t_i}^{(f-1)})) = \bigoplus_{i=1}^{g-1-f} S^f H^0(t_i)$ (by [5, Appendix 6.1]) has dimension $g-1-f$. The tangent space to $C^{(g-1)}$ at $D_{g-1} \in C^{(g-1)}$ can canonically be identified with $\mathcal{O}_{D_{g-1}}(D_{g-1})$. For all $D_{g-1} \in C_D^{(f)} \subset C^{(g-1)}$, we have $\mathcal{O}_D(D) \subset \mathcal{O}_{D_{g-1}}(D_{g-1})$. So $\mathcal{O}_D(D) \subset H^0(T_{C^{(g-1)}}|_{C_D^{(f)}})$ and the two spaces are equal since they have the same dimension. The image of $\mathcal{O}_D(D)$ in $H^0(T_A) = H^1(\mathcal{O}_C)$ by the differential of the map $\psi_0 : C^{(g-1)} \to A$ is $V(D)$. So the image of $H^0(T_{C^{(g-1)}}|_{C_D^{(f)}})$ in $H^1(\mathcal{O}_C)$ is $V(D)$. Therefore the image of $H^0(T_{C^{(g-1)}}|_{C_D^{(f)}}) \otimes H^1(\mathcal{O}_C)$ in $H^1(T_A|_{C_D^{(f)}}) = H^1(\mathcal{O}_C)^{\otimes 2}$ is $V(D) \otimes H^1(\mathcal{O}_C)$. □

5.3. Therefore, if $Z_{g-1} \cap C_D^{(f)}$ has codimension 2 in $C_D^{(f)}$, then the kernel of $S^2 H^1(\mathcal{O}_C) \to H^0(\mathcal{O}_{Z_{g-1} \cap C_D^{(f)}})$ is the span of $H^1(T_C)$ and $S^2 V(D)$.

Since we have assumed $h^0(D) = 1$, the codimension of $Z_{g-1} \cap C_D^{(f)}$ is at least 1. If the codimension of $Z_{g-1} \cap C_D^{(f)}$ in $C^{(f)}$ is 1, then we have the exact sequence

$$0 \longrightarrow \mathcal{K}_Y \longrightarrow \mathcal{I}_{Z_{g-1}}(\Theta)|_{C_D^{(f)}} \longrightarrow \mathcal{I}_{Z_{g-1} \cap C_D^{(f)}}(\Theta) \longrightarrow 0$$

where $\mathcal{K}_Y$ is defined by the exact sequence and Y is the support of $\mathcal{K}_Y$. We have $Y \subset Z_{g-1} \cap C_D^{(f)}$ since outside $Z_{g-1} \cap C_D^{(f)}$ the two sheaves $\mathcal{I}_{Z_{g-1}}(\Theta)|_{C_D^{(f)}}$ and $\mathcal{I}_{Z_{g-1} \cap C_D^{(f)}}(\Theta)$ are isomorphic (to $\mathcal{O}_{C_D^{(f)}}(\Theta)$). If $f = 1$, then Y has dimension $\leqslant 0$, $h^1(\mathcal{K}_Y) = 0$, $H^1(\mathcal{I}_{Z_{g-1}}(\Theta)|_{C_D^{(f)}}) = H^1(\mathcal{I}_{Z_{g-1} \cap C_D^{(f)}}(\Theta))$ and the kernel of

$$S^2 H^1(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_{Z_{g-1}}(\Theta)) \longrightarrow H^0(\mathcal{O}_{Z_{g-1} \cap C_D^{(f)}}(\Theta)) \longrightarrow H^1(\mathcal{I}_{Z_{g-1} \cap C_D^{(f)}}(\Theta))$$

is again the intersection of the image of $H^1(T_{C^{(g-1)}}|_{C_D^{(f)}})$ with $S^2 H^1(\mathcal{O}_C)$.

Suppose from now on that $2 \leqslant f \leqslant g-1$. We have

LEMMA 5.4. *The codimension of $Z_{g-1} \cap C_D^{(f)}$ is 1 in $C_D^{(f)}$ only in the following cases*

(1) *The intersection $Z_{g-1} \cap C_D^{(f)}$ contains $C_{t+D}^{(f-1)}$ for some $t \in C$. This happens if and only if $\langle D \rangle \cap C$ contains $D+t$, meaning the base divisor, in C, of the linear subsystem $|\omega_C(-D)| \subset |\omega_C|$ contains $D+t$.*
(2) *The restriction to C of the projection from $\langle D \rangle$ is not birational to its image. Letting C' be the normalization of this image, the projection from $\langle D \rangle$ induces $\kappa: C \to C'$ of degree at least 2. Given κ, there exist a finite number of linear subspaces $L_i (i=1, \dots, l)$ of $|\omega_C|^*$ such that any such $\langle D \rangle$ contains L_i for some i. Furthermore, $Z_{g-1} \cap C_D^{(f)}$ contains the divisor $C_D^{(f-2)} + X(\kappa) \subset C_D^{(f)}$ where*

$$X(\kappa) := \{D_2 \in C^{(2)} : \exists t \in C', h^0(\kappa^*(t) - D_2) > 0\}.$$

Proof. The first case is clear. Assume therefore that $Z_{g-1} \cap C_D^{(f)}$ contains an irreducible divisor $\mathcal{F} \subset C_D^{(f)}$ which is *not* of the form $C_{t+D}^{(f-1)}$. It is easily seen that this is equivalent to the fact that for a general divisor $D_{f-2} \in C^{(f-2)}$, the projection from $\langle D + D_{f-2} \rangle$ is not birational on C. It first follows that the projection from $\langle D \rangle$ is not birational on C. Indeed, if we call C_1 the image of C by the projection from $\langle D \rangle$, then, by the general position theorem ([2, p. 109]), the projection of C_1 from the span of a general effective divisor on it is always birational unless the image of the projection is $\mathbb{P}^1$ or a point. So, if the projection of C from $\langle D \rangle$ is birational, then so is its projection from $\langle D + D_{f-2} \rangle$ for $D_{f-2} \in C^{(f-2)}$ general.

The general divisors $D_f \in \mathcal{F}$ are of the form $D_{f-2} + D_2 + D$ where $D_{f-2} \in C^{(f-2)}$ is general and $D_2 \leqslant \kappa^*(t)$ for some $t \in C'$, i.e.,

$$\mathcal{F} = C^{(f-2)} + X(\kappa) + D.$$

To prove the assertion about the L_i, first suppose that the cover $\kappa: C \to C'$ is Galois and let $\{\sigma_1, \dots, \sigma_n\}$ be a set of generators for its Galois group. Then, since the projection from $\langle D \rangle$ induces κ, the linear space $\langle D \rangle$ is globally invariant under $\sigma_1, \dots, \sigma_n$ and $\sigma_1, \dots, \sigma_n$ induce the identity on $|\omega_C|^*/\langle D \rangle$. Therefore, if we let V be the vector space whose projectivization is $|\omega_C|^*/\langle D \rangle$, then, for all i, σ_i has only one eigenvalue, say λ_i on V. Hence $\langle D \rangle$ contains the eigenspaces of σ_i for all its eigenvalues which are distinct from λ_i. For each choice $\mu_1, \dots, \mu_n$ of eigenvalues of $\sigma_1, \dots, \sigma_n$, we let $L(\mu_1, \dots, \mu_n)$ be the smallest linear subspace of $|\omega_C|^*$ which, for all i, contains all the eigenspaces of σ_i for the eigenvalues distinct from μ_i. Assuming none of the σ_i is the identity on C, all the $L(\mu_1, \dots, \mu_n)$ are non-empty. So we see that $L(\lambda_1, \dots, \lambda_n) \subset \langle D \rangle$. It is immediate that a linear subspace L contains some $L(\mu_1, \dots, \mu_n)$ if and only if the projection from L factors through κ: $C \to C'$. Therefore, the $L(\mu_1, \dots, \mu_n)$ are the minimal subspaces L of $|\omega_C|^*$ such that the projection from L factors through $\kappa: C \to C'$. This description shows that they only depend on κ and not the choice of the generating set $\{\sigma_1, \dots, \sigma_n\}$. We number them to obtain the subspaces $L_i (i=1, \dots, l)$ in the statement.

Now, if the cover $\kappa: C \to C'$ is not Galois, it can be dominated by a Galois cover: in other words, there exists a Galois cover $\widetilde{\kappa}: \widetilde{C} \to C'$ which factors through $\kappa: C \to C'$. The induced map on the Jacobians $J\widetilde{C} \to JC$ induces a projection $|\omega_{\widetilde{C}}|^* \to |\omega_C|^*$ which, composed with the map $|\omega_C|^* \to |\omega_C|^*/\langle D\rangle$, induces $\widetilde{\kappa}$. The subspaces L_i are well-defined for $\widetilde{\kappa}$, denote them by $\widetilde{L}_i$. The images of the $\widetilde{L}_i$ which are not contained in the center of the projection $|\omega_{\widetilde{C}}|^* \to |\omega_C|^*$ give us the subspaces L_i for κ. □

LEMMA 5.5. *Suppose $e \leqslant g-3$ and for all $e' \leqslant e$, the curve $X_{e'}(L)$ is irreducible and reduced. For $D \in X$ general, the intersection $Z_{g-1} \cap C_D^{(g-1-e)}$ has codimension 2 in $C_D^{(g-1-e)}$.*

Proof. We have to prove that neither of the two cases in Lemma 5.4 occur.

If, for D general in X, $\langle D\rangle$ contains one of the linear spaces L_i from Lemma 5.4, then, since X is irreducible, for all $D \in X$, $\langle D\rangle \supset L_i$. Choose now D_{e+1} general in $X_{e+1}(L)$. Then D_{e+1} is reduced and, by Lemma 1.1, $h^0(D_{e+1}) = 1$ so that $\cap_{D \leqslant D_{e+1}} \langle D\rangle = \emptyset$ and it cannot contain any L_i.

Suppose now that, for D general in X, $\langle D\rangle \cap C \supset D + s_D$ for some $s_D \in C$. Choose $D_{e+1} = \sum_{i=1}^{e+1} t_i \in X_{e+1}(L)$ as above and let $D_i := \sum_{j \neq i} t_j = D_{e+1} - t_i, i = 1, \ldots, e+1$ be the subdivisors of degree e of D_{e+1}. Also, for each $i \in \{1, \ldots, e+1\}$ let $s_i := s_{D_i}$ be a point of C such that $D_i + s_i \subset \langle D_i\rangle$. First note that $s_i \neq t_i$ for all i because otherwise $s_i + D_i = D_{e+1}$ in which case $h^0(D_{e+1}) = h^0(s_i + D_i) \geqslant 2$ which contradicts the choice of D_{e+1}. If $s_i = t_j$ for some $j \neq i$, this simply means that $\langle D_i\rangle$ contains the projective tangent line to C at t_j. We see therefore that if the s_i are distinct for the different values of i, then

$$\langle D_{e+1}\rangle \cap C \geqslant D_{e+1} + \sum_{i=1}^{e+1} s_i$$

This gives a contradiction by Clifford's Theorem and the fact that C is not hyperelliptic.

Therefore there exists $i \neq j$ such that $s_i = s_j$, say $s_1 = s_{e+1}$. Then

$$\sum_{i=2}^{e} t_i + s_1 \subset \langle D_1\rangle \cap \langle D_{e+1}\rangle = \left\langle \sum_{i=2}^{e} t_i \right\rangle.$$

So there is a divisor $D_{e-1} \in X_{e-1}(L)$ ($D_{e-1} = \sum_{i=2}^{e} t_i$) such that $\langle D_{e-1}\rangle \cap C \supset D_{e-1} + s$ for some $s \in C$ (here $s = s_1$). Since $X_{e-1}(L)$ is also irreducible and our choices of divisors were general, this is the case for all $D_{e-1} \in X_{e-1}(L)$. Repeating the argument with $e-1$ instead of e and continuing, we arrive at a contradiction. □

5.4. Therefore, by what we saw above, for $D \in X$ general, the kernel of $S^2H^1(\mathcal{O}_C) \to H^0(\mathcal{O}_{Z_{g-1} \cap C_D^{(f)}})$ is the span of $H^1(T_C)$ and $S^2V(D)$. We have

LEMMA 5.6. *Assume $e \leqslant g-3$ and*

(1) *either $d \geqslant 2e+2$,*
(2) *or $d = 2e+1, h^0(L) \leqslant e$,*
(3) *or $d = 2e, h^0(L) \leqslant e-1$.*

Then there is a reduced $D \in X_{e+1}(L)$ such that $h^0(D) = 1$ and $\langle D \rangle \cap C = D$.

Proof. By Lemma 1.1, there exists $E \in X_{e+2}(L)$ such that $h^0(E) = 1$ and E is reduced. We claim that there exists $D \leqslant E$ with $\langle D \rangle \cap C = D$. Suppose not so that the span of every subdivisor of degree $e+1$ of E contains an extra point of C. As in the proof of Lemma 5.5, if two of these points are equal, then we have a subdivisor of degree e of E whose span contains an extra point of C and this is not possible for all such E by the previous two Lemmas. So, as in the proof of Lemma 5.5, we see that have a divisor E' of degree $e+2$ such that $\langle E + E' \rangle$ has dimension $e+1$. Hence $h^0(E+E') = 1+e+2 = e+3$. By Clifford's Theorem, since C is not hyperelliptic, this is possible only if $E + E'$ is a canonical divisor on C. In particular, $e = g-3$.

Put $E = t_1 + \cdots t_{g-1}$ and $E' = s_1 + \cdots + s_{g-1}$, the points being numbered in such a way that for all $j, s_j + \sum_{i \neq j} t_i \leqslant \langle \sum_{i \neq j} t_i \rangle \cap C$, i.e., $h^0(s_j + \sum_{i \neq j} t_i) = 2$. Since $E + E'$ is a canonical divisor, we also have $h^0(t_j + \sum_{i \neq j} s_i) = 2$ for all j, i.e., $t_j + \sum_{i \neq j} s_i \leqslant \langle \sum_{i \neq j} s_i \rangle \cap C$. Choose a basis of $V(E) = V(E + E') \subset H^1(\mathcal{O}_C)$ in which the coordinates of t_j are $(0, \ldots 0, 1, 0 \ldots, 0)$ where 1 is in the j-th slot. Let $(a_{j1}, \ldots, a_{j\,g-1})$ be the coordinates of s_j. Then $a_{jj} = 0$ for all j. Take $j = 1$. Then $a_{11} = 0$ and the condition $t_1 \in \langle \sum_{i=2}^{g-1} s_i \rangle$ means there are scalars $\lambda_2, \ldots, \lambda_{g-1}$ such that

$$1 = \sum_{i=2}^{g-1} \lambda_i a_{i1}$$
$$0 = \sum_{i=2}^{g-1} \lambda_i a_{ik} \quad \text{for all } k \geqslant 2.$$

Since $E + E'$ is a canonical divisor and $h^0(E) = 1$, we also have $h^0(E') = 1$. Therefore the s_j are linearly independant and, a fortiori, the minor $|a_{jk}|_{\substack{2 \leqslant j \leqslant g-1 \\ 2 \leqslant k \leqslant g-1}}$ is not zero and the condition $0 = \sum_{i=2}^{g-1} \lambda_i a_{ik}$ for all $k \geqslant 2$ implies $\lambda_i = 0$ for all i. Then the condition $1 = \sum_{i=2}^{g-1} \lambda_i a_{i1}$ gives a contradiction. □

LEMMA 5.7. *Suppose $1 \leqslant e \leqslant g-3$ and $D := t_1 + \cdots + t_{e+1}$ is a reduced divisor such that $h^0(D) = 1$ and $\langle D \rangle \cap C = D$. Put $E_i := D - t_i$. Then*

$$\bigcap_{i=1}^{e+1} \langle H^1(T_C), S^2 V(E_i) \rangle = H^1(T_C).$$

Proof. We proceed by induction on e. For $e=1$, we have $E_i = t_{3-i}$,

$$S^2V(E_i) \subset H^1(T_C),$$

and the result is trivially true. Suppose $e \geqslant 2$ and the result holds for $e-1$. Let us rewrite

$$\bigcap_{i=1}^{e+1} \langle H^1(T_C), S^2V(E_i)\rangle = \bigcap_{i=1}^{e} (\langle H^1(T_C), S^2V(E_i)\rangle \cap \langle H^1(T_C), S^2V(E_{e+1})\rangle).$$

We will prove that

$$\langle H^1(T_C), S^2V(E_i)\rangle \cap \langle H^1(T_C), S^2V(E_{e+1})\rangle = \langle H^1(T_C), S^2V(E_i')\rangle$$

where $E_i' := D - t_i - t_{e+1} = E_{e+1} - t_i$. Then replacing D with E_{e+1} we are reduced to the statement for $e-1$.

Dually, we will prove that the annihilators in $S^2H^0(\omega_C)$ of the two spaces are equal. The annihilator of $H^1(T_C)$ is $I_2(C)$. That of $\langle H^1(T_C),\ S^2V(E)\rangle$ for any divisor E is the space $I_2(C,E)$ of homogeneous degree 2 forms vanishing on C and the linear span $\langle E\rangle$ of E in $|\omega_C|^*$. The statement we need to prove has now become

$$I_2(C, D - t_i - t_{e+1}) = I_2(C, D - t_i) + I_2(C, D - t_{e+1}).$$

Choose $g-3-e$ general points $t_{e+2}, \dots, t_{g-2}$ on C. Then

$$\left\langle \sum_{i=1}^{g-2} t_i \right\rangle \cap C = \sum_{i=1}^{g-2} t_i$$

and the t_i are linearly independent and distinct. In particular, the t_i impose independent conditions on quadrics.

We claim that the restriction map

$$I_2(C) \longrightarrow S^2V\left(\sum_{i=1}^{g-2} t_i\right)^*$$

induces an isomorphism between $I_2(C)$ and the homogeneous degree 2 forms on $V(\sum_{i=1}^{g-2} t_i)$ vanishing at the points t_i. These two spaces have the same dimension so it is sufficient to prove that the restriction map is injective, i.e., no quadric in $|\omega_C|^*$ containing the canonical curve C contains $\langle \sum_{i=1}^{g-2} t_i\rangle$. Since $\langle \sum_{i=1}^{g-2} t_i\rangle$ has codimension 2, if a quadric contains it, then the quadric has rank $\leqslant 4$. Then $\langle \sum_{i=1}^{g-2} t_i\rangle$ is a member of a ruling of the quadric and by [1] cuts a divisor of a g^1_d on C. This, however, is not possible by our assumptions on the t_i.

It first follows from this that

$$\dim I_2(C, D - t_{e+1}) = \dim I_2(C, D - t_i) = \dim I_2(C) - \binom{e}{2} = \binom{g-2}{2} - \binom{e}{2},$$
$$\dim I_2(C, D - t_{e+1} - t_i) = \dim I_2(C) - \binom{e-1}{2} = \binom{g-2}{2} - \binom{e-1}{2}.$$

So to prove our claim we need to prove that

$$\begin{aligned}\dim (I_2(C, D - t_{e+1}) \cap I_2(C, D - t_i)) &= 2\left(\binom{g-2}{2} - \binom{e}{2}\right) - \\ &\quad - \left(\binom{g-2}{2} - \binom{e-1}{2}\right) \\ &= \binom{g-2}{2} - \binom{e}{2} - (e-1).\end{aligned}$$

This is easily seen to be true from our assumptions on the t_i. □

5.5. Assume $X_{e'}(L)$ irreducible for every $e' \leqslant e$, $3 \leqslant e \leqslant g-3$ and

- if $d = 2e$, then $h^0(L) \leqslant e-1$,
- if $d = 2e+1$, then $h^0(L) \leqslant e$.

So far it follows from our results above that the intersection of the kernels of the maps

$$S^2 H^1(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_{Z_{g-1}}(\Theta)) \longrightarrow H^0(\mathcal{O}_{Z_{g-1} \cap C_E^{(g-1-e)}}(\Theta))$$

as E varies in X is $H^1(T_C)$. Therefore (see 5.1) for a given $\eta \notin H^1(T_C)$, there exists $a = \sum p_i - \sum q_i$ such that η is *not* in the kernel of the map

$$S^2 H^1(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_{Z_{g-1}}(\Theta)) \longrightarrow H^0(\mathcal{O}_{Z_{g-1} \cap X_{-a}}(\Theta)).$$

6. The Kernel of the Map $H^0(\mathcal{O}_{Z_{g-1} \cap X_{-a}}(\Theta)) \to H^1(\mathcal{I}_{Z_{g-1} \cap X_{-a}}(\Theta))$

We continue with the analysis of the kernel of the coboundary map

$$H^0(\mathcal{O}_{Z_{g-1} \cap X_{-a}}(\Theta)) \longrightarrow H^1(\mathcal{I}_{Z_{g-1} \cap X_{-a}}(\Theta))$$

and see that in fact we can choose our a above also in such a way that this map is injective.

6.1. Let $\tilde{Z}(X) \subset C^{(g-1-e)} \times X$ be the closure of the subvariety parametrizing pairs $(\sum q_i, D_e)$ such that $h^0(\sum q_i) = 1$ and $h^0(\sum q_i + D_e) \geqslant 2$. Let $Z(X) \subset C^{(g-1-e)}$ be the image of $\tilde{Z}(X)$ by the first projection.

LEMMA 6.1. (1) *The varieties $\tilde{Z}(X)$ and $Z(X)$ are not empty.*

(2) *Suppose that C is nonbielliptic and nontrigonal of genus $\geqslant 7$ or nontrigonal of genus 6. Then for any $D_e \in X$ and $\sum q_i + D_e$ general in any irreducible component of $C_{D_e}^{(g-1-e)} \cap Z_{g-1}$, we have $h^0\left(\sum q_i\right) = 1$. In other words $Z(X)$ contains all the divisors $\sum q_i \in C^{(g-1-e)}$ such that $h^0\left(\sum q_i + D_e\right) \geqslant 2$ for some $D_e \in X$.*

Proof. If $h^0(D_e) \geqslant 2$, then $C_{D_e}^{(g-1-e)} \cap Z_{g-1} = C_{D_e}^{(g-1-e)}$ contains divisors $\sum q_i$ such that $h^0\left(\sum q_i\right) = 1$.

Suppose now that $h^0(D_e) = 1$.

We prove the second part first. If, for all $\sum q_i \in C^{(g-1-e)}$ with $h^0\left(\sum q_i + D_e\right) \geqslant 2$, we have $h^0\left(\sum q_i\right) \geqslant 2$, then, for some $r \geqslant 1$, the dimension of W^r_{g-1-e} is at least $g-1-e-r-2 = g-e-r-3$. By [12, pp. 348–350], this can only be the case if $r = 1$ and C is either trigonal, bielliptic or isomorphic to a smooth plane quintic.

In the case of a plane quintic or a bielliptic curve of genus 6, we have $g-1-e \geqslant 4 = g-2$. So $e \leqslant 1$ which is excluded.

For C trigonal of genus $\geqslant 6$ or bielliptic of genus $\geqslant 7$ we now prove that $\tilde{Z}(X)$ and $Z(X)$ are not empty.

In the trigonal case $W^1_{g-1-e} = g^1_3 + C^{(g-e-4)}$. For a point t of D_e, the divisor $\sum q_i = g^1_3 - t + D_{g-3-e}$ with $D_{g-3-e} \in C^{(g-3-e)}$ general satisfies $h^0(\sum q_i) = 1$ and $h^0(\sum q_i + D_e) \geqslant 2$.

In the bielliptic case, if $\pi : C \to E$ is the bielliptic cover, then $W^1_{g-1-e} = \pi^* W^1_2(E) + C^{(g-e-5)}$. For two distinct points s and t of D_e, the divisor $\iota s + \iota t + D_{g-3-e}$ with $D_{g-3-e} \in C^{(g-1-3)}$ general and ι the bielliptic involution satisfies $h^0(\sum q_i) = 1$ and $h^0(\sum q_i + D_e) \geqslant 2$. □

Two immediate consequences of the following lemma are that all irreducible components of $Z(X)$ have dimension at least $g-e-2$ (see Section 6.2 below) and that for a general choice of $\sum q_i \in Z(X)$, the curve X_{-a} is *not* contained in Z_{g-1}.

LEMMA 6.2. *The projection $\tilde{Z}(X) \to Z(X)$ is generically finite.*

Proof. If not, then some component of $\tilde{Z}(X)$ maps with one-dimensional fibers into $Z(X)$. Since X is integral, these one-dimensional fibers are all isomorphic to X. Hence, there is a $(g-e-3)$-dimensional family of divisors $q = \sum q_i$ such that $h^0(q) = 1$ and, for every $D_e \in X, h^0(q + D_e) \geqslant 2$. Let e' be the largest integer such that for a general such q and a general $D_{e'} \in X_{e'}(L)$, we have $h^0(q + D_{e'}) = 1$. It is immediate that $1 \leqslant e' \leqslant e-1$. For a fixed general such $D_{e'}$, let D be the divisor of L containing $D_{e'}$ and let t be a point of $D - D_{e'}$. Then $h^0(q + D_{e'} + t) = 2$. Equivalently $h^0(K - q - D_{e'} - t) = h^0(K - q - D_{e'})$, i.e., t is a base point of the linear system $|K - q - D_{e'}|$. Since $D_{e'}$ is general, so is D, hence D is reduced and, furthermore, it has no points in common with q. It follows that all of $D - D_{e'}$ is contained in the base locus of $|K - q - D_{e'}|$. By Riemann–Roch and Serre duality we see that this implies $h^0(q + D) = 1 + d - e'$. So C has a family of dimension $g-e-3$ of linear systems of degree $g-1-e+d$ and dimension $d-e'$. Since C is not hyperelliptic, it follows from [12, pp. 348–350] that $g-e-3\leqslant g-1-e+d-2(d-e')-1$, i.e., $d\leqslant 2e'+1\leqslant 2e-1$ which contradicts the hypothesis $d\geqslant 2e$. □

LEMMA 6.3. *Suppose that for* $\sum q_i\in Z(X)$ *with* $\sum h^0(\sum q_i)=1$ *the coboundary map*

$$H^0(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta))\longrightarrow H^1(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta))$$

is not injective, then

$$H^0\left(K-\sum q_i-L\right)\neq 0.$$

Proof. Using the exact sequence

$$0\longrightarrow H^0(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta))\longrightarrow H^0(\mathcal{O}_{X_{-a}}(\Theta))\longrightarrow H^0(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta))\\ \longrightarrow H^1(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta)),$$

we need to understand the sections of $\mathcal{O}_{X_{-a}}(\Theta)\cong\mathcal{O}_X(\psi_q^*\Theta)$ (see Section 1.1) which vanish on $Z_{g-1}\cap X_{-a}$. For this, we use the embedding of X in $C^{(e)}$:

$$0\longrightarrow\mathcal{I}_{X/C^{(e)}}(\psi_q^*\Theta)\longrightarrow\mathcal{O}_{C^{(e)}}(\psi_q^*\Theta)\longrightarrow\mathcal{O}_X(\psi_q^*\Theta)\longrightarrow 0.$$

By Appendix 6.1 in [5] this gives the exact sequence of cohomology

$$0\longrightarrow H^0\left(\mathcal{I}_{X/C^{(e)}}(\psi_q^*\Theta)\right)\longrightarrow\wedge^e H^0\left(C,K-\sum_{i=1}^{g-e-1}q_i\right)\longrightarrow H^0(X,(\psi_q^*\Theta))\\ \longrightarrow H^1(\mathcal{I}_{X/C^{(e)}}(\psi_q^*\Theta)).$$

By Appendix 6.2 in [5] the elements of $H^0(C^{(e)},\psi_q^*\Theta)=\wedge^e H^0(C,K-\sum_{i=1}^{g-e-1}q_i)$ all vanish on $\psi_q^*Z_{g-1}$, hence they also vanish on $\psi_q^*Z_{g-1}\cap X$. So if the coboundary map is not injective, then there must be elements of $H^0(X,\psi_q^*\Theta)$ which are not restrictions of elements of $H^0(C^{(e)},\psi_q^*\Theta)$. In particular, we must have $H^1(\mathcal{I}_{X/C^{(e)}}(\psi_q^*\Theta))\neq 0$. From sequence (1.2) tensored with $\mathcal{O}_{C^{(e)}}(\psi_q^*\Theta)$ we can write the following short exact sequences

$$0\longrightarrow\Lambda^e V_L^{e*}(\psi_q^*\Theta)\otimes S^{e-2}W(L)\longrightarrow\Lambda^{e-1}V_L^{e*}(\psi_q^*\Theta)\otimes S^{e-3}W(L)\longrightarrow N_{e-2}\longrightarrow 0$$

$$\vdots$$

$$0\longrightarrow N_3\longrightarrow\Lambda^3 V_L^{e*}(\psi_q^*\Theta)\otimes W(L)\longrightarrow N_2\longrightarrow 0$$

$$0\longrightarrow N_2\longrightarrow\Lambda^2 V_L^{e*}(\psi_q^*\Theta)\longrightarrow\mathcal{I}_{X/C^{(e)}}(\psi_q^*\Theta)\longrightarrow 0$$

where the $N_i, i=2,\dots,e-2$ are defined by the exactness of the sequences. From the exact sequences of cohomologies associated to the above short exact sequences it easily follows that $H^1(\mathcal{I}_{X/C^{(e)}}(\psi_q^*\Theta))\neq 0$ implies that there is an integer $j\in$

$\{2, \ldots, e\}$ such that $H^{j-1}(\Lambda^j V_L^{e*}(\psi_q^*\Theta)) \neq 0$. Equivalently, $H^{e-j+1}(\omega_{C^{(e)}}(-\psi_q^*\Theta) \otimes \Lambda^j V_L^e \neq 0)$. By Appendix 8.1, since ([5], Appendix)

$$\pi_e^* \mathcal{O}_{C^{(e)}}(\psi_q^*\Theta) \cong pr_1^* \mathcal{O}_C(K-q) \otimes \cdots \otimes pr_e^* \mathcal{O}_C(K-q)\Big(-\sum_{1 \leqslant k<l<e} \Delta_{k,l}\Big),$$

this implies that

$$H^{e-j+1}\big(pr_1^*\mathcal{O}_C(L+q) \otimes \cdots \otimes pr_j^*\mathcal{O}_C(L+q) \otimes pr_{j+1}^*\mathcal{O}_C(q) \otimes \ldots pr_e^*\mathcal{O}_C(q) \Big(-\sum_{1 \leqslant k<l \leqslant j} \Delta_{k,l}\Big)\big)^{\mathfrak{S}_j \times \mathfrak{S}_{e-j}} \neq 0$$

As in the Appendix of [5] the above cohomology group is equal to the group of elements of

$$H^{e-j+1}(pr_1^*\mathcal{O}_C(L+q) \otimes \cdots \otimes pr_j^*\mathcal{O}_C(L+q) \otimes pr_{j+1}^*\mathcal{O}_C(q) \otimes \ldots pr_e^*\mathcal{O}_C(q))$$

anti-invariant under the action of $\mathfrak{S}_j$ and invariant under the action of $\mathfrak{S}_{e-j}$. Therefore its nonvanishing implies the nonvanishing of $H^1(L+q) = H^1(L+\sum q_i) = H^0(K - \sum q_i - L)^*$. □

COROLLARY 6.4. *Let Z_0 be an irreducible component of $Z(X)$. Assume $3 \leqslant e \leqslant g-3$ and C nonbielliptic and nontrigonal of genus $\geqslant 7$ or nontrigonal of genus 6. If, for all $\sum q_i \in Z_0$, the coboundary map*

$$H^0(\mathcal{O}_{Z_{g-1} \cap X_{-a}}(\Theta)) \longrightarrow H^1(\mathcal{I}_{Z_{g-1} \cap X_{-a}}(\Theta))$$

is not injective, then

- *either $d = 2e$ and $h^0(L) = e$,*
- *or $d = 2e+1$ and $h^0(L) = e+1$.*

Proof. By Lemma 6.3, $h^0(K - L - \sum q_i) > 0$ for all $\sum q_i \in Z_0$. Every component of $\tilde{Z}(X)$ has dimension $\geqslant g-e-2$, hence, by Lemma 6.2, so does every irreducible component of $Z(X)$. Therefore Z_0 has dimension $\geqslant g-e-2$. Hence $h^0(K - L - \sum q_i) > 0$ for a $(g-e-2)$-dimensional family of divisors $\sum_{i=1}^{g-e-1} q_i$. This implies $h^0(K-L) \geqslant g-e-1$. Or, by Riemann–Roch and Serre Duality, $h^0(L) \geqslant d-e$. Therefore, by Clifford's Theorem, since C is not hyperelliptic, we have $2(d-e-1) < d$ or $d \leqslant 2e+1$. Recall that we have also assumed $d \geqslant 2e$ (see Section 1.1). Using Clifford's Theorem again we see that if $d = 2e$, then $h^0(L) = e$ and if $d = 2e+1$, then $h^0(L) = e+1$. □

6.2. SUMMARY: THE PROOF OF THEOREM 1

With the assumptions of Theorem 1 suppose we are given

$$\eta \in \mathrm{Kernel}\{\nu_e : S^2 H^1(\mathcal{O}_C) \longrightarrow H^1(N_{W_e/A}|_X)\}.$$

Then, as we saw at the end of Section 3 and in Section 4, for every a such that $\Theta_a \supset W_e$ or, equivalently, $\Theta_a \supset W_e - a$, we have

$$\eta \in \mathrm{Kernel}\{S^2H^1(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta)) \longrightarrow H^1(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta))\}. \qquad (6.1)$$

Assume that if $d=2e$, then $h^0(L) \leqslant e-1$ and if $d=2e+1$, then $h^0(L) \leqslant e$. Then, by the results of Section 6, for $\sum q_i$ general in any component of $Z(X)$ the coboundary map

$$H^0(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta)) \longrightarrow H^1(\mathcal{I}_{Z_{g-1}\cap X_{-a}}(\Theta))$$

is injective. For such $\sum q_i$, we must then have

$$\eta \in \mathrm{Kernel}\{S^2H^1(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_{Z_{g-1}\cap X_{-a}}(\Theta))\}.$$

Then, by the results of Section 5,

$$\eta \in H^1(T_C).$$

7. The Consequences of Theorem 1

By Theorem 1 and Appendices 8.2 and 8.3 below, if, for $3 \leqslant e \leqslant g-3$ and a given pair (C, L), the corresponding curve $X_e(L)$ deforms with A out of the Jacobian locus, then one of the following holds

(1) $\acute{e} \leqslant e$ such that $X_{e'}(L)$ is reducible,
(2) $\acute{e} \leqslant e$ such that $X_{e'}(L)$ is non-reduced,
(3) $X \cap Z_e \neq \emptyset$ and there exists a codimension $\leqslant 1$ subvariety $T \subset C^{(g-1-e)}$ such that for $\sum q_i \in T$, $X_{-a} \cap Z_q$ and $X_{-a} \cap Z_{g-1}$ do not have the same length.
(4) C is bielliptic of genus $g \geqslant 7$,
(5) C is trigonal of genus $g \geqslant 6$,
(6) C is nonbielliptic and nontrigonal of genus $\geqslant 7$ or nontrigonal of genus 6, $d = 2e$ and $h^0(L) = e$,
(7) C is nonbielliptic and nontrigonal of genus $\geqslant 7$ or nontrigonal of genus 6, $d = 2e+1$ and $h^0(L) = e+1$.

If C has a g^{e-1}_{2e} (resp. g^e_{2e+1}), then C has Clifford index 2 (resp. 1). By a result of Martens ([10, Satz 4, p. 80]), if C is nonbielliptic, has Clifford index 2 (resp. 1) and genus at least 10 (resp. 8), then C has no g^{e-1}_{2e} for $4 \leqslant e \leqslant g-5$ (resp. no g^e_{2e+1} for $3 \leqslant e \leqslant g-5$). The cases of low genus are easily analyzed and we see that cases (6) and (7) above reduce to

(1) C has a g^1_4 and hence also a $g^{g-4}_{2g-6} = |K_C - g^1_4|$, $e = g-3$,
(2) C has a g^2_6, $e = 3$ or $g-4$,

8. Appendix

8.1. THE COHOMOLOGY OF $\mathcal{I}_{X/C^{(e)}} \otimes N$

Here we introduce a method for computing the cohomology of $\mathcal{I}_{X/C^{(e)}} \otimes N$ where N is a locally free sheaf on $C^{(e)}$. One way to approach this calculation is to compute the cohomologies of the pieces $\Lambda^j V_L^{e*} \otimes S^{j-2} W(L) \otimes N$ of the resolution (1.2) of $\mathcal{I}_{X/C^{(e)}} \otimes N$. Or equivalently, the cohomologies of the sheaves $\omega_{C^{(e)}} \otimes N^* \otimes \Lambda^j V_L^e$. Recall that

$$V_L^e = q_{e*}\left(p_e^* \mathcal{O}_C(L)\big|_{D^e}\right),$$

where $D^e \subset C^{(e)} \times C$ is the universal divisor and q_e, p_e are the first and second projections of $C^{(e)} \times C$ onto its two factors. On this model, for $1 \leqslant j \leqslant e$, let

$$Y^{e,j} \subset C^{(e)} \times C^{(j)}$$

be the universal subvariety, i.e.,

$$Y^{e,j} := \left\{ (D_e, D_j) \in C^{(e)} \times C^{(j)} : D_e \geqslant D_j \right\},$$

and let $q_{e,j}$ and $p_{e,j}$ be the first and second projections of $C^{(e)} \times C^{(j)}$ onto its two factors. Then a moment of reflexion will convince the reader that

$$\Lambda^j V_L^e = q_{e,j_*}\left(\left(p_{e,j}^* \mathcal{L}'_{L,j}\right)\Big|_{Y^{e,j}}\right),$$

where, as in [5], $\mathcal{L}'_{L,j}$ is the sheaf on $C^{(j)}$ whose inverse image on C^j is

$$pr_1^* \mathcal{O}_C(L) \otimes \ldots pr_j^* \mathcal{O}_C(L) \left(- \sum_{1 \leqslant k < l \leqslant j} \Delta_{k,l} \right).$$

So

$$H^k(\omega_{C^{(e)}} \otimes N^* \otimes \Lambda^j V_L^e) = H^k\left(\omega_{C^{(e)}} \otimes N^* \otimes q_{e,j_*}\left(\left(p_{e,j}^* \mathcal{L}'_{L,j}\right)|_{Y^{e,j}}\right)\right)$$

and since $q_{e,j}|_{Y^{e,j}} : Y^{e,j} \to C^{(e)}$ is finite, we have

$$H^k(\omega_{C^{(e)}} \otimes N^* \otimes q_{e,j_*}\left(\left(p_{e,j}^* \mathcal{L}'_{L,j}\right)|_{Y^{e,j}}\right) = H^k\left(q_{e,j}^*\left(\omega_{C^{(e)}} \otimes N^*\right) \otimes p_{e,j}^* \mathcal{L}'_{L,j}|_{Y^{e,j}}\right).$$

The morphism

$$\begin{array}{ccc} C^e & \twoheadrightarrow & Y^{e,j} \\ (s_1, \ldots, s_e) & \mapsto & (s_1 + \cdots + s_e, s_1 + \cdots + s_j) \end{array}$$

shows that $Y^{e,j}$ is the quotient of C^e by the action of $\mathfrak{S}_j \times \mathfrak{S}_{e-j}$ which permutes the first j points and the last $e-j$ points. Therefore

$$\begin{aligned}
&H^k(q_{e,j}^*(\omega_{C^{(e)}} \otimes N^*) \otimes p_{e,j}^* \mathcal{L}'_{L,j}|_{Y^{e,j}}) \\
&= H^k(\pi_{e,j}^*(q_{e,j}^*(\omega_{C^{(e)}} \otimes N^*) \otimes p_{e,j}^* \mathcal{L}'_{L,j}|_{Y^{e,j}}))^{\mathfrak{S}_j \times \mathfrak{S}_{e-j}} \\
&= H^k \Bigg(pr_1^* \mathcal{O}_C(K+L) \otimes \cdots \otimes pr_j^* \mathcal{O}_C(K+L) \\
&\otimes pr_{j+1}^* \mathcal{O}_C(K) \otimes \cdots \otimes pr_e^* \mathcal{O}_C(K) \otimes \pi_e^* N^* \\
&\quad \Bigg(- \sum_{1 \leqslant k < l \leqslant j} \Delta_{k,l} - \sum_{1 \leqslant k < l \leqslant e} \Delta_{k,l} \Bigg)\Bigg)^{\mathfrak{S}_j \times \mathfrak{S}_{e-j}}.
\end{aligned}$$

8.2. Here we make some observations about the equality $X_{-a} \cap Z_q = X_{-a} \cap Z_{g-1}$ which we need for Theorem 1.

As we noted in Section 3.4, the scheme Z_q is contained in $Z_{g-1} \cap C^{(e)}_{\sum q_i}$ and

$$Z_q \backslash Z_e = Z_{g-1} \cap C^{(e)}_{\sum q_i} \backslash Z_e$$

as schemes. In particular, $X_{-a} \cap Z_q = X_{-a} \cap Z_{g-1}$ if and only if these two finite schemes have the same length. As in the proof of Lemma 6.1, we see that for C nontrigonal and nonbielliptic of genus $\geqslant 7$ or nontrigonal of genus 6, the dimension of W_e^1 $(3 \leqslant e \leqslant g-3)$ is at most $e-1-3$, meaning that the dimension of Z_e is at most $e-3$. Recall that the dimension of every irreducible component of $Z_{g-1} \cap C^{(e)}_{\sum q_i}$ is at least $e-2$. Therefore the equality $Z_q \backslash Z_e = Z_{g-1} \cap C^{(e)}_{\sum q_i} \backslash Z_e$ and the inclusion $Z_e \subset Z_{g-1} \cap C^{(e)}_{\sum q_i}$ imply that $Z_e \subset Z_q$ as sets. So, letting $(Z_e)_{\mathrm{red}}$ denote the reduced support of Z_e, the equality $X_{-a} \cap (Z_e)_{\mathrm{red}} = X_{-a} \cap Z_{g-1}$ implies $X_{-a} \cap Z_q = X_{-a} \cap Z_{g-1}$.

8.3. The purpose of this section is to show that when $e \leqslant g-3$, for a sufficiently general pair (C, L) the curve X obtained satisfies the condition

$$X_{-a} \cap Z_{g-1} = X_{-a} \cap Z_q$$

for all $\sum q_i \in C^{(g-1-e)}$. Since the schemes $Z_{g-1} \cap (W_e - a)$ and Z_q are equal outside Z_e, this will follow if we show that $X \cap Z_e = \emptyset$.

So we shall prove that for (C, L) sufficiently general, there is no subdivisor D of degree e of a divisor of L such that $h^0(D) \geqslant 2$. If $d \leqslant g+1$, then for a sufficiently general (C, L), $h^0(L) = 2$ and since we can suppose L base-point-free, the assertion is true.

Suppose therefore that $d \geqslant g+2$. In such a case, supposing (C, L) general also means that C is general, since a general curve has g_d^1's. If $e < (g+2)/2$, then by

Brill–Noether theory a general curve does not have any g^1_e and the assertion is true.

So we suppose $e \geqslant (g+2)/2$ (which then implies $d \geqslant g+2$). In this case we prove that in a general linear system G of degree $d \geqslant g+2$ on the general curve C, the family M_G of divisors of the form $D_e + D_{d-e}$ with $h^0(D_e) \geqslant 2$ has codimension at least 2. Then a general pencil L in G will not intersect M_G. It suffices to show that the union $M := \cup_{\deg(G)=d} M_G$ has dimension at most $d-2$ since $\cup_{\deg(G)=d} G = C^{(d)}$ has dimension d. We can rewrite $M = C_1^{(e)} + C^{(d-e)} := \cup_{D \in C_1^{(e)}} C_D^{(d-e)}$. Since C is general, by Brill–Noether theory

$$\dim W^1_e = g - 2(g-e+1) = 2e - g - 2.$$

Hence $\dim C_1^{(e)} = 2e - g - 1$ and $\dim M = 2e - g - 1 + d - e = d + e - g - 1$. We have $e - g - 1 \leqslant -2$ which concludes our proof.

Acknowledgement

I wish to thank the referee for many useful suggestions and comments.

References

1. Andreotti, A. and Mayer, A.: On period relations for Abelian integrals on algebraic curves, *Ann. Scuola Norm. Sup. Pisa* **21**(3) (1967), 189–238.
2. Arbarello, E., Cornalba, M., Griffiths, P. A. and Harris, J.: *Geometry of Algebraic Curves,* vol. 1, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985.
3. Eagon, J. A. and Northcott, D. G.: Ideals defined by matrices and a certain complex associated with them, *Proc. Roy. Soc.* **269** (1962), 188–204.
4. Green, M. L.: Quadrics of rank four in the ideal of a canonical curve, *Invent. Math.* **75**(1) (1984), 85–104.
5. Izadi, E.: Deforming curves in Jacobians to non-Jacobians I, http://xxx.lanl.gov/math.AG/0103204, to appear.
6. Izadi, E.: Deforming curves in Jacobians to non-Jacobians III, in preparation.
7. Izadi, E.: Subvarieties of Abelian varieties, *Applications of Algebraic Geometry to Coding Theory, Physics and Computation (Eilat, 2001)*, NATO Sci. Ser. II Math. Phys. Chem., 36, Kluwer Acad. Publ., Dordrecht, 2001, pp. 207–214.
8. Izadi, E. and van Straten, D.: The intermediate jacobians of the theta-divisors of four-dimensional principally polarized Abelian varieties, *J. Algebraic Geom.* **4**(3) (1995), 557–590.
9. Kollàr, J.: *Rational Curves on Algebraic Varieties,* Ergeb. Math. Grenzgeb (3) 32, Springer-Verlag, Berlin, 1996.
10. Martens, G.: Funktionen von vorgegebener ordnung auf komplexen kurven, *J. Reine Angew. Math.* **320** (1980), 68–85.
11. Matsusaka, T.: On a characterization of a Jacobian variety, *Mem. Coll. Sci. Univ. Kyoto* **32** (1959), 1–19.
12. Mumford, D.: Prym varieties I, In: L.V. Ahlfors, I. Kra, B. Maskit, and L. Niremberg, (eds.), *Contributions to Analysis*, Academic Press, New York, 1974, pp. 325–350.

13. Recillas, S.: Jacobians of curves with a g^1_4 are Prym varieties of trigonal curves, *Bol. Soc. Math. Mexicana* **19** (1974), 9–13.
14. Smith, R. and Varley, R.: Deformations of theta divisors and the rank 4 quadrics problem, *Compositio Math.* **76**(3) (1990), 367–398.