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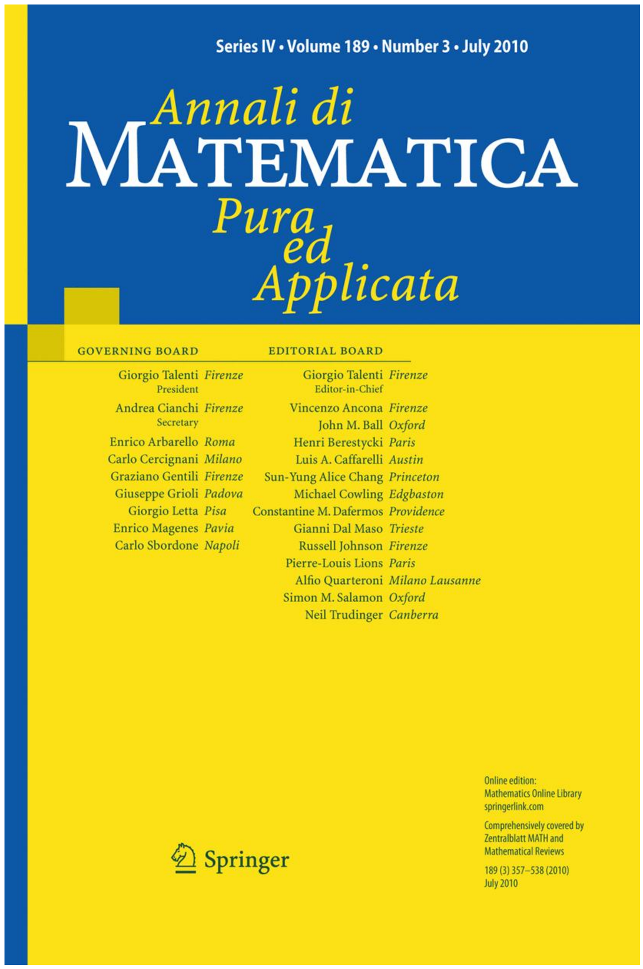

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Some remarks on the Hodge conjecture for abelian varieties

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Abstract We show how the classical Hodge conjecture for the middle cohomology of an abelian variety is equivalent to the general Hodge conjecture for the middle cohomology of a smooth ample divisor in the abelian variety. This is best suited to abelian varieties with actions of imaginary quadratic fields.

Keywords Abelian varieties · Hodge conjecture · Weil type · Imaginary quadratic field

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Introduction

Let X be a smooth complex projective variety of dimension g. A Hodge class of degree $2d$ on X is, by definition, an element of $H^{2d}(X, \mathbb{Q}) \cap H^{d,d}(X)$. The cohomology class of an algebraic subvariety of codimension d of X is a Hodge class of degree $2d$. The classical Hodge conjecture states that any Hodge class on X is algebraic, i.e., a $\mathbb{Q}$-linear combination of classes of algebraic subvarieties of X. Lefschetz' Theorem says that Hodge classes of degree 2 are always algebraic.

The classical Hodge conjecture is a special case of the general Hodge conjecture, corrected by Grothendieck to the form below (see Steenbrink [11], p. 166). To fix some notation, we will always designate a Hodge structure by its rational vector space V, the splitting $V \otimes \mathbb{C} = \oplus_{p+q=m} V^{p,q}$ being implicit. We say that V is effective if $V^{p,q} = 0$ when either

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p or q is negative. Recall that the level of a Hodge structure is the integer

$$\mathrm{Max}\{|p-q| : V^{p,q} \neq 0\}.$$

So to say that V has level l means that after a Tate twist, V will be an effective Hodge structure of weight l with non-vanishing $(l, 0)$ component.

Below m is a positive integer and p is a positive integer less than or equal to $\frac{m}{2}$.

Conjecture 1 *$GHC(X, m, p)$: For every $\mathbb{Q}$-Hodge substructure V of $H^m(X, \mathbb{Q})$ with level $\leq m - 2p$, there exists a subvariety Z of X of pure codimension p such that $V \subset Ker\{H^m(X, \mathbb{Q}) \to H^m(X \backslash Z, \mathbb{Q})\}$.*

Given V and Z as in the conjecture, we say that V is supported on Z or that Z supports V. Letting $\widetilde{Z} \to Z$ be a generically finite morphism from a non-singular variety $\widetilde{Z}$, we have the Gysin map $H^{m-2p}(\widetilde{Z}, \mathbb{Q}) \to H^m(X, \mathbb{Q})$ obtained by Poincaré Duality from push-forward on homology. The above conjecture is equivalent to (see e.g., [11], p. 166)

Conjecture 2 *$GHC(X, m, p)$: For every $\mathbb{Q}$-Hodge substructure V of $H^m(X, \mathbb{Q})$ with level $\leq m - 2p$, there exists a subvariety Z of X of pure codimension p such that V is contained in the image of the Gysin map $H^{m-2p}(\widetilde{Z}, \mathbb{Q}) \to H^m(X, \mathbb{Q})$.*

Henceforth we shall refer to the general Hodge conjecture simply as the Hodge conjecture. Let Y be a smooth ample divisor in X. The primitive part $K(Y, \mathbb{Q})$ of the cohomology of Y can be defined as the kernel of the Gysin map $H^{g-1}(Y, \mathbb{Q}) \longrightarrow H^{g+1}(X, \mathbb{Q})$.

Definition 1 We shall say that $GHC'(Y, p)$ holds if every Hodge substructure of level $\leq g - 1 - 2p = \dim X - 1 - 2p$ of $K(Y, \mathbb{Q})$ is supported on a pure codimension p subvariety of Y.

We prove

Theorem 1 *If $m \leq g - 2 = \dim X - 2$, then*

$$GHC(Y, m, p) \Longleftrightarrow GHC(X, m, p),$$

and,

$$GHC(Y, g-1, p) \Longleftrightarrow GHC(X, g-1, p) \quad and \quad GHC'(Y, p).$$

Now assume there is a smooth and complete curve C, a smooth variety Y' of dimension $g-1$ and a surjective morphism $C \times Y' \to X$ such that there exists $o \in C$ such that the image of $\{o\} \times Y'$ in X is Y. Our main examples of varieties with such a property are symmetric powers of curves and abelian varieties:

For a curve T of any genus, the symmetric power $T^{(g-1)}$ embeds as a smooth ample divisor in $X = T^{(g)}$ via addition of a point of T. We can then take $Y = Y' = T^{(g-1)}$ and $C = T$.

For an abelian variety $A = X$, we can take $Y = Y'$ to be any smooth ample divisor in A and C to be a smooth curve with a non-constant map $C \to A$ whose image generates A as a group. The map $C \times Y \to A$ is then induced by addition.

With the above assumptions on X, Y, Y' and C we have (compare to Theorem 2.2 in [11], p. 167):

Theorem 2 *Suppose $GHC(Y', g-1, p-1)$ (where $g = \dim X = \dim Y + 1 = \dim Y' + 1$) and $GHC(Y', g-2, p-1)$ hold and, for any subvariety Z of Y' of pure codimension $p-1$, $GHC(C \times \widetilde{Z}, g-2p+2, 1)$ holds, then $GHC(X, g, p)$ holds. Moreover, if $g = 2p+2$, then $GHC(X, g, p+1)$ (i.e., the usual Hodge conjecture for the middle cohomology of X) also holds.*

Since the Hodge conjecture is known for degree two classes, it is also known for Hodge classes that are polynomial combinations of degree 2 Hodge classes. Hodge classes which are *not* combinations of degree 2 classes are usually called *exceptional*.

Perhaps the first example of an abelian variety with an exceptional Hodge class was that of an abelian fourfold with complex multiplication given by Mumford [9, p. 166]. Weil [13] observed that the presence of the exceptional Hodge classes was not due to A being of CM-type but to the fact that the $\mathbb{Q}$-endomorphism ring of A contained an imaginary quadratic field K, stable under all Rosati involutions, whose action on the tangent space of A had eigenspaces of equal dimension, say n. Note that the dimension of the abelian variety is then $g = 2n$. These are now called abelian varieties of Weil type. Weil proved that $W^0 := \wedge_K^{2n} H^1(A, \mathbb{Q}) \subset H^{2n}(A, \mathbb{Q}) \cap H^{n,n}(A)$, i.e., consists of Hodge classes (see [13] and [1] Lemma 5.2 p. 238). The elements of W^0 are called Weil classes.

More generally, we make the following

Definition 2 An abelian variety A is of Weil type $k \geq 0$ if the action of $K \hookrightarrow End_0(A)$ is stable under all Rosati involutions and the eigenspaces of the action of K on $T_0 A$ have dimension n and $n + k$.

We observe

Lemma 1 *Suppose A is of Weil type k. For each $m \leq g = \dim A$, the exterior power $W^m := \wedge_K^{g-m} H^1(A, \mathbb{Q})$ over K is a Hodge substructure of $H^{g-m}(A, \mathbb{Q})$ of level $m + k$.*

So the cohomology of an abelian variety of Weil type k contains many extra Hodge substructures (when compared to a generic abelian variety).

Weil also showed that any abelian variety of Weil type 0 is a member of a family of dimension n^2 of such abelian varieties. We observe that one can similarly construct families of dimension $n(n + k)$ of abelian varieties of Weil type k. By a generic abelian variety of Weil type k we mean a generic member of such a family. Weil proved that Weil classes on generic abelian varieties of Weil type 0 are exceptional [13].

As before let Y be a smooth ample divisor on A. The spaces W^m are Hodge substructures of level $m + k$ of $H^{g-m}(A, \mathbb{Q})$ and also $H^{g-m}(Y, \mathbb{Q})$ via the embedding $H^{g-m}(A, \mathbb{Q}) \hookrightarrow H^{g-m}(Y, \mathbb{Q})$. We have the following

Theorem 3 *Suppose $m \leq n - 1 \left(= \frac{g}{2} - 1 = \frac{\dim A}{2} - 1\right)$.*

If the Hodge conjecture holds for $W^m \subset H^{g-m}(A, \mathbb{Q})$, then the Hodge conjecture holds for $W^{m+1} \subset H^{g-m-1}(A, \mathbb{Q})$ and also for $W^{m+1} \subset H^{g-m-1}(Y, \mathbb{Q})$.

If the Hodge conjecture is true for $W^{m+1} \subset H^{g-m-1}(Y, \mathbb{Q})$ and, for every subvariety Z of pure codimension $n - m - 1$ of Y, the conjecture $GHC(C \times \widetilde{Z}, m + k + 2, 1)$ holds, then the Hodge conjecture holds for $W^m \subset H^{g-m}(A, \mathbb{Q})$.

In particular, using Lefschetz' Theorem in the case $k = m = 0$, the Hodge conjecture for $W^0 \subset H^g(A, \mathbb{Q})$ is equivalent to that for $W^1 \subset H^{g-1}(A, \mathbb{Q})$ and to that for $W^1 \subset H^{g-1}(Y, \mathbb{Q})$.

For our next result, we use the following formulation of the Hodge conjecture.

Conjecture 3 *For every $\mathbb{Q}$-Hodge substructure V of $H^m(X, \mathbb{Q})$ with level $\leq m - 2p$, there exists a non-singular projective family of subvarieties of pure dimension $g - m + p = \dim X - m + p$ of X*

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & X \\ \downarrow^{r} & & \\ S & & \end{array}$$

whose base is a non-singular projective variety S of dimension $m-2p$ such that the image of $H^{m-2p}(S,\mathbb{Q})$ by the Abel–Jacobi map q_*r^* of the family contains V.

Conjecture 3 is a more geometric way of looking at the Hodge conjecture. For the convenience of the reader, we include a proof of the equivalence of this conjecture with Conjecture 2 at the end of Section 1.

We say that a variety D has an ordinary double locus of dimension $n-3$ when either $n \geq 3$ and, locally analytically, D is isomorphic to the product of a smooth variety of dimension $n-3$ and a threefold with an ordinary double point, or $n=2$ and D is smooth.

Theorem 4 *Suppose A is a general abelian variety of Weil type $k=0$ and suppose that the Hodge conjecture holds for $W^1 \subset H^{2n-1}(Y,\mathbb{Q})$. Then there is a one-parameter family of subvarieties of dimension $n-1$ of Y* $\left(\text{where } n=\frac{g}{2}=\frac{\dim A}{2}\right)$

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & Y \\ \downarrow & & \\ X & & \end{array}$$

whose Abel–Jacobi map is a solution to the Hodge conjecture for W^1 and satisfies the following property. For a general point $x \in X$, any $N \gg 0$, any choice of $n-1$ general divisors $D_1,\dots,D_{n-1}$ of the linear system $|\mathcal{I}_{Z_x}(NY)|_Y|$, the singularities of the intersection $D := D_1 \cap \dots \cap D_{n-1}$ consist exactly of an ordinary double locus of dimension $n-3$ and the homology class of Z_x in D is not a multiple of $[Y]_D$. Here, Z_x is the fiber of $\mathcal{Z}$ at x and $\mathcal{I}_{Z_x}$ is the ideal sheaf of Z_x in A.

Theorem 4 shows, for instance when $n=2$, that if the Hodge conjecture is true for $W^1 \subset H^3(Y,\mathbb{Q})$, then it has a solution given by a family of curves that can be embedded, fiber by fiber, in a family of smooth surfaces in Y such that the cohomology class of each curve in its surface is linearly independent from the restriction of the class of Y. So, to solve the Hodge conjecture for $W^1 \subset H^3(Y,\mathbb{Q})$ or, equivalently, for $W^0 \subset H^4(A,\mathbb{Q})$, we need to look for families of smooth surfaces in Y with Picard number >1 (i.e., points of Noether-Lefschetz loci). In higher dimensions, we would look for families of n-dimensional complete intersections in Y that contain Weil divisors homologically independent of the restriction of Y and whose singular loci are ordinary double loci of dimension $n-3$.

1 The proofs in the general case and some remarks

Proof of Theorem 1 Suppose $m \leq g-1$ and let us prove the implication $GHC(Y,m,p) \Rightarrow GHC(X,m,p)$. Let $V \subset H^m(X,\mathbb{Q})$ be a Hodge substructure of level $\leq m-2p$. Then, since by the weak Lefschetz theorem, $H^m(X,\mathbb{Q}) \hookrightarrow H^m(Y,\mathbb{Q})$, we can consider V to be a Hodge substructure of $H^m(Y,\mathbb{Q})$. So, by assumption, there is a subvariety Z of Y, of pure codimension p, such that $V \subset Ker\{H^m(Y,\mathbb{Q}) \to H^m(Y\backslash Z,\mathbb{Q})\}$. Let W be a pure codimension p subvariety of X containing Z such that $Y \cap W$ has pure codimension $p+1$ in X. Write $Y\backslash W$ for $Y\backslash(Y\cap W)$. Then, $H^m(Y,\mathbb{Q}) \to H^m(Y\backslash W,\mathbb{Q})$ factors through $H^m(Y,\mathbb{Q}) \to H^m(Y\backslash Z,\mathbb{Q})$ and hence its kernel contains V. Consider the following commutative diagram of pull-back maps on cohomology

$$\begin{array}{ccc} H^m(X,\mathbb{Q}) & \hookrightarrow & H^m(Y,\mathbb{Q}) \\ \downarrow & & \downarrow \\ H^m(X\backslash W,\mathbb{Q}) & \longrightarrow & H^m(Y\backslash W,\mathbb{Q}). \end{array}$$

□

Lemma 1.1 *The map* $H^m(X\backslash W, \mathbb{Q}) \to H^m(Y\backslash W, \mathbb{Q})$ *in the above diagram is injective.*

Proof In a projective embedding of X given by some multiple of Y, let H be a hyperplane cutting a multiple of Y on X. Then, by [5, Theorem p. 170] and, e.g., [4, Proposition A.5 p. 523], $Y\backslash W = H \cap (X\backslash W)$ (set-theoretic equality) is a deformation retract of a δ-neighborhood of $H \cap (X\backslash W)$ in $X\backslash W$. Hence the homotopy, homology and cohomology groups of $Y\backslash W = H \cap (X\backslash W)$ are isomorphic to those of its δ-neighborhood in $X\backslash W$. The lemma is then a consequence of [2, Theorem pp. 150–151] via the general Hurewicz Theorem [4, p. 371] and the Universal Coefficient Theorem [4, p. 201]. □

Therefore, $V \subset Ker\{H^m(X, \mathbb{Q}) \to H^m(X\backslash W, \mathbb{Q})\}$.

Now, suppose $m \leq g-2$ and let us prove the implication $GHC(Y, m, p) \Leftarrow GHC(X, m, p)$. In this case, by the weak Lefschetz theorem, the pull-back map $H^m(X, \mathbb{Q}) \to H^m(Y, \mathbb{Q})$ is an isomorphism. Let $V \subset H^m(Y, \mathbb{Q})$ be a Hodge substructure of level $\leq m-2p$. Let $W \subset X$ be a pure codimension p subvariety such that $V \subset Ker\{H^m(X, \mathbb{Q}) \to H^m(X\backslash W, \mathbb{Q})\}$. Then, a commutative diagram such as the above shows that $Z := Y \cap W$ has the property $V \subset Ker\{H^m(Y, \mathbb{Q}) \to H^m(Y\backslash Z, \mathbb{Q})\}$. If Z has pure codimension p in Y, then we are done. If not, let Y' be a smooth ample divisor in X such that $Y \cap Y'$ is smooth and Y' intersects W transversely. Then, every component of $Z' := Y \cap Y' \cap W$ has codimension at least p in Y. Replace the components of codimension $> p$ of Z' by irreducible subvarieties of codimension p containing them. Then, since $m \leq g-2$, as above we have the commutative diagram with top horizontal injective map

$$\begin{array}{ccc} H^m(X, \mathbb{Q}) & \hookrightarrow & H^m(Y \cap Y', \mathbb{Q}) \\ \downarrow & & \downarrow \\ H^m(X\backslash W, \mathbb{Q}) & \longrightarrow & H^m(Y \cap Y'\backslash Z', \mathbb{Q}) \end{array}$$

which shows

$$V \subset Ker\{H^m(Y \cap Y', \mathbb{Q}) \longrightarrow H^m(Y \cap Y'\backslash Z', \mathbb{Q}).$$

Now using the diagram

$$\begin{array}{ccc} H^m(Y, \mathbb{Q}) & \hookrightarrow & H^m(Y \cap Y', \mathbb{Q}) \\ \downarrow & & \downarrow \\ H^m(Y\backslash Z', \mathbb{Q}) & \hookrightarrow & H^m(Y \cap Y'\backslash Z', \mathbb{Q}) \end{array}$$

where the injectivity of the bottom horizontal map is proved as in Lemma 1.1, we obtain

$$V \subset Ker\{H^m(Y, \mathbb{Q}) \longrightarrow H^m(Y\backslash Z', \mathbb{Q})\}$$

and we are done.

Finally, the proof of the implication $GHC(Y, g-1, p) \Leftarrow (GHC(X, g-1, p)$ & $GHC'(Y, p))$ is similar to the above proofs.

Proof of Theorem 2 The pull-back

$$\begin{aligned} H^g(X, \mathbb{Q}) \longrightarrow H^g(C \times Y', \mathbb{Q}) &\cong H^0(C, \mathbb{Q}) \otimes_{\mathbb{Q}} H^g(Y', \mathbb{Q}) \\ &\oplus H^1(C, \mathbb{Q}) \otimes_{\mathbb{Q}} H^{g-1}(Y', \mathbb{Q}) \oplus H^2(C, \mathbb{Q}) \otimes H^{g-2}(Y', \mathbb{Q}) \end{aligned}$$

is injective. Using cup product to identify $H^g(X, \mathbb{Q})$ and $H^g(C \times Y', \mathbb{Q})$ with their duals, the transpose of pull-back is Gysin push-forward and is surjective. Let V be a Hodge substructure of level $\leq g-2p$ of $H^g(X, \mathbb{Q})$. Let V_0, V_1 and V_2 be the images of V by the compositions

$$V \longrightarrow H^g(X, \mathbb{Q}) \longrightarrow H^g(C \times Y', \mathbb{Q}) \longrightarrow H^0(C, \mathbb{Q}) \otimes_{\mathbb{Q}} H^g(Y', \mathbb{Q}),$$
$$V \longrightarrow H^g(X, \mathbb{Q}) \longrightarrow H^g(C \times Y', \mathbb{Q}) \longrightarrow H^1(C, \mathbb{Q}) \otimes_{\mathbb{Q}} H^{g-1}(Y', \mathbb{Q})$$

and

$$V \longrightarrow H^g(X,\mathbb{Q}) \longrightarrow H^g(C\times Y',\mathbb{Q}) \longrightarrow H^2(C,\mathbb{Q})\otimes H^{g-2}(Y',\mathbb{Q}),$$

respectively. If each of V_0, V_1 and V_2 is supported on a subvariety of codimension p of $C\times Y'$, then V is supported on the union of the images of these subvarieties in X and the Hodge conjecture will follow for V. First, consider

$$V_2 \subset H^2(C,\mathbb{Q})\otimes H^{g-2}(Y',\mathbb{Q}) \cong H^{g-2}(Y',\mathbb{Q}).$$

Since $g-2p = g-2-2(p-1)$, by assumption there is a subvariety Z_2 of pure codimension $p-1$ of Y' that supports V_2. Let t be a general point of C. Then $\{t\}\times Z_2 \subset C\times Y'$ has pure codimension p and supports V_2.

Next, consider

$$V_0 \subset H^0(C,\mathbb{Q})\otimes H^g(Y',\mathbb{Q}) \cong H^g(Y',\mathbb{Q}).$$

Choose an ample divisor D on Y'. Cup product with the class of D induces the isomorphism $\cup[D] : H^{g-2}(Y',\mathbb{Q}) \xrightarrow{\sim} H^g(Y',\mathbb{Q})$. Let Z_0' be a subvariety of pure codimension $p-1$ of Y' supporting the inverse image of V_0 under the isomorphism $\cup[D]$. For m large enough, a sufficiently general divisor $E \in |\mathcal{O}_{Y'}(mD)|$ is transverse to every element of the image of $H^{g-2p}(\widetilde{Z}_0',\mathbb{Q})$. Hence, $V_0 \subset H^g(Y',\mathbb{Q})$ is supported on $Z_0 := E\cap Z_0'$ that has pure codimension p in Y'. The map $H^{g-2p}(C\times\widetilde{Z}_0) \to H^g(C\times Y')$ preserves Künneth components hence $V_0 \subset H^g(C\times Y')$ is supported on $C\times Z_0$.

Finally, we consider

$$V_1 \subset H^1(C,\mathbb{Q})\otimes_{\mathbb{Q}} H^{g-1}(Y',\mathbb{Q}).$$

Using the intersection pairing to identify $H^1(C,\mathbb{Q})$ with its dual, we obtain the map

$$V_1\otimes H^1(C,\mathbb{Q}) \longrightarrow H^{g-1}(Y',\mathbb{Q})$$

whose image W is a Hodge substructure of level $\le g-2p+1 = g-1-2(p-1)$. So W is supported on a subvariety Z_1 of pure codimension $p-1$ of Y'. Then $V_1 \subset H^1(C,\mathbb{Q})\otimes W$ is supported on $C\times Z_1$. If $GHC(C\times\widetilde{Z}_1, g-2p+1, 1)$ holds, then V_1 is supported on a subvariety of pure codimension 1 of $C\times\widetilde{Z}_1$ whose image in $C\times Y'$ has pure codimension p. The image of this subvariety in X is the solution to the Hodge conjecture in this case.

The last assertion now follows from Lemma 2.1 on page 167 of [11]. □

Remark 1.2 Note that Theorem 2 is only interesting if $p\ge 2$. For $g=4$, the theorem is especially interesting since the hypothesis $GHC(C\times\widetilde{Z}, g-2p+2, 1)$ for all $Z\subset Y'$ of pure codimension $p-1$ is simply Lefschetz' Theorem for $(1,1)$-classes and is automatically true.

Remark 1.3 Not every variety can be dominated by a product of lower-dimensional varieties as shown by Chad Shoen in [10, Theorem 1.1].

Remark 1.4 For g even and $p=\frac{g}{2}$, Theorem 2 is reminiscent of the celebrated approach of Poincaré-Lefschetz-Griffiths to the classical Hodge conjecture via the theory of normal functions (see for instance [14], [15, Theorems 2.13 and 9.2] and [7, Chaps. 12 and 14]).

Proof of the equivalence of Conjectures 2 and 3 First, note that Conjecture 3 immediately implies Conjecture 2 as we can take Z to be the image of $\mathcal{Z}$.

To see the converse, let Z be as in Conjecture 2. Choose a very ample linear system L on $\widetilde{Z}$ and let $S' \subset \widetilde{Z}$ be the complete intersection of $g - m + p$ general elements of L. Then S' is smooth of dimension $m - 2p$ and the pull-back $H^{m-2p}(\widetilde{Z}, \mathbb{Q}) \to H^{m-2p}(S', \mathbb{Q})$ is injective. So we have embeddings $V \subset H^{m-2p}(\widetilde{Z}, \mathbb{Q})$ and $V \subset H^{m-2p}(S', \mathbb{Q})$. Using the polarization on $H^{m-2p}(S', \mathbb{Q})$, choose a projection $H^{m-2p}(S', \mathbb{Q}) \twoheadrightarrow V$ and compose this with the embedding $V \subset H^{m-2p}(\widetilde{Z}, \mathbb{Q})$ to obtain a strict morphism of Hodge structures $\varphi : H^{m-2p}(S', \mathbb{Q}) \to H^{m-2p}(\widetilde{Z}, \mathbb{Q})$ with image V. Again, using the polarization on $H^{m-2p}(S', \mathbb{Q})$, identify $H^{m-2p}(S', \mathbb{Q})$ with its dual Hodge structure $H^{m-2p}(S', \mathbb{Q})^*$, so that we can think of φ as an element of $H^{m-2p}(S', \mathbb{Q}) \otimes H^{m-2p}(\widetilde{Z}, \mathbb{Q})$. The Künneth isomorphism gives an embedding

$$H^{m-2p}(S', \mathbb{Q}) \otimes H^{m-2p}(\widetilde{Z}, \mathbb{Q}) \subset H^{2m-4p}(S' \times \widetilde{Z}, \mathbb{Q}).$$

Since φ is a strict morphism of Hodge structures (i.e., $\varphi(H^{p,q}(S')) \subset H^{p,q}(\widetilde{Z})$), φ is a Hodge class in $H^{2m-4p}(S \times \widetilde{Z}, \mathbb{Q})$. Hence, φ is the class of a cycle A of pure dimension $g - p$ on $S \times \widetilde{Z}$. Let Z' be the underlying variety of A. Now, standard arguments using resolutions of singularities show that there are birational morphisms $\mathcal{Z} \to Z'$, $S \to S'$ and a projection $r : \mathcal{Z} \to S$ that is a solution of Conjecture 3. □

2 The proofs for abelian varieties with an action of an imaginary quadratic field

We first note that the proof of Lemma 1 is entirely similar to the proof of Lemma 5.2 on page 238 of [1].

Proof of Theorem 3 We first assume that the Hodge conjecture holds for W^m and prove it for W^{m+1}. There is then an irreducible subvariety, say Z_0, of pure codimension $n - m$ of A such that W^m is contained in the image of Gysin push-forward

$$H^{m+k}(\widetilde{Z}_0, \mathbb{Q}) \longrightarrow H^{g-m}(A, \mathbb{Q}).$$

Let now C be a smooth irreducible curve with a non-constant map $C \to A$ such that the image of C generates A as a group. Then, the pull-back map $H^1(A, \mathbb{Q}) \to H^1(C, \mathbb{Q})$ is injective. The translates of Z_0 in A by points of C form the family of subvarieties

$$\begin{array}{ccc} C \times Z_0 & \xrightarrow{q} & A \\ \downarrow^{p} & & \\ C & & \end{array}$$

whose image in A we denote by Z_1. We choose $\widetilde{Z}_1$ to be $C \times \widetilde{Z}_0$. Pontrjagin product and a straightforward linear algebra computation now show that the image of

$$H^1(A, \mathbb{Q}) \otimes H^{m+k}(\widetilde{Z}_0, \mathbb{Q}) \subset H^1(C, \mathbb{Q}) \otimes H^{m+k}(\widetilde{Z}_0, \mathbb{Q}) \subset H^{m+k+1}(C \times \widetilde{Z}_0, \mathbb{Q})$$

by Gysin push-forward is a Hodge substructure of $H^{g-m-1}(A, \mathbb{Q})$ containing W^{m+1}. So, we obtain the Hodge conjecture for $W^{m+1} \subset H^{g-m-1}(A, \mathbb{Q})$. The Hodge structure $W^{m+1} \subset H^{g-m-1}(Y, \mathbb{Q})$ is now supported on $Z_1 \cap Y$ that is a subvariety of pure codimension $n - m - 1$ of Y after replacing Z_1 by a general translate if necessary to ensure that the intersection of Z_1 with Y is proper.

Assume now that the Hodge conjecture holds for $W^{m+1} \subset H^{g-m-1}(Y, \mathbb{Q})$ and let $Z \subset Y$ be a subvariety of pure codimension $m - n - 1$ supporting W^{m+1}. Consider the addition map $C \times Y \to A$. The Hodge structure $H^1(A, \mathbb{Q}) \otimes W^{m+1}$ is supported on the subvariety

$C \times Z$ of $C \times Y$. It is immediately seen that pull-back on cohomology from A induces an embedding

$$W^m \hookrightarrow H^1(A, \mathbb{Q}) \otimes W^{m+1} \hookrightarrow H^1(C, \mathbb{Q}) \otimes H^{m+k+1}(\widetilde{Z}, \mathbb{Q}).$$

Now, we use $GHC(C \times \widetilde{Z}, m+k+2, 1)$ to deduce that there is a codimension 1 subvariety Z' of $C \times \widetilde{Z}$ supporting W^m. The push-forward Z_0 to A of Z' will then support W^m. This proves the Hodge conjecture for $W^m \subset H^{g-m}(A, \mathbb{Q})$. □

Proof of Theorem 4 By Theorem 3, the Hodge conjecture also holds for W^0. Let Z_0 be a subvariety of A whose class is $\alpha + \lambda[Y]^n$ (for some $\alpha \in W^0$, $\alpha \neq 0$ and $\lambda \in \mathbb{Q}$) and let C be a smooth curve with a non-constant map $C \to A$ whose image generates A as a group. Then, as in the proof of Theorem 3, using Pontrjagin product and a straightforward linear algebra argument, one can see that the family $\{Z_x := (Z_0 + x) \cap Y : x \in C\}$ gives an answer to the Hodge conjecture for a summand of $[Y]^{n-1} \wedge H^1(A, \mathbb{Q}) \oplus W^1 \subset H^{2n-1}(Y, \mathbb{Q})$ that projects onto W^1.

By [6], we can assume Z_0 to be smooth. For N sufficiently large, the variety Z_0 is cut out scheme theoretically by divisors in the linear system $|\mathcal{I}_{Z_0}(NY)|$. In particular, any $n-1$ general divisors $E_1, \ldots, E_{n-1}$ in $|\mathcal{I}_{Z_0}(NY)|$ intersect properly where $\mathcal{I}_{Z_0}$ is the ideal sheaf of Z_0 in A. Furthermore, since $\mathcal{I}^2_{Z_0}$ is a coherent sheaf and Y an ample divisor, for N large enough (see [3] p. 229),

$$H^1\left(\mathcal{I}^2_{Z_0}(NY)\right) = 0,$$

hence the restriction map

$$H^0(\mathcal{I}_{Z_0}(NY)) \longrightarrow H^0\left(\frac{\mathcal{I}_{Z_0}}{\mathcal{I}^2_{Z_0}}(NY)\right)$$

is surjective so that any general n sections of $\mathcal{I}_{Z_0}(nY)$ map to n general sections of $\frac{\mathcal{I}_{Z_0}}{\mathcal{I}^2_{Z_0}}(nY)$. Let $s_1, \ldots, s_{n-1}$ be sections with divisors of zeros equal to $E_1, \ldots, E_{n-1}$, respectively. As in the proof of Theorem 4.2 in [12], by our hypotheses of genericity, the singular locus of the intersection $E_1 \cap \cdots \cap E_{n-1}$ is either empty or is an ordinary double locus and is the locus where the images of $s_1, \ldots, s_{n-1}$ in $H^0(\frac{\mathcal{I}_{Z_0}}{\mathcal{I}^2_{Z_0}}(NY))$ fail to be independent. Again, for sufficiently large N and because $\frac{\mathcal{I}_{Z_0}}{\mathcal{I}^2_{Z_0}}(NY)$ has rank n and the sections are general, this locus either is empty or has pure dimension $n-2$. The intersection of $E_1 \cap \cdots \cap E_{n-1}$ with a general translate of Y will therefore either have an ordinary double locus of dimension $n-3$ or be smooth. By [2] p. 199, the push-forward map

$$H_2(E_1 \cap \cdots \cap E_{n-1} \cap Y, \mathbb{Q}) \longrightarrow H_2(E_1 \cap \cdots \cap E_{n-1}, \mathbb{Q})$$

is surjective. From this, it follows that the homology class of $(Z_0+x) \cap Y$ in $E_1 \cap \cdots \cap E_{n-1} \cap Y$ is independent of the homology class of $Y|_{E_1 \cap \cdots \cap E_{n-1} \cap Y}$. The fact $H_2(E_1 \cap \cdots \cap E_{n-1} \cap Y, \mathbb{Q}) \neq H_2(A, \mathbb{Q})$ and the weak Lefschetz Theorem imply that $E_1 \cap \cdots \cap E_{n-1} \cap Y$ is *not* smooth if $n \geq 3$.

Now, choosing our curve C general and putting $D_i := E_i \cap Y$, the family will have the desired property for at least one choice of $D_1, \ldots, D_{n-1}$ and therefore for all sufficiently general choices of such divisors. □

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