1.5.1 a. The set $\{ x \in \mathbb{R} \mid 0 < x \leq 1 \}$ is neither open nor closed: the point 1 is in the set, but $1 + \epsilon$ is not for every ϵ, showing it isn't open, and 0 is not but $0 + \epsilon$ is for every $\epsilon > 0$, showing that the complement is also not open, so the set is not closed.

Similarly, one proves the follwoing:

b. open c. neither d. closed e. closed f. neither

g. both. That the empty set $\emptyset$ is closed is obvious. Showing that it is open is an "eleven-legged alligator" argument (see Section 0.2). Is it true that for every point of the empty set, there exists $\epsilon > 0$ such that . . . ? Yes, because there are no points in the empty set.

1.5.2 a. The (x, y)-plane in $\mathbb{R}^3$ is not open; you cannot move in the z direction and stay in the (x, y)-plane. It is closed because its complement is open: any point in $\{\mathbb{R}^3 - (x, y)\text{-plane}\}$ can be surrounded by an open 3-dimensional ball in $\{\mathbb{R}^3 - (x, y)\text{-plane}\}$.

b. The set $\mathbb{R} \subset \mathbb{C}$ is not open: the ball of radius $\epsilon > 0$ around a real number x always contains the non-real number $x + i\epsilon/2$. It is closed because its complement is open; if $z = x + iy \in \{\mathbb{C} - \mathbb{R}\}$, i.e., if $y \neq 0$, then the ball of radius $|y|/2$ around z is contained in $\{\mathbb{C} - \mathbb{R}\}$.

c. The line $x = 5$ in the (x, y)-plane is closed; any point in its complement can be surrounded by an open ball in the complement.

d. The set $(0, 1) \subset \mathbb{C}$ is not open since (for example) the point $0.5 \in \mathbb{R}$ cannot be surrounded by an open ball in $\mathbb{R}$. It is not closed because its complement is not open. For example, the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}$, cannot be surrounded by an open ball in $\{\mathbb{C} - (0, 1) \subset \mathbb{C}\}$.

e. $\mathbb{R}^n \subset \mathbb{R}^n$ is open. It is also closed, because its complement, the empty set, is trivially open.

f. The unit 2-sphere $S \subset \mathbb{R}^3$ is not open: if $\mathbf{x} \in S^2$ and $\epsilon > 0$, then the point $(1 + \epsilon/2)\mathbf{x}$ is in $B_\epsilon(\mathbf{x})$ but not on S^2. It is closed, since its complement is open: if $\mathbf{y} \notin S^2$, i.e., if $|\mathbf{y}| \neq 1$, then the open ball $B_{||\mathbf{y}|-1|/2}(\mathbf{y})$ does not intersect S^2.

1.5.3 a. Suppose $A_i, i \in I$ is some collection (probably infinite) of open sets. If $\mathbf{x} \in \bigcup_{i \in I} A_i$, then $\mathbf{x} \in A_j$ for some j, and since A_j is open, there exists $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset A_j$. But then $B_\epsilon(\mathbf{x}) \subset \bigcup_{i \in I} A_i$.

Solution 1.5.3, part b: There is a smallest ϵ_i, because there are finitely many of them, and it is positive. If there were infinitely many, then there would be a greatest lower bound, but it could be 0.

Part c: In fact, every closed set is a countable intersection of open sets.

b. If $A_1, \dots, A_j$ are open and $\mathbf{x} \in \cap_{i=1}^k A_i$, then there exist $\epsilon_1, \dots, \epsilon_k > 0$ such that $B_{\epsilon_i}(\mathbf{x}) \subset A_i$, for $i = 1, \dots, k$. Set ϵ to be the smallest of $\epsilon_1, \dots, \epsilon_k$. Then $B_\epsilon(\mathbf{x}) \subset B_{\epsilon_i}(\mathbf{x}) \subset A_i$.

c. The infinite intersection of open sets $(-1/n, 1/n)$, for $n = 1, 2, \dots$, is not open; as $n \to \infty$, $-1/\infty \to 0$ and $1/\infty \to 0$; the set $\{0\}$ is not open.

1.5.4 a. Suppose $U \subset A$ is open. Then for all $\mathbf{x} \in U$, there exists $r > 0$ such that

$$B_r(\mathbf{x}) \subset U, \quad \text{hence} \quad B_r(\mathbf{x}) \subset A,$$

so $\mathbf{x}$ is in the interior of A. Clearly the interior of A is open, and since it contains as a subset every open set contained in A, it is the biggest open set contained in A.

b. We are assuming that A is a subset of $\mathbb{R}^n$. The definition of the closure says that $\mathbb{R}^n - \overline{A}$ is open, so $\overline{A}$ is the complement of an open set, so it is closed. Moreover, if $B \subset \mathbb{R}^n$ is any closed set containing A, then $\mathbb{R}^n - B$ is an open set not intersecting A, so $\mathbb{R}^n - B$ is contained in the interior of $\mathbb{R}^n - A$, so it is contained in $\mathbb{R}^n - \overline{A}$ by part a, so $B \supset \overline{A}$.

c. Suppose $\mathbf{x} \in \overline{A}$ but $\mathbf{x} \notin A$. Then every neighborhood of $\mathbf{x}$ contains parts of A (because $\mathbf{x} \in \overline{A}$) and parts not in A (for instance, $\mathbf{x}$ itself), so $\mathbf{x}$ is in the boundary of A.

d. If $\mathbf{x} \in \partial A$, then certainly $\mathbf{x} \in \overline{A}$, but $\mathbf{x} \notin \mathring{A}$, since for all $r > 0$, we have $B_r(\mathbf{x}) \cap (\mathbb{R}^n - A) \neq \phi$. Conversely, if $\mathbf{x}$ is in $\overline{A} - \mathring{A}$, then for all $r > 0$, we have $B_r(\mathbf{x}) \not\subset A$, since $\mathbf{x} \notin \mathring{A}$, and $B_r(\mathbf{x}) \cap A \neq \phi$, since $\mathbf{x} \in \overline{A}$.

1.5.7 a. The natural domain is $\mathbb{R}^2$ minus the union of the two axes; it is open.

b. The natural domain is that part of $\mathbb{R}^2$ where $x^2 > y$ (i.e., the area "outside" the parabola of equation $y = x^2$). It is open since its "fence" x^2 belongs to its neighbor.

c. The natural domain of $\ln \ln x$ is $\{x | x > 1\}$, since we must have $\ln x > 0$. This domain is open.

d. The natural domain of arcsin is $[-1, 1]$. Thus the natural domain of $\arcsin \frac{3}{x^2+y^2}$ is $\mathbb{R}^2$ minus the open disc $x^2 + y^2 < 3$. Since this domain is the complement of an open disc it is closed (and not open, since it isn't $\mathbb{R}^2$ or the empty set).

e. The natural domain is all of $\mathbb{R}^2$, which is open.

f. The natural domain is $\mathbb{R}^3$ minus the union of the three coordinate planes of equation $x = 0$, $y = 0$, $z = 0$; it is open.

1.5.14 a. The functions x and y both are continuous on $\mathbb{R}^2$, so they have limits at all points. Hence so does $x+y$ (the sum of two continuous functions is continuous), and x^2 (the product of two continuous functions is continuous). The quotient of two continuous functions is continuous wherever the denominator is not 0, and $x+y=3$ at $\begin{bmatrix}1\\2\end{bmatrix}$. So the limit exists, and is $1/3$.

b. The denominator vanishes at $\begin{pmatrix}0\\0\end{pmatrix}$. If we let $x=y=t\neq 0$, the function becomes

$$\frac{t\sqrt{|t|}}{2t^2}=\frac{1}{2\sqrt{|t|}}.$$

Evidently this can be made arbitrarily large by taking $|t|$ sufficiently small. Strictly speaking, this shows that the limit does not exist, but sometimes one allows infinite limits. Is the limit ∞? No, because $f\begin{pmatrix} t \\ 0 \end{pmatrix} = 0$, so there also are points arbitrarily close to the origin where the function is zero. So there is no value, even ∞, which the function is close to when $\left|\begin{pmatrix} x \\ y \end{pmatrix}\right|$ is small (i.e., the distance from $\begin{pmatrix} x \\ y \end{pmatrix}$ to the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\left|\left[\begin{matrix} x-0 \\ y-0 \end{matrix}\right]\right|$, is small).

c. This time, if we approach the origin along the diagonal, we get

$$f\begin{pmatrix} t \\ t \end{pmatrix} = \frac{|t|}{\sqrt{2}|t|} = \frac{1}{\sqrt{2}},$$

whereas if we approach the origin along the axes, the function is zero, and the limit is zero. Thus the limit does not exist.

d. This is no problem: x^2 is continuous everywhere, y^3 is continuous everywhere, -3 is continuous everywhere, the sum is continuous everywhere, and the limit exists, and is 6.

1.5.19 a. Suppose $I - A$ is invertible, and write

$$I - A + C = (I - A) + C(I - A)^{-1}(I - A) = \big(I + C(I - A)^{-1}\big)(I - A),$$

so

$$\begin{aligned}(I - A + C)^{-1} &= (I - A)^{-1}\Big(I + C(I - A)^{-1}\Big)^{-1} \\ &= (I - A)^{-1}\Bigg(\underbrace{I - (C(I - A)^{-1}) + (C(I - A)^{-1})^2 - (C(I - A)^{-1})^3 + \cdots}_{\text{geometric series if } |C(I-A)^{-1}|<1}\Bigg)\end{aligned}$$

so long as the series is convergent. By Proposition 1.5.38, this will happen if

$$|C(I - A)^{-1}| < 1, \quad \text{in particular if} \quad |C| < \frac{1}{|(I - A)^{-1}|}.$$

Thus every point of U is the center of a ball contained in U.

For the second part of the question, the matrices

$$C_n = \begin{bmatrix} 1 - 1/n & 0 \\ 0 & 1 - 1/n \end{bmatrix}, \quad n = 1, 2, \dots$$

converge to I, and C_n is in U since $I - C_n = \begin{bmatrix} 1/n & 0 \\ 0 & 1/n \end{bmatrix}$ is invertible.

b. Simply factor: $(A + I)(A - I) = A^2 + A - A - I = A^2 - I$, so

$$(A^2 - I)(A - I)^{-1} = (A + I)(A - I)(A - I)^{-1} = A + I,$$

which converges to $2I$ as $A \to I$.

c. Showing that V is open is very much like showing that U is open (part a). Suppose $B - A$ is invertible, and write

$$B - A + C = (I + C(B - A)^{-1})(B - A),$$

so

$$\begin{aligned}(B - A + C)^{-1} &= (B - A)^{-1}\big(I + C(B - A)^{-1}\big)^{-1} \\ &= (B - A)^{-1}\Big(I - (C(B - A)^{-1}) + (C(B - A)^{-1})^2 - (C(B - A)^{-1})^3 + \cdots\Big)\end{aligned}$$

so long as the series is convergent. This will happen if

$$|C(B - A)^{-1}| < 1, \quad \text{in particular, if} \quad |C| < \frac{1}{|(B - A)^{-1}|}.$$

Thus every point of V is the center of a ball contained in V. Again, the matrices

$$\begin{bmatrix} 1+1/n & 0 \\ 0 & -1+1/n \end{bmatrix}, \quad n = 1, 2, \dots$$

do the trick.

d. This time, the limit does not exist. Note that you cannot factor $A^2 - B^2 = (A+B)(A-B)$ if A and B do not commute.

First set

$$A_n = \begin{bmatrix} 1/n+1 & 1/n \\ 0 & -1+1/n \end{bmatrix}.$$

Solution 1.5.19, part d: You may wonder how we came by the matrices A_n; we observed that

$$B\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

so these matrices do not commute.

Then

$$A_n^2 - B^2 = \begin{bmatrix} 2/n + 1/n^2 & 2/n^2 \\ 0 & -2/n + 1/n^2 \end{bmatrix} \quad \text{and} \quad (A-B)^{-1} = \begin{bmatrix} n & -n \\ 0 & n \end{bmatrix}.$$

Thus we find

$$(A_n^2 - B^2)(A_n - B)^{-1} = \begin{bmatrix} 2+1/n & -2+1/n \\ 0 & -2+1/n \end{bmatrix} \to \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix}$$

as $n \to \infty$.

Do the same computation with $A'_n = \begin{bmatrix} 1/n+1 & 0 \\ 0 & -1+1/n \end{bmatrix}$. This time we find

$$({A'}_n^2 - B^2)({A'}_n - B)^{-1} = \begin{bmatrix} 2+1/n & 0 \\ 0 & -2+1/n \end{bmatrix} \to \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = 2B$$

as $n \to \infty$.

Since both sequences $n \mapsto A_n$ and $n \mapsto A'_n$ converge to B, this shows that there is no limit.

1.5.20 a. The powers of A are

$$A^2 = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix}, \; A^3 = \begin{bmatrix} 4a^3 & 4a^3 \\ 4a^3 & 4a^3 \end{bmatrix}, \; \dots, \; A^n = \begin{bmatrix} 2^{n-1}a^n & 2^{n-1}a^n \\ 2^{n-1}a^n & 2^{n-1}a^n \end{bmatrix}.$$

For this sequence of matrices to converge to the zero matrix, each entry must converge to 0. This will happen if $|a| < 1/2$ (see Example 0.5.6). The sequence will also converge if $a = 1/2$; in that case the sequence is constant.

b. Exactly as above,

$$\begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}^n = \begin{bmatrix} 3^{n-1}a^n & 3^{n-1}a^n & 3^{n-1}a^n \\ 3^{n-1}a^n & 3^{n-1}a^n & 3^{n-1}a^n \\ 3^{n-1}a^n & 3^{n-1}a^n & 3^{n-1}a^n \end{bmatrix},$$

so the sequence converges to the 0 matrix if $|a| < 1/3$; it converges when $a = 1/3$ because it is a constant sequence. For an $m \times m$ matrix filled with a's, the same computation shows that A^n will converge to 0 if $|a| < 1/m$. It will converge when $a = 1/m$ because it is a constant sequence.