A function of class C^ω, also called a real analytic function, is a function which equals its Taylor expansion

$$f(x) = f(a) + (x-a)f'(a) + (x-a)^2\frac{f''(a)}{2!} + \ldots + (x-a)^n\frac{f^{(n)}(a)}{2!} + \ldots.$$

We sometimes incorrectly learn that every function of class C^∞ equals its Taylor series, but this is not the case as we will show below.

(a) Consider the function

$$f(x) = \begin{cases} exp(\frac{1}{x}) & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that $f'(0) = 0$, directly from the definition.

(b) Write down a formula for $f'(x)$ which is valid for all values of x.

(c) Now consider the formula you found in (ii) and differentiate once again. First, show that $f''(0) = 0$ by using the definition of the derivative. Then, write down a formula for $f''(x)$ which is valid for all x.

(d) Do the same for third derivatives. You should be able to prove $f'''(0) = 0$.

(e) In fact, continuing in this fashion you will see that $f^{(n)}(0) = 0$ for all n. Explain that this implies that f is not real analytic, i.e., it does not equal its Taylor series at $a = 0$.