(a) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz if there exists a constant $M > 0$ such that, for all real numbers x, y, we have

$$| f(x) - f(y) | \quad \leq M \mid x - y \mid .$$

Use the mean value theorem to show that if f is differentiable and f' is bounded, then f is Lipschitz. Show that any Lipschitz function is uniformly continuous but not necessarily differentiable.

(b) Extra credit: Prove that a subset $C \subset \mathbb{R}^n$ is compact if and only if (one of) the following equivalent statements hold

   (i) If there exists a collection $\{U_i \mid i \in \mathbb{N}\}$ of open sets such that $C \subset \cup_i U_i$, then there exists a finite collection $\{U_{i_1}, \ldots, U_{i_l}\}$ such that $C \subset \cup_j U_{i_j}$.

   (ii) If there exists a collection $\{B_{r_i}(v_i) \mid i \in \mathbb{N}\}$ of open balls such that $C \subset \cup_i B_{r_i}(v_i)$, then there exists a finite collection $\{B_{r_{i_1}}(v_{i_1}), \ldots, B_{r_{i_l}}(v_{i_l})\}$ such that $C \subset \cup_j B_{r_{i_j}}(v_{i_j})$.