MATH 31BH

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January 27 2023: Preparation for the first midterm

The first midterm will cover the material of Section 1.5, Section 1.6 with the exception of the Fundamental Theorem of Algebra, and Section 1.7 up to (not including) directional derivatives.

Please explain or prove all your assertions and show your work. You are allowed to use all results proved in Sections 1.5, 1.6 and 1.7 or in class, unless the problem specifically says you cannot use a particular result. When using a result, please state it precisely before using it.

(1) Give the following definitions
   (a) an open ball in $\mathbb{R}^n$
   (b) a closed ball in $\mathbb{R}^n$
   (c) an open subset of $\mathbb{R}^n$
   (d) a closed subset of $\mathbb{R}^n$
   (e) the boundary of a subset of $\mathbb{R}^n$
   (f) the interior of a subset of $\mathbb{R}^n$
   (g) a bounded subset of $\mathbb{R}^n$
   (h) a compact subset of $\mathbb{R}^n$
   (i) a neighborhood of a point of $\mathbb{R}^n$
   (j) the closure of a subset of $\mathbb{R}^n$
   (k) a convergent sequence of points of $\mathbb{R}^n$
   (l) a subsequence of a sequence
   (m) limit of a function
   (n) a continuous function
   (o) a uniformly continuous function
   (p) the supremum of a function
   (q) the maximum of a function
   (r) partial derivatives
   (s) Jacobian matrix
   (t) total derivative

(2) Prove the following

(a) The limit of a sequence, if it exits, is unique. The limit of a function at any given point v_0 is unique.

(b) A set C is closed if and only if the limits of all convergent sequences in C belong to C.

(c) If a sequence $i \mapsto v_i$ converges to a, then any subsequence of $i \mapsto v_i$ also converges to a.

(d) For $X \subset \mathbb{R}^n$, a function $f : X \to \mathbb{R}^m$ is continuous at $v_0 \in X$ if and only if, for all sequences $i \mapsto v_i$ with limit v_0, we have

$$\lim_{i\to\infty} f(v_i) = f(v_0).$$

(e) Prove that a continuous function on a compact set has a maximum and a minimum.

(3) Prove that if a function $f : X \to \mathbb{R}^m$ has a limit when v approaches $v_0 \in X$, then there exists a neighborhood of v_0 (or an open ball centered at v_0) on which $|f(v)|$ is bounded.

(4) Prove that if a sequence has a limit, then it is bounded.

(5) Prove by induction that for any strictly increasing function $g : \mathbb{N} \to \mathbb{N}$, we have $g(m) \geq m$ for all $m \in \mathbb{N}$.

(6) Prove that if a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point v with derivative L, then the Jacobian matrix of f at v represents the linear transformation L.

(7) State and prove the mean value theorem.

(8) Review all homework problems.