Examples: $M_a :=$ sphere of radius a in $\mathbb{R}^3$ centered at 0. $a>0$

$$M_a = Z(x^2+y^2+z^2-a^2)$$

$f(x,y,z) = xyz$ continuous on M_a, M_a is compact and f is continuous (in fact C^∞), so f has a global minimum and a global maximum on M_a.

At these points we have $D(f|_{M_a}) = 0$

The theorem about Lagrange multipliers tells us that $D(f|_{M_a}) = 0$ exactly when $Df = \lambda DF$ where $F = x^2+y^2+z^2-a^2$ is the equation of M_a.

We compute the Jacobian matrices:

$$Jf(x,y,z) = \begin{pmatrix} yz & xz & xy \end{pmatrix}$$

$$JF(x,y,z) = \begin{pmatrix} 2x & 2y & 2z \end{pmatrix}$$

$Jf = \lambda JF$ means $\begin{pmatrix} yz & xz & xy \end{pmatrix} = \lambda \begin{pmatrix} 2x & 2y & 2z \end{pmatrix}$

we obtain 3 equations $\begin{cases} yz = 2\lambda x \\ xz = 2\lambda y \\ xy = 2\lambda z \end{cases}$

$$xyz = 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$$

if $\lambda = 0$ (when we have a critical point of f in $\mathbb{R}^3$)

then $yz = xz = xy = 0$

If $z = 0$, then either $x = 0$ or $y = 0$

If $z \neq 0$, then $x = y = 0$

So we have $x = y = 0$ or $y = z = 0$ or $x = z = 0$

So the three axes are critical points for f.

The intersections of these axes with M_a will give us 6 critical points of f on the sphere.

Now assume $\lambda \neq 0$, then we have $x^2 = y^2 = z^2$

plug into F: $x^2 + y^2 + z^2 = a^2 \Rightarrow 3x^2 = a^2$

$$\Rightarrow \quad x = \pm \frac{1}{\sqrt{3}} a, \quad y = \pm \frac{1}{\sqrt{3}} a, \quad z = \pm \frac{1}{\sqrt{3}} a$$

Now plug the critical points into f to find the maximum and minimum of f on M_a:

$$f(0,0,\pm a) = f(0,\pm a,0) = f(\pm a,0,0) = 0$$

$$f\left(\pm\frac{1}{\sqrt{3}}a, \pm\frac{1}{\sqrt{3}}a, \pm\frac{1}{\sqrt{3}}a\right) = \pm\frac{1}{3\sqrt{3}}a^3$$

So the minimum of f on M_a is $\frac{-1}{3\sqrt{3}}a^3$

and the maximum of f on M_a is $\frac{1}{3\sqrt{3}}a^3$.

The minimum is reached at $\left(\frac{-1}{\sqrt{3}}a, \frac{1}{\sqrt{3}}a, \frac{1}{\sqrt{3}}a\right)$

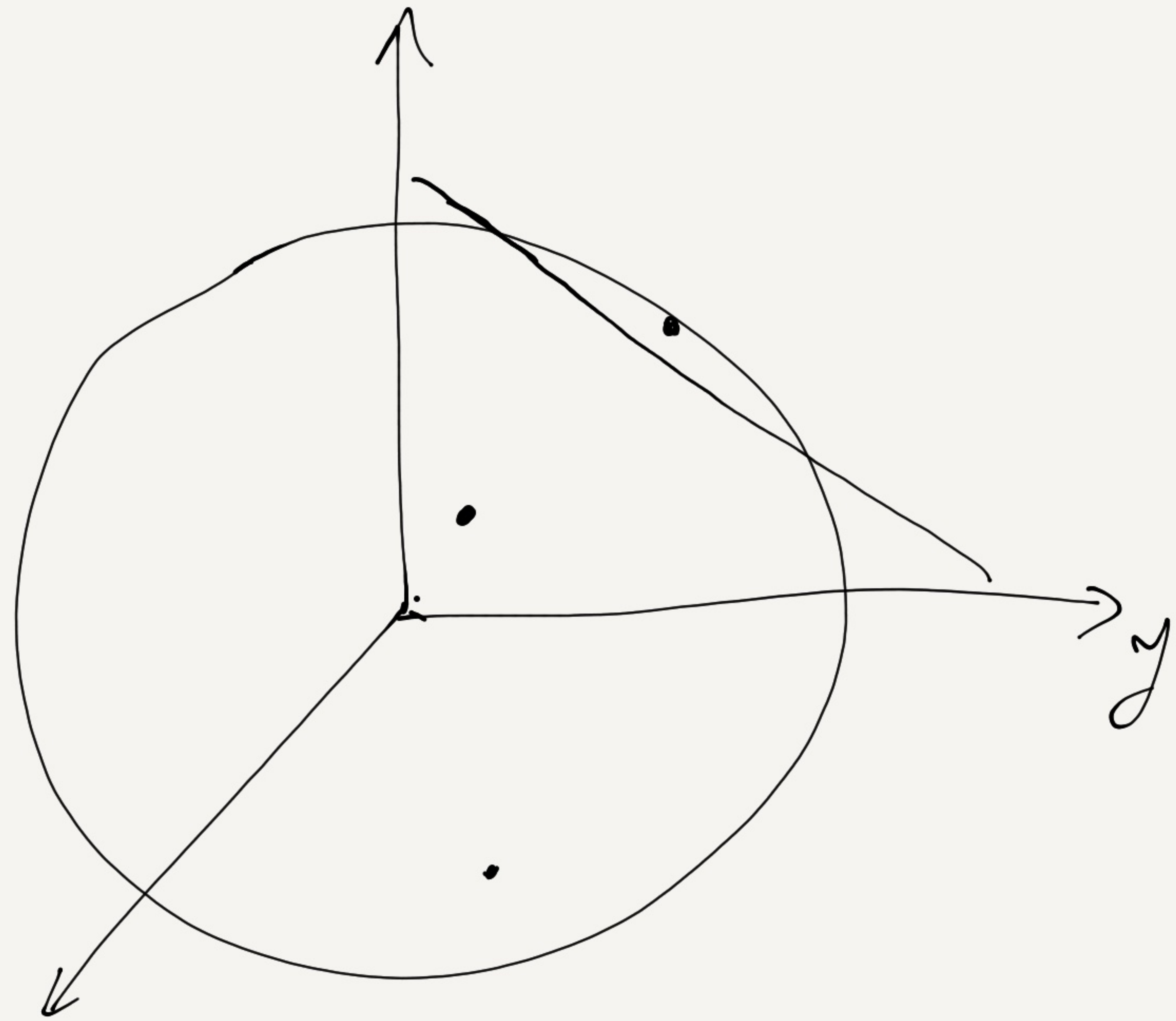
$\left(\frac{1}{\sqrt{3}}a, -\frac{1}{\sqrt{3}}a, \frac{1}{\sqrt{3}}a\right)$ $\left(\frac{1}{\sqrt{3}}a, \frac{1}{\sqrt{3}}a, -\frac{1}{\sqrt{3}}a\right)$

$\left(-\frac{1}{\sqrt{3}}a, -\frac{1}{\sqrt{3}}a, -\frac{1}{\sqrt{3}}a\right)$

And similarly for the maximum

Second example:

$f(x,y,z) = xyz$ find the extrema on a different region described below.



cut the sphere with the plane of equation $x+y+z=b$ look at the part of the sphere which is above this plane. This region is defined by two conditions:

$$X_{a,b} := \begin{cases} x^2+y^2+z^2-a^2=0 \\ x+y+z \geq b \end{cases}$$

still compact

So f has a minimum and a maximum on $X_{a,b}$.

Take $b = \frac{3}{2}a$ to make the problem interesting.

We need to distinguish two cases:

$$\begin{cases} \text{extrema in the interior } \mathring{X}_{a,\frac{3a}{2}} \\ \text{extrema on the boundary } \partial X_{a,\frac{3a}{2}} \end{cases}$$

$\mathring{X}_{a,\frac{3a}{2}}$ = open set $\cap$ X_a is a manifold
so we find the critical points of f on $\mathring{X}_{a,\frac{3a}{2}}$
to find the extrema.

$\partial X_{a,\frac{3a}{2}}$ is defined by two equations:

$$\begin{cases} x^2+y^2+z^2-a^2=0 \\ x+y+z-\frac{3a}{2}=0 \end{cases}$$

$$G : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x^2+y^2+z^2-a^2 \\ x+y+z-\frac{3a}{2} \end{pmatrix} = \begin{pmatrix} G_1 = F \\ G_2 \end{pmatrix}$$

critical points of f on $\partial X_{a,\frac{3a}{2}}$ can be found using Lagrange multipliers for G and f:

$$Jf = \lambda_1 JG_1 + \lambda_2 JG_2$$

$$JG_2 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} yz & xz & xy \end{pmatrix} = \lambda_1 \begin{pmatrix} 2x & 2y & 2z \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$\begin{cases} yz = 2\lambda_1 x + \lambda_2 \\ xz = 2\lambda_1 y + \lambda_2 \\ xy = 2\lambda_1 z + \lambda_2 \end{cases}$$

subtract the last equation from the first two:

$$yz - xy = 2\lambda_1 (x - y)$$

$$\begin{cases} y(z-x) = 2\lambda_1 (x-z) \\ z(x-y) = 2\lambda_1 (y-x) \\ x(y-z) = 2\lambda_1 (z-y) \end{cases} \quad \text{or} \quad \begin{cases} (2\lambda_1 + y)(x-z) = 0 \\ (2\lambda_1 + z)(y-x) = 0 \\ (2\lambda_1 + x)(z-y) = 0 \end{cases}$$

So $\begin{cases} \text{either } & 2\lambda_1 + y = 0 \text{ or } x - z = 0 \\ \quad '' & 2\lambda_1 + z = 0 \text{ or } y - x = 0 \\ \quad '' & 2\lambda_1 + x = 0 \text{ or } z - y = 0 \end{cases}$ eight cases.

For simplicity, we take $a = 1$.

We also have the two equations $\begin{cases} x + y + z = \frac{3}{2} & (A) \\ x^2 + y^2 + z^2 = 1 & (B) \end{cases}$

Own 8 cases:

(1) $x = y = z = -2\lambda_1$ equation (A) gives $x = \frac{1}{2}$

$x = y = z = \frac{1}{2}$ this does not satisfy equation (B)

So this case does not happen.

(2) $x = y = z$, as above (A) gives $x = y = z = \frac{1}{2}$ and this does not satisfy (B)

(3) $y = z = -2\lambda_1$ and $y = z$

equation (A) gives $x + 2y = \frac{3}{2}$, (B) gives $x^2 + 2y^2 = 1$

substitute $x = \frac{3}{2} - 2y$ into $x^2 + 2y^2 = 1$:

$$\left(\frac{3}{2} - 2y\right)^2 + 2y^2 = 1 \qquad \frac{9}{4} - 6y + 4y^2 + 2y^2 - 1 = 0$$

$$6y^2 - 6y + \frac{5}{4} = 0 \qquad y^2 - y + \frac{5}{24} = 0 \qquad \left(y - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{5}{24} = 0$$

$$\left(y - \frac{1}{2}\right)^2 - \frac{1}{24} = 0 \qquad y = \frac{1}{2} \pm \frac{1}{2\sqrt{6}}$$

So. $x = \frac{3}{2} - 2y$, $y = z = \frac{1}{2} \pm \frac{1}{2\sqrt{6}}$

we have two solutions, $x = \frac{1}{2} - \frac{1}{\sqrt{6}}$, $y = \frac{1}{2} + \frac{1}{2\sqrt{6}} = z$

$x = \frac{1}{2} + \frac{1}{\sqrt{6}}$, $y = \frac{1}{2} - \frac{1}{2\sqrt{6}} = z$

At the first point, called P_1

$$f(P_1) = \left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right)\left(\frac{1}{2} + \frac{1}{2\sqrt{6}}\right)^2$$

At the second point, Q_1: $f(Q_1) = \left(\frac{1}{2} + \frac{1}{\sqrt{6}}\right)\left(\frac{1}{2} - \frac{1}{2\sqrt{6}}\right)^2$

Case 4: $y = -2\lambda_1$, $x = y = z$ this is the same as case 1 and does not happen.

The remaining cases are similar to the above, we permute the variables x, y, z. This will produce 4 more points: $P_2 = \left(\frac{1}{2} + \frac{1}{2\sqrt{6}}, \frac{1}{2} - \frac{1}{\sqrt{6}}, \frac{1}{2} + \frac{1}{2\sqrt{6}}\right)$, $Q_2 = \left(\frac{1}{2} - \frac{1}{2\sqrt{6}}, \frac{1}{2} + \frac{1}{\sqrt{6}}, \frac{1}{2} - \frac{1}{2\sqrt{6}}\right)$

$P_3 = \left(\frac{1}{2} + \frac{1}{2\sqrt{6}}, \frac{1}{2} + \frac{1}{2\sqrt{6}}, \frac{1}{2} - \frac{1}{\sqrt{6}}\right)$ $Q_3 = \left(\frac{1}{2} - \frac{1}{2\sqrt{6}}, \frac{1}{2} - \frac{1}{2\sqrt{6}}, \frac{1}{2} + \frac{1}{\sqrt{6}}\right)$

One of the points on the sphere where f reaches a maximum is on the region $X_{1,\frac{3}{2}}$ that we are interested in.

So the maximum of f on $X_{1,\frac{3}{2}}$ is equal to its maximum on the whole sphere M_1.

The minimum of f on $X_{1,\frac{3}{2}}$ is reached on the boundary $\partial X_{1,\frac{3}{2}}$ at one of the points P_i or Q_i.

So the minimum is the smaller of

$$f(P_i) = \left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right)\left(\frac{1}{2} + \frac{1}{2\sqrt{6}}\right)^2 \quad \text{and} \quad \left(\frac{1}{2} + \frac{1}{\sqrt{6}}\right)\left(\frac{1}{2} - \frac{1}{2\sqrt{6}}\right)^2 = f(Q_i)$$

$$\begin{array}{c|c}
\| & \| \\
\frac{1}{8}\left(1 - \frac{2}{\sqrt{6}}\right)\left(1 + \frac{1}{\sqrt{6}}\right)^2 & \frac{1}{8}\left(1 + \frac{2}{\sqrt{6}}\right)\left(1 - \frac{1}{\sqrt{6}}\right)^2 \\
\frac{1}{8}\left(1 - \frac{2}{\sqrt{6}}\right)\left(1 + \frac{2}{\sqrt{6}} + \frac{1}{6}\right) & \frac{1}{8}\left(1 + \frac{2}{\sqrt{6}}\right)\left(1 - \frac{2}{\sqrt{6}} + \frac{1}{6}\right) \\
\frac{1}{8}\left(1 - \frac{2}{\sqrt{6}}\right)\left(\frac{7}{6} + \frac{2}{\sqrt{6}}\right) & \frac{1}{8}\left(1 + \frac{2}{\sqrt{6}}\right)\left(\frac{7}{6} - \frac{2}{\sqrt{6}}\right) \\
\frac{1}{8}\left(\frac{7}{6} + \frac{2}{\sqrt{6}} - \frac{7}{3\sqrt{6}} - \frac{2}{3}\right) & \frac{1}{8}\left(\frac{7}{6} - \frac{2}{\sqrt{6}} + \frac{7}{3\sqrt{6}} - \frac{2}{3}\right)
\end{array}$$

$$\frac{1}{8}\left(1 - \frac{1}{3\sqrt{6}}\right) < \frac{1}{8}\left(1 + \frac{1}{3\sqrt{6}}\right)$$

This minimum is reached at the points P_1, P_2, P_3.